Polarized radiative transfer in optically active light scattering media

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1. Introduction

The disperse media composed of non-spherical particles (say, dust aerosols layers, and ice crystal clouds) can appear both optically isotropic and optically anisotropic, depending on local optical characteristics of turbid medium in question and also on the orientation of particles.

Chiral media belong to the type of optically anisotropic media that is characterized by circular birefringence and circular dichroism (different medium refractive index and different absorption of left-handed and right-handed circularly polarized radiation). The media can be composed either of spherical particles consisting of optically active matter or of particles of special shape (non-spherical shape with broken mirror symmetry). In the Earth atmosphere remote sensing problems the anisotropic media can be produced by ensembles of non-spherical aerosol particles, and ice crystals. The anisotropic media can also arise in the situations when aerosol contains a mixed combination of organic and inorganic particles, and the biological aerosol component dominates.

Polarization characteristics of scattered radiation can provide a valuable information on medium optical properties and medium miscrostructure. The adequate interpretation of optical and scattering characteristics of optically anisotropic media can be done on the solid ground of the polarized radiation transport theory in optically anisotropic media (using the vector radiation transport equation, the VRTE, for anisotropic media). For the retrieval algorithms, based on the inverse radiation transfer problem solutions, the data of multi-angular and multi-spectral measurements of the Stokes vector of back-scattered solar radiation are used. Usually a kind of statistically optimized problem solution is applied (relied either on the usage of look-up-tables or on direct radiative transfer calculations). An accurate accounting of terrestrial underlying surface reflectance is also quite essential. Realistic models of disperse media play a significant role in the construction of the retrieval algorithms.

A detailed overview of recently developed aerosol retrieval algorithms, based on measurements of back-scattered polarized radiation, is given in (Kokhanovsky, 2015). The proper computational algorithms and codes for accurate VRTE solution are necessary for realization of the developed retrieval procedures. Among a variety of the developed RT-codes, the codes developed in (Katsev, 2009; Kokhanovsky, 2010; Cairns, 2010; Cheng, 2010; Dubovik, 2011; Hasekamp, 2011; Knobelspiesse, 2011) should be marked. The TR-code, developed in (Nikolaeva et al, 2007; Bass et al., 2009; 2010) for the VRTE solution in 3D cylindrical geometry and successfully tested in a number of atmosphere remote sensing problems, could be also mentioned as one of available codes for extension to radiation transfer problems for optically anisotropic media.

The overview of main topics considered in this review is given below. They are related to various aspects of polarized radiation transfer processes in optically anisotropic media

In section 2 the essential steps of vector transport equation deriving from the system of Maxwell equations for the problem of electromagnetic radiation multiple scattering by an ensemble of discrete isolated scatterers are outlined. The attention is paid on the set of restrictions imposed on the system ensemble of scatterers – radiation field in the process of transport equation deriving. The work on the topic was started long ago (Foldy, 1945; Lax, 1951; Watson, 1969) and finally allowed to obtain the matrix and the vector transport equations for optically anisotropic media (Gnedin et al, 1969, 1970; Dolginov et al.,1970; Kuzmina, 1976, 1986, 1987, 1989, 1991; Zege et al, 1984; Kokhanovsky, 1999 a), b); Kokhanovsky, 2000). Subsequently the strict and detailed way of the VRTE derivation, realized in (Mishchenko et al, 2006), allowed to additionally study the phenomenon of medium coherent backscattering (CB) (otherwise known as weak localization of electromagnetic waves). Because the four-component vector transport equation can be correctly used for radiation transport problems only in the case of weakly anisotropic media, some necessary information on quasi-isotropic approximation of geometrical optics for weakly anisotropic media is included as well (section 2.3).

In section 3 the peculiarities of radiation transfer processes in anisotropic optically active media are discussed. The characteristic features of the vector transport equation for optically active media are the matrix extinction operator (that can be expressed in terms of medium refraction indices), and the integral operator of scattering, defined by the non-block-diagonal phase matrix of special type), (section 3.1). The main properties of radiation transport problems for slabs of optically active media (including boundary conditions) are marked in sections 3.2 and 3.3. The polarization characteristics of coherently scattered (refracted and attenuated) radiation propagating in slabs of optically active media, that can be obtained analytically, are presented in sections 3.4 and 3.5. The transport problems for slabs with reflecting boundaries are discussed as well (section 3.6).

In section 4 the perturbation method developed for transport problems in slabs of weakly anisotropic optically active media is presented. The method can be used for the estimation of the total Stokes vector perturbation due to medium optical anisotropy (section 4.1). The example of estimation of transport problem solution perturbation for a slab of optically isotropic medium with scattering operator, specified by non-block-diagonal phase matrix, is given in section 4.3. Similarly the polarization characteristics perturbation due to utilizing of the transport equation with scalar extinction operator (instead of the matrix one, valid for optically anisotropic medium) could be estimated. The situation has already been encountered earlier in the study of multi-scattered polarized infrared radiation transport in anisotropic media formed by horizontally oriented ice crystals (Takano et al., 1993). As it was pointed out. (Mishchenko, 1994), the utilization of the VRTE with scalar extinction operator could provide a significant error in solution of the transport problems. The comparison of the exact and the approximate solutions of similar transport problem for another type of anisotropic medium model (composed of perfectly aligned prolate and oblate spheroids) has been fulfilled previously in (Tsang et al., 1991). And a significant discrepancy in solutions was demonstrated.

In section 5 the results on radiative transfer problems in anisotropic media related to the Earth atmosphere remote sensing are presented. First of all these were the problems for ice clouds (cirrus and cirrostratus), where disperse anisotropic media can be formed by spatially oriented suspended tiny ice crystals. The well-known atmospheric optical phenomenon of halo is just created by light reflection from these anisotropic media. Another familiar phenomenon is light pillars that is produced by light reflection from anisotropic media formed by column-shaped ice crystals. Modeling of radiative transfer in turbid anisotropic media requires for construction of the matrix extinction operator and the scattering phase matrix of the VRTE, governing radiation transport in anisotropic medium. The section 5.1 contains an overview of the papers where various models of disperse anisotropic media were designed and the operators of the VRTE were constructed. In particular, the disperse medium models, composed of chiral particles, were considered, and the extinction matrices for the media were constructed (Ablitt et al., 2006; Liu et al., 2013). The multiply scattered light transfer in the chiral anisotropic medium was studied via Monte Carlo simulations, and the effects of medium chirality were elucidated (Ablitt et al., 2006). The models allowed to study the dependence of medium scattering macro-characteristics on the medium micro-structure parameters. For some medium models the backscattering efficiencies were estimated as well (Mishchenko et al, 1992; Gao et al., 2012).

The Monte-Carlo simulations of radiation transfer processes in various optically anisotropic media models of ice clouds were performed (including the simulations of halos). Particularly, these results demonstrated, that the anisotropy of cloud medium can strongly affect the optical properties of crystalline clouds (Prigarin et al., 2005; 2007; 2008).

Densely packed disperse media (where the assumption concerning scatterer locations in far zones of each other is violated) are shortly reviewed in section 5.1 B). The ice and snow cover media, representing the interest for the Earth remote sensing problems, are usually modelled as random disperse media with densely packed particles (Kokhanovsky, 1998). Sometimes (mainly in the visible wavelength range) the snow layers may be also modelled as ice clouds consisting of separated fractal particles (Kokhanovsky, 2003; Liou et al., 2011). The deriving of the VRTE for densely packed disperse media is not a simple task. For that reason the exact computer studies (in terms of the Maxwell equations) of multiply scattered radiation transfer in densely packed media have been undertaken to clarify the conditions of the VRTE applicability. As it turned out, in a number of situations the qualitative agreement with the results of RT-calculations takes place (the corresponding references are included).

The features of radiative transfer phenomena in magnetoactive plasma are shortly reflected in section 5.2. The polarization states of the normal waves in this kind of anisotropic media are not orthogonal, in general. But normal waves are reduced to circular polarized waves (for lengthwise propagation) and to linear polarized waves (for transverse propagation). In appropriate parametric domain the magnetoactive plasma can possess a strong optical anisotropy.

Optically active media occurring in bio-medical field of research are touched in section 5.3. These media can be divided into two main classes – strongly scattering (turbid) and weakly scattering (transparent). The analysis of polarization characteristics of multiply scattered radiation in biological media is one the most important instruments for estimation of internal structure of the media. Chiral molecules are often enclosed in bio-tissues, and multi-scattered light depolarization measurements are widely used for estimation the concentrations of optically active molecules (such as glucose). So the design of adequate mathematical models of disperse bio-tissues is of importance. A closely related area of research concerns the application of radiation transport theory to the problems of non-invasive medical diagnostics of heterogeneities in biological tissues. The deterministic method of numerical transport equation solution for calculating the characteristics of multiply scattered light in the 3D-regions (instead of Monte-Carlo simulations) has been proposed in (Bass et al., 2009; Bass et al., 2010).

The sections 5.4 and 5.5 are devoted to some radiation transport problems for optically anisotropic media that are encountered in rather less popular application areas, such as multi-scattered radiation transport in bio-medical anisotropic media, liquid crystals, layered anisotropic media, in two-dimensional periodic optically anisotropic structures known as photonic crystals. Interesting phenomena of resonant radiation interaction with the medium can arise in the layered structures. The liquid crystals can demonstrate phase transitions of the second order and spontaneous symmetry breaking. Besides, the studying of radiative transfer processes in the multilayered anisotropic structures is of value in view of increasingly wide applications of these anisotropic media. Thus, the short overview of some topics in the research field can provide a glance at a new class of radiation transport problems where both theoretical and computational work are still at the very beginning.

2. Radiation transfer problems for disperse optically anisotropic media

2.1 The radiation transport equation for sparse disperse media derived from the Maxwell equations for an ensemble of scatterers

As well known, the radiation transport equation is the basis for calculation of various problems of radiation transfer in scattering media. Initially it was derived phenomenologically via considerations of energy balance for volume element of scattering and absorbing medium through which the radiation propagates (see, for instance, (Chandrasekhar, 1960), (Van de Hulst, 1957, 1980)). At the same time it was clearly understood, that for correct and comprehensive derivation of transport equation describing electromagnetic (polarized) radiation transfer it is necessary to study the underlying problem of classical statistical electrodynamics – the problem of multiple scattering of electromagnetic waves in disperse media, formed by ensembles of discrete isolated scatterers, the scattered radiation behavior being described by the Maxwell equations. The approach allows to formulate the full system of restrictions on the medium microstructure and the radiation field properties, and thus allows to reveal the relation between classical statistical electrodynamics and phenomenological radiative transfer theory. The applicability conditions of the classical radiative transfer equation can be also clarified in the way. The versions of the program have been successfully fulfilled in a variety of well known papers and monographs (see, for example, (Foldy, 1945; Lax, 1951; Watson, 1969; Gnedin et al, 1969; Borovoi, 1966 a),b), 1967 a),b), 1983, 2005, 2013; Barabanenkov et al, 1970; Dolginov et al, 1970; 1975; 1979; Barabanenkov, 1975; Kuzmina, 1976; Rytov et al, 1978; Ishimaru, 1978; Apresyan et al., 1983; Tsang et al., 2001; Mishchenko, 2002, 2003; Mishchenko et al., 2006; 2014)).

The problem of multiple scattering of classical electromagnetic radiation in a sparse disperse medium (an ensemble of N, $N \gg 1$, sparsely randomly distributed isolated macroscopic scatterers) is considered as the base problem for deriving of polarized radiation transport equation (the VRTE). The following natural set of restrictions is often admitted: 1) quasi-monochromatic incident radiation field is considered, and radiation scattering is supposed to occur without frequency redistribution; 2) the inequalities $\lambda \ll l$, $d_s \ll l$ are fulfilled, where l is the length of free radiation path between the acts of scattering, λ being the radiation wavelength, d_s being the average scatterer diameter; 2) each scatterer is located in the far-field zone of all the other scatterers, and so electromagnetic wave travelling from scatterer to scatterer can be assumed quasi-spherical at the vicinity of each scatterer; 3) the scatterer velocities are small in comparison with phase velocity of incident electromagnetic wave (in continuous transparent medium into which all the scatterers are embedded); 4) the scattering characteristics of all scatterers are stationary.

The system of Maxwell equations governing the process of multiple scattering of classical electromagnetic field by statistical ensemble of N macroscopic scatterers has been written down in various forms in many papers, listed above. If the scatterer centers are located at the spatial points specified by the radius-vectors $\mathbf{r}_1$, ... $\mathbf{r}_N$, the starting system of equation for the vector of electric field of the electromagnetic wave at spatial point $\mathbf{r}$ the can be written as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{inc}(\mathbf{r}) + \sum_{j=1}^{N} \mathbf{E}^{sc}(\mathbf{r}; \mathbf{r}_j), \tag{2.1.1}$$

where $\mathbf{E}^{inc}(\mathbf{r})$ is the vector of the electric field of incident wave, and $\mathbf{E}^{sc}(\mathbf{r}; \mathbf{r}_j)$ is the vector of the electric field of electromagnetic wave, scattered by j-th scatterer. It can be calculated via integration over the volume of space, occupied with the j-th scatterer, of the function $U_j(\mathbf{r})\mathbf{E}^{exc}(\mathbf{r}; \mathbf{r}_j)$, where $U_j(\mathbf{r})$ is the "potential" function for the j-th scatterer, defined by its refractive index, and $\mathbf{E}^{exc}(\mathbf{r}; \mathbf{r}_j)$ is the vector of electric field, acting on the scatterer (the "exciting" field). The accurate calculation of $\mathbf{E}^{exc}(\mathbf{r}; \mathbf{r}_j)$ requires the attraction of Lippmann–Schwinger integral equation. However, a simplified approach was developed in (Foldy, 1945; Lax, 1951; Watson, 1969) and was further used in (Gnedin et al, 1969; Barabanenkov et al, 1970, 1975; Kuzmina, 1976). Besides the natural restrictions on the ensemble of scatterers and radiation field, listed above, two additional significant approximations for radiation field were usually admitted:

- $\langle \mathbf{E}^{sc}(\mathbf{r}; \mathbf{r}') \rangle_{\mathbf{r}'} \approx \langle \mathbf{E}^{sc}(\mathbf{r}; \mathbf{r}') \rangle$ (the ensemble averaged radiation field in the vicinity of the scatterer, located at the point $\mathbf{r}'$, only slightly differs from the field which would exist there in the case if the scatterer is absent at the $\mathbf{r}'$);

- It is assumed, that the coherent scattering of electromagnetic waves in sparse disperse media takes place only in the exactly forward-scattering direction (the inequality $D_s \ll l$ is supposed to be satisfied, where D_s is the diameter of the whole medium volume, and l is the value of the radiation free path in the disperse medium).

These two approximations allow to obtain the system of equations for $\mathbf{E}(\mathbf{r}; \mathbf{r}_j)$ in the form (2.1.1), where $\mathbf{E}^{sc}(\mathbf{r}; \mathbf{r}_j)$ are expressed in terms of operators of scattering amplitude, $\hat{A}(\mathbf{r}; \mathbf{s}, \mathbf{s}')$, and the free-space Green function operators $\hat{G}(|\mathbf{r}-\mathbf{r}'|)$. Performing the configuration ensemble averaging, analogous to that performed in (Lax, 1951; Watson, 1969), the VRTE can be derived for the Stokes vector $\mathbf{I}(\mathbf{r},\mathbf{s}) = [I(\mathbf{r},\mathbf{s}), Q(\mathbf{r},\mathbf{s}), U(\mathbf{r},\mathbf{s}), V(\mathbf{r},\mathbf{s})]^{\mathrm{T}}$ in the form

$$(\mathbf{s}\cdot\nabla)\mathbf{I}(\mathbf{r},\mathbf{s}) + \hat{\sigma}(\mathbf{r},\mathbf{s})\mathbf{I}(\mathbf{r},\mathbf{s}) = (\hat{P}\mathbf{I})(\mathbf{r},\mathbf{s}) + \mathbf{F}(\mathbf{r},\mathbf{s}), \tag{2.1.2}$$

where $\hat{\sigma}(\mathbf{r})$ is the extinction matrix of medium volume element, $(\hat{P}\mathbf{I})(\mathbf{r},\mathbf{s})$ is the integral operator of scattering, defined by scattering phase matrix of the medium volume element, and $\mathbf{F}(\mathbf{r},\mathbf{s})$ is the Stokes vector of internal sources of radiation in the medium. The extinction operator $\hat{\sigma}(\mathbf{r},\mathbf{s})$ of the VRTE (2.1.2), can be expressed in terms of the refractive index operator, $\hat{n}(\mathbf{r},\mathbf{s})$, corresponding to the effective continuous transparent medium (in general, optically anisotropic), in which coherently scattered (i.e. refracted by the disperse medium) radiation propagates (see section 3.1). The operator $\hat{n}(\mathbf{r},\mathbf{s})$ is expressed in terms of the ensemble averaged operator of scattering amplitude in forward-scattering direction, $\langle \hat{A}(\mathbf{r};\, \mathbf{s},\mathbf{s})\rangle$ (we use the notations from (Kuzmina, 1976)):

$$\hat{n}^{2}(\mathbf{r},\mathbf{s}) = \hat{I} + 4\pi(\frac{\omega}{c})^{-2}\langle \hat{A}(\mathbf{r};\, \mathbf{s},\, \mathbf{s})\rangle. \qquad (2.1.3)$$

Since the effective medium, specifying the non-scattered radiation propagation, is usually turned out to be weakly optically anisotropic (due to the disperse medium sparsity), the quasi-isotropic approximation of geometrical optics for optically anisotropic media is applicable (see section 2.3). The main features of the VRTE for optically active media are described in section 3.

It is necessary to mention that the most consecutive, strict and detailed VRTE derivation has been later performed in the monograph (Mishchenko et al, 2006) (see also (Mishchenko, 2014; 2016). In (Mishchenko et al, 2006) the radiative transfer theory (RTT) is presented as a branch of classical macroscopic electromagnetics, and the detailed theory of multiple scattering of electromagnetic radiation in random discrete media composed of sparsely distributed particles is developed. The Foldy–Lax equations for *N*-particle ensemble of scatterers (which can be viewed as the basic equations of modern theory of multiple scattering in random particle ensembles) figure as a starting point for strict development of the phenomenological VRTE. The diagrammatic interpretation of the order-of-scattering expansion for the scattered radiative field is exploited. The Twersky approximation for the coherent radiation field (in the diagrammatic interpretation the approximation means that all scattering paths going through a particle two or more times are neglected) is used. Theoretically justified, ebsemble averaging procedures are used in the process of obtaining of statistically averaged (macroscopic) scattering and absorption characteristics of the disperse media.

As a result, the integral and the integro-differential versions of the VRTE are obtained in (Mishchenko et al, 2006). The coherency extinction matrix of the VRTE for the Stokes vector (that corresponds to $\hat{\sigma}(\mathbf{r},\mathbf{s})$ in Eq. (2.1.2)) is expressed in terms of the ensemble averaged components of the amplitude scattering matrix (denoted by $\hat{S}$) in forward-scattering direction. The scattering phase matrix of the VRTE is expressed in terms of sums of pairwise products of the $\hat{S}$ components. The properties of both coherent and diffuse (multiply scattered) radiation field are extensively discussed. In particular, the attention is paid to ladder approximation of diagrammatic approach (consisting in keeping of only a certain class of diagrams) and its physical interpretation. The diagrams with crossing connectors and their cumulative contribution are analyzed as well. In addition the helpful discussion on many problems concerning optical polarization measurements in the context of phenomenological RTT approach is provided.

The coherent backscattering (CB) (weak localization of electromagnetic waves), one of the most remarkable effects caused by multiply scattered radiation in a disperse medium, has been studied in detail in (Mishchenko et al, 2006) in the frames of theory of electromagnetic radiation multiple scattering by an ensemble of particles. The CB belongs to radiation interference phenomena. Arising as a result of interference of scattered waves in the exactly backscattering direction, the CB reveals itself in a narrow interference peak of intensity and is characterized by a specific behavior of polarization. In fact, the CB should be qualified as a mesoscopic physical phenomenon emerging as a result of correlation of multi-particle scatterer groups of a disperse medium (Sheng, 2006).

It is a significantly more difficult task to derive the VRTE for densely packed disperse media composed of large scatterers, where the assumption concerning scatterer locations in wave zones of each other is violated. Among a variety of approaches to treatment of radiation transport problems in dense particulate media we would like to mention the three ones. In some parametrical domains (for instance, in the case of media composed of moderate size scatterers studied in microwave spectral range) different procedures of replacement of the dispersed medium by a continuous one with an effective refractive index can be used ((Kokhanovsky, 1999 b); 2004; Tsang et al., 1985; 2001). A quasi-crystalline approximation represents another approach developed for radiative transfer problems in dense media (Lax, 1952; Tsang et al., 2001). Being applied to problems of remote sensing of snow in microwave spectral range, the quasi-crystalline approximation provides taking into account the coherent interaction among the scatterers, located at the vicinity of each other, via weighted pair distribution function of particle positions. It permits to calculate the coherently transmitted radiation and radiation absorption in densely packed media composed of moderate size non-spherical particles (Tsang at al., 2001).

An approach to analysis of macro-characteristics of densely packed disperse media composed of large non-spherical particles (including faceted particles imitating ice crystals of cirrus clouds) has been developed in a series of papers (Borovoi et al,, 2003; 2006; 2007;2010; Borovoi 2005; 2006; 2013). The approach is based on accurate estimation of scattered field in the near zone of single scatterer via introduction of the so-called shadow-forming field. The really existing shadow-forming field can be determined at any distance from the scattering particle in the near zone in the frames of physical optics, and there is a number of advantages to operate with it. For example, both Fresnel and Fraunhofer diffraction can be taken into account by the method without tedious calculations (Borovoi, 2013). In the paper (Borovoi et al., 2010) a treatment of a disperse medium composed of large faceted particles has been fulfilled by the approach of shadow-forming field analysis. The scattered field is succeded to present in the form of a set of plane parallel beams, each beam being a clearly defined as the physical object with finite transverse size, known shape and spatial location. The polarization of each beam is described analytically (in terms of 2D electric field vector and the Jones polarization matrix). The shadow-forming plane parallel beam is included into the superposition of scattered beams as the additional beam. It is taken into account that all the beams undergo the Fraunhofer diffraction in the wave zone of each particle. The diffracted field is calculated via the vector Fraunhofer diffraction equation. As a result the analytical expression for the scattered field in the wave zone of the particle has been obtained in terms of shadow functions, containing all the parameters of near-zone plane-parallel beams. Finally, the polarization characteristics of radiation, scattered by the particle, are expressed in terms of the Mueller matrix (the interference of all the diffracted beams being taken into account). In the case of statistical ensemble of scatterers having certain sizes, shapes, and spatial orientations the scattered field is naturally expressed through the ensemble average of shadow functions. In addition, the method of calculation of diffraction contribution in near forward-scattering direction has been developed as well.

To elucidate the issue to what extent the VRTE can be applied to densely packed media the numerical solutions of various problems of multiply scattered electromagnetic radiation in densely packed ensembles of discrete scatterers have been performed. In a number of situations the qualitative agreement with the results of RT calculations (with the CB accounting) takes place (Mishchenko et al. 2007, 2009; Okada, Kokhanovsky, 2009; Dlugach et al., 2011; Muinonen et al, 2012; Mackowski et al, 2013).

2.2 Optically active anisotropic media

Optically active (gyrotropic, chiral) media are optically anisotropic media characterized by elliptical birefringence and elliptical dichroism. The more general type of optically anisotropic media, to which the optically active media belong – bianisotropic media – can be specified by the following general form of constitutive equations (constitutive relations, or material equations), reflecting the magneto-electric cross- coupling:

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} + \hat{\xi}\mathbf{H}, \quad \mathbf{B} = \hat{\mu}\mathbf{H} + \hat{\eta}\mathbf{E}, \tag{2.2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are applied electric and magnetic fields, $\mathbf{D}$ and $\mathbf{B}$ are the corresponding vectors of electric and magnetic induction, $\hat{\varepsilon}$ is the electric permittivity tensor, $\hat{\mu}$ is the magnetic permittivity tensor, $\hat{\xi}$ and $\hat{\eta}$ are tensors, defining the magneto-electric cross-coupling (Kong, 1974, 1990; Landau & Lifshitz, 1960). Being placed in an electric or magnetic field bianisotropic media become both polarized and magnetized. The tensors $\hat{\xi}$ and $\hat{\eta}$ in the equations (2.2.1) are not independent tensors: the relation between them should be obtained from the condition of energy conservation for electromagnetic field in the concrete bianisotropic medium. Magnetoelectrical materials were theoretically predicted by L.E.Dzyaloshinsky (Dzyaloshinsky,1959) and observed experimentally in 1960 by D.N.Astrov (Astrov, 1960). Bianisotropic media can be divided into the so-called reciprocal and nonreciprocal bianisotropic media, that are usually studied separately. For example, the reciprocal aniaxially anisotropic chiral media are characterized by the tensors (Kong, 1990)

$$\hat{\varepsilon} = \varepsilon_0\, diag\{\varepsilon, \varepsilon, \varepsilon_1\},\ \hat{\mu} = \mu_0\, diag\{\mu, \mu, \mu_1\},$$
$$\hat{\xi} = c^{-1}\, diag\{0, 0, -i\xi_0\}, \hat{\eta} = c^{-1}\, diag\{0, 0, i\xi_0\}.$$

For nonreciprocal aniaxially anisotropic medium the tensors $\hat{\xi}$ and $\hat{\eta}$ are defined as

$$\hat{\xi} = c^{-1}\, diag\{0, 0, \xi_0\}, \quad \hat{\eta} = c^{-1}\, diag\{0, 0, \xi_0\}.$$

Phenomenological theory of gyrotropic media, belonging to reciprocal bianisotropic media, was developed by F.I.Fedorov (Fedorov, 1976). It was shown that for gyrotropic media the equations (2.2.1) can be rewritten in the form (Fedorov, 1976)

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} + i\hat{\gamma}\mathbf{H}, \quad \mathbf{B} = \hat{\mu}\mathbf{H} - i\hat{\gamma}^T\mathbf{E}, \tag{2.2.1*}$$

where the gyration tensor $\hat{\gamma}$ (the $\hat{\gamma}$ is a real-valued pseudo-tensor) defines medium optical activity (gyrotropy) (the symbol T denotes matrix transposition operation). Thus, gyrotripic media can be specified by a single gyration tensor $\hat{\gamma}$. For problems related to propagation of quasi-monochromatic plane electromagnetic waves $\mathbf{E}(\mathbf{r}) = \mathbf{E}\exp(-i\mathbf{k}\cdot\mathbf{r})$ in non-magnetic gyrotropic media the first of the constitutive equations (2.2.1*) can be also written as

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} + \hat{\gamma}(\nabla \times \mathbf{E}). \tag{2.2.1**}$$

In this case the gyration vector $\mathbf{g}$ can be defined as $\mathbf{g} = \zeta\mathbf{k}$, $\mathbf{g} = (0, 0, g)$, where ζ is a pseudo-scalar (changing sign depending on the handedness of the coordinate system), and the equations (2.2.1**) can be represented in the $\mathbf{k}$ – dependent form

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} + i\varepsilon_0(\zeta\mathbf{k}\times\mathbf{g})\mathbf{E}. \tag{2.2.2}$$

If the electromagnetic wave propagates in the z-direction, so that $\mathbf{k} = (0, 0, k)$, $\mathbf{g} = (0, 0, g)$, one can present the Eq. (2.2.2) in the matrix form

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \varepsilon_0 \begin{bmatrix} n^2 & -ig & 0 \\ ig & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \tag{2.2.3}$$

where $n^2 = \varepsilon / \varepsilon_0$. Obviously, the diagonal elements of the matrix in the Eq. (2.2.3) correspond to the phase velocity of electromagnetic wave in the isotropic medium with refractive index n, whereas the off-diagonal elements, proportional to g, define medium optical activity. The normal modes of the chiral medium (right-hand and left-hand circularly polarized electromagnetic waves that transmit through the media without distortion at different phase velocities $n_{\pm}c$) are obtained as the eigen vectors of the matrix, entering to the Eq. (2.2.3), corresponding to the eigen values $n_{\pm}$,

$$n_{\pm} = \sqrt{n^2 \pm g}. \tag{2.2.4}$$

The Faraday effect – a magneto-optical phenomenon of the polarization plane rotation – is caused by the circular birefringence inherent to chiral medium (different phase velocities for right-hand and left-hand circularly polarized waves). The rotatory power of the medium is proportional to $(n_- - n_+)$, the difference of refractive indices for the normal modes. Natural optical activity is inherent to materials with intrinsically helical microstructure. Examples include selenium, tellurium oxide (TeO_2), quartz ($\alpha - SiO_2$), and cinnabar (HgS). Many materials act as polarization rotators at the presence of acting magnetic field. Magnetoactive plasma and liquid crystals present the other examples of gyrotropic media. At the absence of external magnetic fields three types of weakly damping waves can exist in an isotropic plasma: a transverse electromagnetic wave and two types of longitudinal waves – a high-frequency plasma (Langmuir) wave and a low-frequency ion-sound wave.

The double refraction (birefringence) is one of the most significant phenomena inherent to all types of anisotropic media. In the simplest case of a dielectric (non-magnetic) anisotropic medium the birefringence is described in terms of the electric permittivity tensor $\hat{\varepsilon}$. In general case the tensor $\hat{\varepsilon}$ has six independent components in an arbitrary coordinate system. For crystals of certain symmetries the tensor $\hat{\varepsilon}$ possesses fewer independent components. The refractive indices for the normal modes and their polarization states are traditionally determined via an analysis of refractive index ellipsoid. In the simplest case of an uniaxial crystal the refractive index ellipsoid is a spheroid (the ellipsoid of rotation with $n_1 = n_2 = n_o,$ $n_3 = n_e$). A monochromatic plane wave incident on the boundary between the isotropic and anisotropic media generates two refracted waves inside the anisotropic medium with different directions of propagation and different polarizations. At the boundary between two isotropic media with different refractive indices the angles of incidence θ and refraction θ_1 are related by the Snell law

$$k_0 \sin\theta_1 = k \sin\theta.$$

In the case of a uniaxial crystal two refracted waves arise inside the anisotropic medium: an ordinary wave of orthogonal polarization (TE) at the an angle $\theta = \theta_o$ for which

$$\sin\theta_1 = n_o \sin\theta_o,$$

and an extraordinary wave of parallel polarization (TM) at an angle $\theta = \theta_e$ for which

$$\sin\theta_1 = n(\theta_a + \theta_e)\sin\theta_e.$$

The values n_o and n_e can be determined from the relation

$$\frac{1}{n^2(\theta)} = \frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2(\theta)}{n_e^2},$$

where $n_a = n(\theta)$ (see Fig. 2.2.1).

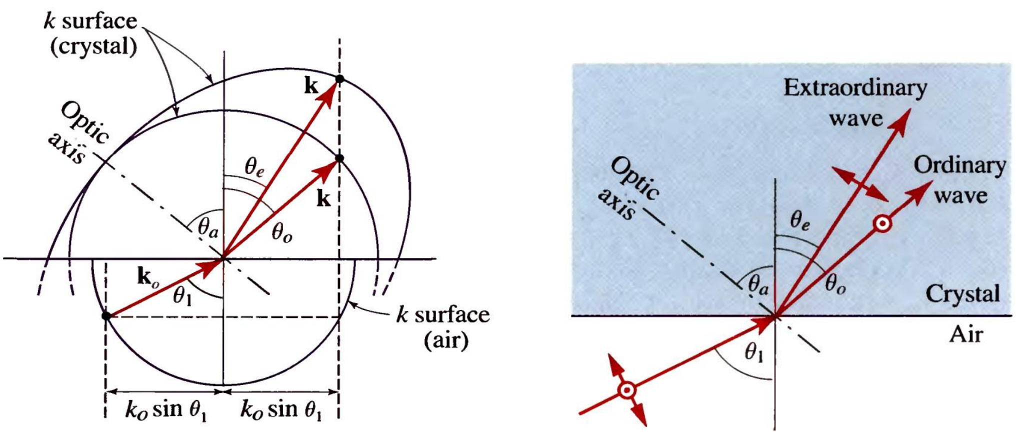


Fig 2.2.1. Double refraction in an uniaxial crystal

Thus, if the incident wave carries two polarizations and the wave vector $\mathbf{k}$ is not normal to the boundary between the isotropic and uniaxial anisotropic media, two refracted waves emerge at the boundary (as shown in Fig. 2.2.1). If the wave vector $\mathbf{k}$ of the incident wave is normal to the boundary, in addition to the ordinary wave with the wave vector $\mathbf{k}_o$, parallel to $\mathbf{k}$, the refracted extraordinary wave at $\mathbf{k} = \mathbf{k}_s$ is emerges as well. That is, the normal incidence on the boundary between isotropic and anisotropic media creates oblique refraction.

It should be stressed that the elliptic birefringence and elliptic dichroism are inherent to optically active media of general type (Fedorov, 1976). But in the frames of present paper we concentrate attention mainly on radiative transfer problems (and the VRTE) for isotropic optically active media so far (see sections 3 and 4). For these media the circular birefringence and polarization plane rotation take place for any direction in the medium. On the contrary, for gyrotropic crystals the polarization plane rotation takes place only along the directions of optical axes (Fedorov, 1976). Another feature of any optically anisotropic media (gyrotropic media including) is that the formulation of boundary conditions for radiative transfer problems requires of special attention. Before consideration this peculiarity in more detail it is worth to shortly remind of the quasi-isotropic approximation of geometrical optics that is used in quasi-monochromatic radiation transfer problems in weakly anisotropic media (Kravtsov et al., 2007).

2.3 Quasi-isotropic approximation of geometrical optics

The coherent component of the radiation field in a scattering medium (non-scattered radiation, refracted by the medium) can be considered as the radiation propagating in the effective continuous transparent refractive medium which optical characteristics can be directly derived from the Maxwell equations (Kravtsov, Orlov, 1990). In a weakly anisotropic medium the monochromatic non-scattered radiation propagates in the form of a transverse electromagnetic waves. The polarization of the transverse electromagnetic wave can be calculated in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, so as $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{s} = \mathbf{e}_2 \cdot \mathbf{s} = 0$ where $\mathbf{s} = \dot{\mathbf{r}}$ (the unit vector tangent to the ray). The complex-valued basis of circularly polarized waves can also be used:

$$\mathbf{e}^- = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}, \quad \mathbf{e}^+ = (\mathbf{e}_1 - i\mathbf{e}_2)/\sqrt{2}. \tag{2.3.1}$$

Polarization evolution of a partially polarized electromagnetic wave propagating in a weakly aisotropic inhomogeneous medium without scattering is governed by the equation for the four-component Stokes vector

$$\mathbf{I}(\mathbf{s}) = [I(\mathbf{s}), Q(\mathbf{s}), U(\mathbf{s}), V(\mathbf{s})]^{\mathrm{T}} \equiv col[I(\mathbf{s}), Q(\mathbf{s}), U(\mathbf{s}), V(\mathbf{s})] \tag{2.3.2}$$

which can be written as

$$\dot{\mathbf{I}}(\mathbf{s}) = \hat{M}\,\mathbf{I}(\mathbf{s}), \tag{2.3.3}$$

where $\hat{\mathbf{M}}$ is the differential Mueller matrix for the anisotropic medium. The matrix $\hat{\mathbf{M}}$ can be expressed in terms of three-component vector $\mathbf{G} = (G_1, G_2, G_3)$ (Kravtsov, Orlov, 1990):

$$\hat{\mathbf{M}} = \begin{bmatrix} \operatorname{Im} G_0 & \operatorname{Im} G_1 & \operatorname{Im} G_2 & \operatorname{Im} G_3 \\ \operatorname{Im} G_1 & \operatorname{Im} G_0 & -\operatorname{Re} G_3 & \operatorname{Re} G_2 \\ \operatorname{Im} G_2 & \operatorname{Re} G_3 & \operatorname{Im} G_0 & -\operatorname{Re} G_1 \\ \operatorname{Im} G_3 & -\operatorname{Re} G_2 & \operatorname{Re} G_1 & \operatorname{Im} G_0 \end{bmatrix} \tag{2.3.4}$$

where the Mueller matrix can be presented as the sum of three terms (Azzam et al.,1989). The first one, $\hat{\mathbf{M}}_a = \mathrm{Im}\, G_0\, diag[1, 1, 1, 1]$ describes attenuation common for all components of the Stokes vector. The second term, the dichroic one,

$$\hat{\mathbf{M}}_d = \begin{bmatrix} 0 & \mathrm{Im}\, G_1 & \mathrm{Im}\, G_2 & \mathrm{Im}\, G_3 \\ \mathrm{Im}\, G_1 & 0 & 0 & 0 \\ \mathrm{Im}\, G_2 & 0 & 0 & 0 \\ \mathrm{Im}\, G_3 & 0 & 0 & 0 \end{bmatrix} \qquad (2.3.5)$$

corresponds to the attenuation, responsible for dichroism (i.e., selective attenuation of the normal modes). Finally, the matrix

$$\hat{\mathbf{M}}_b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Re}\, G_3 & \mathrm{Re}\, G_2 \\ 0 & \mathrm{Re}\, G_3 & 0 & -\mathrm{Re}\, G_1 \\ 0 & -\mathrm{Re}\, G_2 & \mathrm{Re}\, G_1 & 0 \end{bmatrix} \qquad (2.3.6)$$

describes the birefringence.

Thus, for multiply scattered polarized radiation transport problems in weakly optically anisotropic chiral media the four-component VRTE with the matrix extinction operator (making sense the differential Mueller matrix) and the integral operator of scattering, defined by the phase matrix of a non-block-diagonal form, can be adequately used.

3. Radiation transport problems for optically active media

3.1 The radiation transport equation for isotropic optically active media

Recall that the Stokes parameters of a radiation beam, propagating in the direction $\mathbf{s}$,

$$\mathbf{I}(\mathbf{s}) = [I(\mathbf{s}),\ Q(\mathbf{s}),\ U(\mathbf{s}),\ V(\mathbf{s})]^{\mathrm{T}} \qquad (3.1.1)$$

are related to the time-averaged bilinear products of components of electric field vector $\mathbf{E}$ of plane quasi-monochromatic electromagnetic wave, $\mathbf{E} = E_1\mathbf{e}_1 + E_2\mathbf{e}_2$, $\mathbf{e}_1\mathbf{s} = \mathbf{e}_2\mathbf{s} = 0$, accordingly to the expressions

$$I = \langle |E_1|^2 + |E_2|^2 \rangle,\ Q = \langle |E_1|^2 - |E_2|^2 \rangle,\ U = 2\,\mathrm{Re}\langle\left(E_1 E_2^*\right)\rangle,\ V = -2\,\mathrm{Im}\langle\left(E_1 E_2^*\right)\rangle. \qquad (3.1.2)$$

The Stokes parameters define the beam intensity I, the polarization degree p, the shape and orientation of the polarization ellipse in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ (see Fig. 3.1.1):

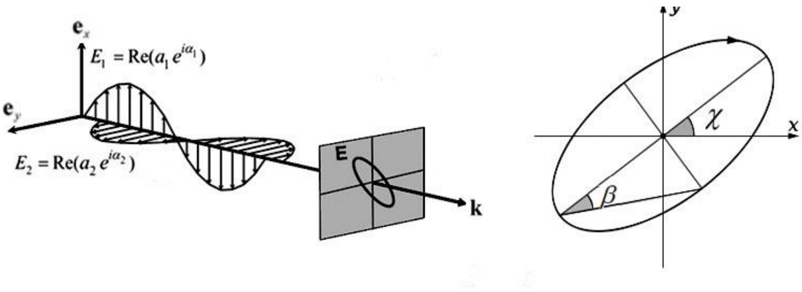


Fig. 3.1.1. The polarization ellipse.

$$p = \frac{(Q^2+U^2+V^2)^{1/2}}{I}; \quad \chi = \frac{1}{2}\arctan\frac{U}{Q}; \qquad \beta = \frac{1}{2}\arcsin(\frac{V}{(Q^2+U^2+V^2)^{1/2}}). \qquad (3.1.3)$$

The radiation transport equation governing polarized radiation transfer in a scattering and absorbing medium is usually written in terms of the Stokes vector $\mathbf{I}(\mathbf{r},\mathbf{s})$, defined by (3.1.1), (3.1.2) and depending on the coordinate of spatial point $\mathbf{r}$ in the medium and the unit vector $\mathbf{s}$, defining the direction of radiation propagation. The matrix transport equation, written in terms of the coherence matrix $\hat{\rho}$,

$$\hat{\rho}(\mathbf{r},\mathbf{s}) = \begin{bmatrix} |E_1|^2 & E_1 E_2^* \\ E_2 E_1^* & |E_1|^2 \end{bmatrix} = 0.5 \begin{bmatrix} I+Q & U-iV \\ U-iV & I+Q \end{bmatrix}, \qquad (3.1.4)$$

is exploited as well (see, for example, (Dolginov et al.,1970; Gnedin et al,, 1970; Kokhanovsky, 2000). We will further use the integro-differential vector transport equation in the form

$$(\mathbf{s}\cdot\nabla)\mathbf{I}(\mathbf{r},\mathbf{s}) + \hat{\sigma}(\mathbf{r})\mathbf{I}(\mathbf{r},\mathbf{s}) = (\hat{P}\mathbf{I})(\mathbf{r},\mathbf{s}) + \mathbf{F}(\mathbf{r},\mathbf{s}), \qquad (3.1.5)$$

where $\hat{\sigma}(\mathbf{r})$ is the extinction operator, $(\hat{P}\mathbf{I})(\mathbf{r},\mathbf{s})$ is the integral scattering operator, defining multiple scattering of radiation in the medium, and $\mathbf{F}(\mathbf{r},\mathbf{s})$ is the Stokes vector of internal sources of radiation in the medium. For weakly anisotropic optically active media the extinction operator $\hat{\sigma}$ can be expressed in terms of the medium refractive index operator $\hat{n}$ by the formula (Kuzmina, 1976, 1986):

$$\hat{\sigma} = -i\frac{\omega}{c}\hat{T}\{\hat{n}\otimes\hat{I}_2 - \hat{I}_2\otimes\hat{n}^*\}\hat{T}^{-1}, \qquad (3.1.6)$$

where

$$\hat{n} = \begin{bmatrix} n^- + i\kappa^- & 0 \\ 0 & n^+ + i\kappa^+ \end{bmatrix}, 2\frac{\omega}{c}\kappa^{\pm} = \sigma_t^{\pm} = \sigma_s^{\pm} + \sigma_a^{\pm}, \ \hat{T} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix}. \quad (3.1.7)$$

Here $n^{\pm}$ are the values of refractive indices for the radiation in the polarization states of the normal waves (i.e., the states of right and the left circular polarization for the chiral medium), $\kappa^{\pm}$ are the attenuation coefficients for the radiation in the same polarization states, $\sigma_s^{\pm}$, $\sigma_a^{\pm}$, $\sigma_t^{\pm}$ are the cross sections of scattering, absorption and extinction for the radiation in the mentioned polarization states, $\hat{I}_2 = diag[1, \ 1]$, and $\otimes$ is the symbol of tensor product (see, for example, (Dullemond, 1991-2010)).

For isotropic optically active medium the extinction operator $\hat{\sigma}(\mathbf{r})$ in the VRTE (3.1.5), is defined by the matrix (the notations from (Kuzmina, 1986, 1989, 1991) are further used):

$$\hat{\sigma} = \begin{bmatrix} \bar{\sigma}_t & 0 & 0 & \Delta\bar{\sigma}_t/2 \\ 0 & \bar{\sigma}_t & \frac{\omega}{c}\Delta n & 0 \\ 0 & -\frac{\omega}{c}\Delta n & \bar{\sigma}_t & 0 \\ \Delta\bar{\sigma}_t/2 & 0 & 0 & \bar{\sigma}_t \end{bmatrix} \quad (3.1.8)$$

where

$$\bar{\sigma}_t = \frac{\sigma_t^+ + \sigma_t^-}{2}, \quad \Delta\,\sigma_t = \sigma_t^+ - \sigma_t^-, \quad \Delta n = n^+ - n^-. \quad (3.1.9)$$

The integral operator of scattering in the VRTE (3.1.5) possesses the same structure as that in the case of optically isotropic medium (due to the medium geometrical isotropy). It can be written as

$$(\hat{P}\mathbf{I})(\mathbf{r},\mathbf{s}) = \frac{\bar{\sigma}_s}{4\pi}\int_{\Omega} \hat{L}(\mathbf{s},\mathbf{s}')\hat{\Gamma}(\mathbf{r};\,\mathbf{s}\cdot\mathbf{s}')\hat{L}^{+}(\mathbf{s}',\mathbf{s})d\mathbf{s}', \quad (3.1.10)$$

where $\hat{\Gamma}(\mathbf{r};\,\mathbf{s}\cdot\mathbf{s}')$ is the scattering phase matrix, defining the law of scattering by the medium volume element and depending on variables $\mathbf{s}$ and $\mathbf{s}'$ through the scalar product $\mathbf{s}\cdot\mathbf{s}' \equiv cos\,\theta_s \equiv \gamma$ (θ_s being the angle of scattering), Ω is the unit sphere in three-dimensional vector space $\mathbb{R}^3$. The matrix $\hat{L}(\mathbf{s},\,\mathbf{s}')$ in Eq. (3.1.10) is the known matrix of the Stokes vector transformation at the transition from one polarization basis to another, and $\hat{L}^{+}$ is the Hermitian conjugate to $\hat{L}$. Geometrically isotropic optically active media are characterized by the special type of phase matrices:

$$\hat{\Gamma}(\gamma) = \begin{bmatrix} a_1(\gamma) & b_1(\gamma) & c_2(\gamma) & b_2(\gamma) \\ b_1(\gamma) & a_2(\gamma) & c_3(\gamma) & b_3(\gamma) \\ -c_2(\gamma) & -c_3(\gamma) & a_3(\gamma) & c_1(\gamma) \\ b_2(\gamma) & b_3(\gamma) & -c_1(\gamma) & a_4(\gamma) \end{bmatrix} \qquad (3.1.11)$$

In addition to the usual normalization condition for the scattering phase function (the indicatrix of scattering) $\Gamma_{11}(\mathbf{r};\gamma) \equiv a_1(\mathbf{r};\gamma)$,

$$\frac{1}{2}\int_{-1}^{1} \hat{\Gamma}_{11}(\mathbf{r};\gamma)\,d\gamma = 1,$$

there is another normalization condition for the element $\Gamma_{14}(\mathbf{r};\gamma) \equiv b_2(\mathbf{r};\gamma)$, that should be fulfilled:

$$\frac{1}{2}\int_{-1}^{1} \hat{\Gamma}_{14}(\mathbf{r};\gamma)\,d\gamma = \frac{\sigma_s^+ - \sigma_s^-}{\sigma_s^+ + \sigma_s^-} = \frac{\Delta\sigma_s}{2\overline{\sigma}_s} \equiv \Delta_s. \qquad (3.1.12)$$

The relation (3.1.12) can be derived as the consequence of the energy conservation law for volume element of the medium in the situation when it is illuminated by a mono-directed beam of circular polarized radiation. The value $\Delta_s(\mathbf{r})$ determines an essential macro-characteristics of scattering medium, that might be called the medium dichroism due to scattering. The condition $\Delta_s(\mathbf{r}) \equiv 0$ may be considered as the necessary condition of the medium optical isotropy, whereas the condition $\Delta_s(\mathbf{r}) \neq 0$ can figure as the sufficient condition of the medium optical anisotropy. Using the natural relation between the cross sections of extinction, scattering and absorption, one can obtain the relation between the full medium dichroism $\Delta(\mathbf{r}) = (\sigma_t^+ - \sigma_t^-)/2\overline{\sigma}_t$ and the dichroism contributions $\Delta_s(\mathbf{r})$ and $\Delta_a(\mathbf{r})$ due to scattering and absorption, respectively:

$$\Delta(\mathbf{r}) = \lambda(\mathbf{r})\Delta_s(\mathbf{r}) + (1-\lambda(\mathbf{r}))\Delta_a(\mathbf{r}), \qquad (3.1.13)$$

where

$$\Delta_s(\mathbf{r}) = (\sigma_s^+ - \sigma_s^-)/2\overline{\sigma}_s,\ \Delta_a(\mathbf{r}) = (\sigma_a^+ - \sigma_a^-)/2\overline{\sigma}_a,\ \overline{\lambda}(\mathbf{r}) = \overline{\sigma}_s/\overline{\sigma}_t.$$

So, as one can see, instead of the two cross sections, $\sigma_t(\mathbf{r})$ and $\sigma_s(\mathbf{r})$, that in combination with the scattering phase matrix $\hat{\Gamma}(\mathbf{r};\gamma)$ are sufficient to completely specify the VRTE for optically isotropic media, the six cross sections $\sigma_t^{\pm}(\mathbf{r})$, $\sigma_s^{\pm}(\mathbf{r})$, $\sigma_a^{\pm}(\mathbf{r})$, and additionally $n^{\pm}(\mathbf{r})$, are needed to specify the VRTE for optically active media. Surely, the equivalent collection of functions can also be used, for instance: $\overline{\sigma}_t(\mathbf{r})$, $\overline{\lambda}(\mathbf{r}) = \overline{\sigma}_s(\mathbf{r})/\overline{\sigma}_t(\mathbf{r})$, $\overline{n}(\mathbf{r})$, $\Delta(\mathbf{r}) = (\sigma_t^+(\mathbf{r}) - \sigma_t^-(\mathbf{r}))/2\overline{\sigma}_t(\mathbf{r})$, $\Delta_s(\mathbf{r}) = (\sigma_s^+(\mathbf{r}) - \sigma_s^-(\mathbf{r}))/2\overline{\sigma}_s(\mathbf{r})$, and $\delta(\mathbf{r})=(\omega/c)(n^{+}-n^{-})/\bar{\sigma}_t$. We will use further just the last collection of functions and rewrite the extinction operator (3.1.8) in the form

$$\hat{\sigma}_{ext}(\mathbf{r})=\bar{\sigma}_t(\mathbf{r})\begin{bmatrix}1 & 0 & 0 & \Delta(\mathbf{r})\\ 0 & 1 & \delta(\mathbf{r}) & 0\\ 0 & -\delta(\mathbf{r}) & 1 & 0\\ \Delta(\mathbf{r}) & 0 & 0 & 1\end{bmatrix} \qquad (3.1.8^*)$$

where

$$\bar{\sigma}_t=\frac{\sigma_t^{+}+\sigma_t^{-}}{2},\quad \Delta=\frac{\sigma_t^{+}-\sigma_t^{-}}{2\bar{\sigma}_t},\quad \delta=\frac{\omega}{c}\cdot\frac{n^{+}-n^{-}}{\bar{\sigma}_t}, \qquad (3.1.14)$$

$\sigma_t^{\pm}(\mathbf{r})$ being the extinction cross sections, and $n^{\pm}$ being the refractive indices for right and left circularly polarized radiation, respectively. The integral operator of scattering (3.1.10) is specified by the phase matrix of scattering $\hat{\Gamma}(\mathbf{r};\gamma)$, that is defined accordingly Eq. (3.1.11). The additional normalization conditions, defined via Eq. (3.1.12) and (3.1.13), can be rewritten in the form

$$\bar{\sigma}_t=\bar{\sigma}_a+\bar{\sigma}_s=\bar{\sigma}_a+\frac{\bar{\sigma}_s}{2}\int_{-1}^{1}\Gamma_{11}(\gamma)d\gamma, \qquad (3.1.15)$$

$$\Delta=\Delta_a+\Delta_s=\Delta_a+\frac{1}{2}\int_{-1}^{1}\Gamma_{14}(\gamma)d\gamma. \qquad (3.1.16)$$

The relation (3.1.16) determines Δ_s, an essential macro-characteristics of the scattering medium (the medium dichroism due to scattering). The condition $\Delta_s\equiv 0$ can figure as the necessary condition of optical isotropy of the medium, whereas the condition $\Delta_s\neq 0$ represents the sufficient condition of its optical anisotropy (i.e., the optical activity).

3.2 Radiation transport problems for slabs of isotropic optically active medium

The problem of polarized radiation transfer in a slab of isotropic optically active medium is a boundary value problem for the VRTE. Let us consider the slab $0\le z\le H$, z being the coordinate along the unit normal $\mathbf{n}$ to the plane $z=0$. Let $\mathbf{s}$ be the unit vector of radiation transfer direction ($\mathbf{s}\in\Omega$, Ω being the unit sphere in 3D vector space $\mathbb{R}^3$), and $\mathbf{I}(\mathbf{r},\mathbf{s})$ be the four-component Stokes vector, defined in (3.1.1), (3.1.2). Then we have the following boundary value problem for the VRTE:

$$\mu\frac{\partial\mathbf{I}(z,\mathbf{s})}{\partial z}+\hat{\sigma}(z)\mathbf{I}=(\hat{P}\mathbf{I})(z,\mathbf{s})+\mathbf{F}(z,\mathbf{s}), \qquad (3.2.1)$$

$$\mathbf{I}^{+}(0,\mathbf{s})=\mathbf{f}_0^{+}(\mathbf{s}), \qquad (3.2.2)$$

$$\mathbf{I}^{-}(H,\mathbf{s})=\mathbf{f}_H^{-}(\mathbf{s}). \qquad (3.2.3)$$

Here $\mu = \mathbf{s}\cdot\mathbf{n} = \cos\theta$, while

$$\mathbf{I}^{+}(z,\mathbf{s}) = \begin{cases} \mathbf{I}(z,\mathbf{s}), & \mu \geq 0 \\ 0, & \mu < 0 \end{cases} \tag{3.2.4}$$

$$\mathbf{I}^{-}(z,\mathbf{s}) = \begin{cases} 0, & \mu \geq 0 \\ \mathbf{I}(z,\mathbf{s}), & \mu < 0. \end{cases} \tag{3.2.5}$$

The functions $\mathbf{f}_0^{+}$ and $\mathbf{f}_H^{-}$ define the Stokes vectors of the external radiation at the boundaries $z=0$ and $z=H$, while $\mathbf{F}(z,\mathbf{s})$ defines the Stokes vector of internal volume sources.

It should be noted that we do not concern here an interesting and noteworthy issue on the relation between vector space, affine space and point Euclidean space. For interested reader the monograph (Faure et al., 1964) could be recommended. See also (Rogovtsov, 2015 a); Rogovtsov et al., 2016).

The operator $\hat{\sigma} = \hat{\sigma}_{ext}$ for slabs is specified by the matrix (see (3.1.8*)):

$$\hat{\sigma}(z) = \bar{\sigma}_t(z)\begin{bmatrix} 1 & 0 & 0 & \Delta(z) \\ 0 & 1 & \delta(z) & 0 \\ 0 & -\delta(z) & 1 & 0 \\ \Delta(z) & 0 & 0 & 1 \end{bmatrix} \tag{3.2.6}$$

where $\bar{\sigma}_t$, Δ and δ are defined through $\sigma_t^{\pm}$ and $n^{\pm}$ accordingly to (3.1.14). The integral scattering operator $(\hat{P}\mathbf{I})(z;\,\mathbf{s})$ is specified by formulas (3.1.10), (3.1.11) with $\hat{\Gamma} = \hat{\Gamma}(z;\,\gamma)$.

3.3 Boundary conditions

The transport problem with the given mono-directed monochromatic radiation beam, defined at the slab boundary, is a typical model problem in Earth remote sensing. In the case of oblique beam incidence at the boundary of a slab of optically anisotropic medium the two geometrically separated refracted beams arise inside the slab due to the birefringence phenomenon. For fully polarized mono-directed beam the amplitudes and the polarization states of these two refracted electromagnetic waves can be exactly calculated (accordingly to the Fresnel formula generalization (see, for instance, (Fedorov, 1976; Fedorov, Philippov, 1976)). For weakly anisotropic media the angles of refraction of the two beams and their polarizations can be estimated in the approximation of weak medium anisotropy.

Let the incident non-polarized monochromatic mono-directed beam (plane quasi-monochromatic electromagnetic wave) is defined by the Stokes vector

$$\mathbf{I}^{inc}(0,\mathbf{s}_0) = [I^0,\,0,\,0,\,0]^T\,\tilde{\delta}(\mathbf{s}-\mathbf{s}_0) = [I^0,\,0,\,0,\,0]^T\,\tilde{\delta}(\theta-\theta_0)\tilde{\delta}(\varphi-\varphi_0) \tag{3.3.1}$$

Due to the geometrical medium isotropy according to Snell's law one has

$$\sin\theta^{+} = \sin\theta_0 / n^{+}, \quad \sin\theta^{-} = \sin\theta_0 / n^{-}, \quad \varphi^{+} = \varphi^{-} = \varphi_0 . \tag{3.3.2}$$

Putting (for certainty) $n^{+} > n^{-}$ $(\Delta n > 0)$, we obtain

$$\sin\theta^{+} = \sin\theta_0 (1 - \Delta n / 2\overline{n}), \quad \sin\theta^{-} = \sin\theta_0 (1 + \Delta n / 2\overline{n}). \tag{3.3.3}$$

The polarization states of two refracted beams can be easily calculated (Kuzmina, 1986 a)). In the case of mono-directed non-polarized monochromatic beam (3.3.1), incident on the slab boundary $z = 0$ of homogeneous chiral medium ($\sigma_t^{\pm} = const,\ n^{\pm} = const$), the refracted beam represents the superposition of two fully circularly polarized beams:

$$\begin{aligned}\mathbf{I}^{refr}(0;\mathbf{s}_0^{+},\mathbf{s}_0^{-}) &= \overline{\sigma}_t \frac{1+\Delta}{2} I^0 [1,\ 0,\ 0,\ 1]^T\, \tilde{\delta}(\theta - \theta_0^{+})\delta(\varphi - \varphi_0) \\ &+ \overline{\sigma}_t \frac{1-\Delta}{2} I^0 [1,\ 0,\ 0,\ -1]^T\, \tilde{\delta}(\theta - \theta_0^{-})\tilde{\delta}(\varphi - \varphi_0).\end{aligned} \tag{3.3.4}$$

So, the mono-directed beam of non-polarized external radiation of intensity I^0, incident to the boundary of homogeneous chiral medium, is transformed inside the medium into superposition of two geometrically separated fully circularly polarized beams of intensities $I_0^{+} = \overline{\sigma}_t \frac{1+\Delta}{2} I^0$ (right-hand circularly polarized beam) and $I_0^{-} = \overline{\sigma}_t \frac{1-\Delta}{2} I^0$ (left-hand circularly polarized beam). The angles $\theta_0^{\pm}$ are needed to be obtained.

3.4 Coherently scattered radiation in a slab of chiral medium

The Stokes vector of non-scattered radiation in a slab, $\mathbf{I}_c(z,\mathbf{s}) = \mathbf{I}_c(\tau,\mathbf{s})$, can be found as the solution of the following boundary value problem:

$$\mu \frac{\partial \mathbf{I}_c(\tau,\mathbf{s})}{\partial \tau} + \hat{\sigma}(\tau)\mathbf{I}_c = \tilde{\mathbf{F}}(\tau,\mathbf{s}), \tag{3.4.1}$$

$$\mathbf{I}_c^{+}(0,\mathbf{s}) = \mathbf{I}_c^{-}(\tau_H,\mathbf{s}) = 0 \tag{3.4.2}$$

where $\mu = \mathbf{s}\cdot\mathbf{n} = \cos\theta$ and

$$\tilde{\mathbf{F}}(\tau,\mathbf{s}) = \mathbf{F}(\tau,\mathbf{s}) + \mu \mathbf{f}_0^{+}(\mathbf{s})\tilde{\delta}(\tau) + |\mu|\mathbf{f}_H^{-}(\mathbf{s})\tilde{\delta}(\tau - \tau_H), \tag{3.4.3}$$

$$\tau \equiv \tau(0,z) = \int_0^z \overline{\sigma}_t(x)dx, \qquad \tau_H \equiv \tau(0,H), \tag{3.4.4}$$

and $\tilde{\delta}(\tau)$ is Dirac's delta-function. The solution of the problem (3.4.1) – (3.4.4) may be written in the form

$$\mathbf{I}_c(\tau,\mathbf{s}) = \mu^{-1}\int_0^{\tau} \hat{G}(\tau/\mu,\ \tau'/\mu)\tilde{\mathbf{F}}^{+}(\tau',\mathbf{s})d\tau' + |\mu|^{-1}\int_{\tau}^{\tau_H} \hat{G}(\tau'/\mu,\ \tau/\mu)\tilde{\mathbf{F}}^{-}(\tau',\mathbf{s})d\tau', \tag{3.4.5}$$

$\hat{G}(z,z',\mu) \equiv \hat{G}(\tau,\tau',\mu)$ is the known Green function for a slab of non-scattering geometrically isotropic optically active medium (Kuzmina, 1989, 1991; Kokhanovsky, 1999 a)):

$$\hat{G}(z,z',\mu) = e^{-\frac{\overline{\tau}(z,z')}{\mu}} \begin{bmatrix} ch\frac{\Delta\tau(z,z')}{2\mu} & 0 & 0 & -sh\frac{\Delta\tau(z,z')}{2\mu} \\ 0 & \cos\frac{\Delta\phi(z,z')}{\mu} & -\sin\frac{\Delta\phi(z,z')}{\mu} & 0 \\ 0 & \sin\frac{\Delta\phi(z,z')}{\mu} & \cos\frac{\Delta\phi(z,z')}{\mu} & 0 \\ -sh\frac{\Delta\tau(z,z')}{2\mu} & 0 & 0 & ch\frac{\Delta\tau(z,z')}{2\mu} \end{bmatrix}, \quad (3.4.6)$$

$$\overline{\tau}(z,z') = \int_{z'}^{z} \overline{\sigma}_t(x)dx, \quad (3.4.7)$$

$$\Delta\tau(z,z') = \int_{z'}^{z} [\sigma_t^+(x) - \sigma_t^-(x)]dx, \quad \Delta\phi(z,z') = \int_{z'}^{z} [n^+(x) - n^-(x)]dx. \quad (3.4.8)$$

If internal volume sources of radiation are absent, the coherently scattered radiation is obviously expresses by

$$\mathbf{I}_c(z,\mathbf{s}) = \hat{G}(z,0,\mu)\mathbf{f}_0^+(\mathbf{s}) + \hat{G}(H,z,|\mu|)\mathbf{f}_H^-(\mathbf{s}). \quad (3.4.9)$$

Further consideration easily shows, that the dynamical system (3.4.1) can be represented in the form of four independent one-dimensional equations. To find the equations one should calculate the eigen values and eigen vectors of the extinction operator $\hat{\sigma}$ of the equation (3.2.6):

$$\begin{aligned} \lambda_1 &= \overline{\sigma}_t(z)[1+\Delta(z)] \equiv \sigma_t^+(z), & \mathbf{\Psi}^{(1)} &= [1, \ 0, \ 0, \ 1]^T, \\ \lambda_2 &= \overline{\sigma}_t(z)[1-i\delta(z)], & \mathbf{\Psi}^{(2)} &= [0, \ 1, \ -i, \ 0]^T, \\ \lambda_3 &= \overline{\sigma}_t(z)[1+i\delta(z)], & \mathbf{\Psi}^{(3)} &= [0, \ 1, \ i, \ 0]^T, \\ \lambda_4 &= \overline{\sigma}_t(z)[1-\Delta(z)] \equiv \sigma_t^-(z), & \mathbf{\Psi}^{(4)} &= [1, \ 0, \ 0, \ -1]^T. \end{aligned} \quad (3.4.10)$$

In the eigen basis $\{\mathbf{\Psi}^{(1)},\mathbf{\Psi}^{(2)},\mathbf{\Psi}^{(3)},\mathbf{\Psi}^{(4)}\}$ the extinction operator $\hat{\sigma}$ is diagonal:

$$\hat{\sigma}^{(0)} = \hat{L}^{-1}\hat{\sigma}\,\hat{L}, \qquad \hat{L} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad \hat{L}^{-1} = \frac{1}{2}\,\hat{L}^{+},$$

$$\hat{\sigma}^{(0)}(z) = \bar{\sigma}_t(z)\ diag\,\{1+\Delta(z),\ 1-i\delta(z),\ 1+i\delta(z),\ 1-\Delta(z)\}. \qquad (3.4.11)$$

Thus, if instead of the vector $\mathbf{I} = [I,\ Q,\ U,\ V]^T$ we use the vector

$$\mathbf{\Phi} = \frac{1}{2}[I+V,\quad Q+iU,\quad Q-iU,\quad I-V\,]^T, \qquad (3.4.12)$$

we will have the system of independent equations for the components of the vector $\mathbf{\Phi}$ with the diagonal extinction operator defined by (3.4.11). To write the expression for the Green function $\hat{G}(z,z',\mu)$, providing the solution to the problem for non-scattered radiation transfer in terms of the vector $\mathbf{\Phi}$, it is convenient to introduce the optical thicknesses

$$\tau_k(z,z') = \int_{z'}^{z} \lambda_k(x)dx\,, \qquad (3.4.13)$$

where λ_k are defined in (3.4.10). Finally we have

$$\tau_1 = \tau^+(z,z') = \int_{z'}^{z} \bar{\sigma}_t^+(x)dx = \bar{\tau} + \Delta\tau, \qquad \tau_2 = \bar{\tau}(z,z') + i\,\Delta\phi(z,z'),$$

$$\tau_3 = \bar{\tau}(z,z') - i\,\Delta\phi(z,z'), \qquad \tau_4 = \tau^-(z,z') = \int_{z'}^{z} \bar{\sigma}_t^-(x)dx = \bar{\tau} - \Delta\tau, \qquad (3.4.14)$$

where

$$\bar{\tau}(z,z') = \int_{z'}^{z} \bar{\sigma}_t(x)dx = \frac{1}{2}[\tau^+(z,z') + \tau^-(z,z')],$$

$$\Delta\phi(z,z') = \phi^+(z,z') - \phi^-(z,z'), \qquad \phi^{\pm}(z,z') = \frac{\omega}{c}\int_{z'}^{z} n^{\pm}(x)dx. \qquad (3.4.15)$$

So, for the Green function $\hat{G}^{(0)}(z,z',\mu)$, providing the solution to the problem (3.4.1)-(3.4.2) in terms of $\mathbf{\Phi}$, we find the expression in the form of the diagonal matrix:

$$\hat{G}^{(0)}(z,z',\mu) = diag\,\{G_1,\ G_2,\ G_3,\ G_4\},$$
$$G_k(z,z',\mu) = \exp[-\mu^{-1}\tau_k(z,z')],\ \ k = 1,\,2,\,3,\,4. \qquad (3.4.16)$$

It remains to note, that the optical thicknesses τ^+ and τ^- characterize spatial attenuation of radiation beams in the states of right and left circular polarizations, whereas the function $\Delta\phi$ defines the phase incursion between the beams in same polarization states.

3.5 The equivalent system of equations for parameters of polarization ellipse

In order to study the behavior of polarization state of non-scattered radiation in a slab $0 \le z \le H$ of isotropic chiral medium we consider the simplest radiation transport problem with homogeneous monodirected radiation beam incident to the boundary $z = 0$:

$$\begin{aligned} &\mu \frac{\partial \mathbf{I}(z,\mathbf{s})}{\partial z} + \hat{\sigma}(z)\mathbf{I} = 0, \\ &\mathbf{I}^+(0,\mathbf{s}) = \mathbf{I}_0(\mathbf{s}_0) = \mathbf{I}_0 \tilde{\delta}(\mu - \mu_0)\tilde{\delta}(\varphi - \varphi_0), \\ &\mathbf{I}^-(H,\mathbf{s}) = 0. \end{aligned} \tag{3.5.1}$$

It is convenient to introduce new variables

$$\begin{aligned} &\chi = 0.5\arctan(U/Q), \qquad Y = V/I = \sin(2\beta), \\ &Z = 1 - p^2 = I^{-2}(I^2 - Q^2 - U^2 - V^2), \end{aligned} \tag{3.5.2}$$

where (χ,β) are the parameters of polarization ellipse, p is the polarization degree (see (3.1.3)). From the system of ordinary differential equations for the Stokes parameters the following system of the ODE equations for the variables (3.5.2) can be easily derived (Kuzmina, 1986 b)):

$$\begin{aligned} &\mu \frac{\partial \chi}{\partial z} = \frac{\omega}{c}\frac{\Delta n(z)}{2}, \\ &\mu \frac{\partial Y}{\partial z} = -\frac{\Delta\sigma_t(z)}{2}(1 - Y^2), \\ &\mu \frac{\partial Z}{\partial z} = \Delta\sigma_t(z) Y \cdot Z, \\ &\mu \frac{\partial I}{\partial z} = -\bar{\sigma}_t(z) I - \frac{\Delta\sigma_t(z)}{2} Y \cdot I, \end{aligned} \tag{3.5.3}$$

$$\chi(0) = \chi_0,\; Y(0) = Y_0,\; Z(0) = 1 - p_0^2,\; I(0) = I_0. \tag{3.5.4}$$

As one can see, the equations for χ and Y are independent on the other equations of the system (3.5.3)-(3.5.4), and each of the equations can be exactly integrated:

$$\chi(z,\mu) = \chi_0 + \mu^{-1}\frac{\omega}{c}\int_0^z \frac{\Delta n(z')}{2} dz', \tag{3.5.5}$$

$$Y = \tanh(ar\tanh Y_0 - \mu^{-1}\frac{\omega}{c}\int_0^z \frac{\Delta\sigma_t(z')}{2} dz'). \tag{3.5.6}$$

When Y has been obtained from (3.5.6), the function Z can be found from the third equation of the system (2.5.3)-(2.5.4) (under Y known the equation is also exactly integrated). Finally we obtain the polarization degree p, dependent on Z accordingly to (3.5.2):

$$p^2 = 1 - \frac{1-p_0^2}{1-Y_0^2} sech^2 (ar\tanh Y_0 - \frac{\Delta\tau(z)}{2\mu}). \tag{3.5.7}$$

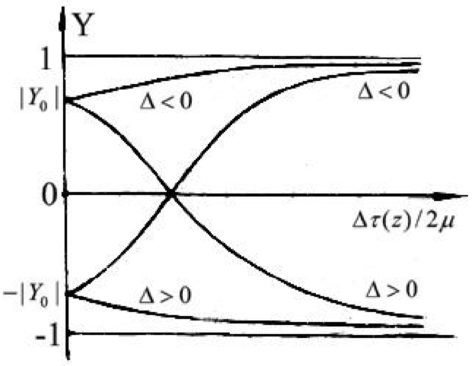
Now the intensity I can be found from the last equation of the system (3.5.3):

$$I = \frac{1}{2} I_0 (e^{-\tau^+(z)/\mu} + e^{-\tau^-(z)/\mu})[1 - \frac{V_0}{I_0}\tanh(\frac{\Delta\tau(z)}{2\mu})]. \tag{3.5.8}$$

In the case of homogeneous medium ($\Delta\sigma_t(z) \equiv \Delta = const,\quad \Delta n(z) \equiv \delta = const$) the first two equations of system (3.5.3)-(3.5.4) are reduced to

$$2\dot{\chi} = \delta,$$
$$\dot{Y} = -\Delta(1 - Y^2). \tag{3.5.9}$$

As one can see from simple analysis of the system (3.5.9), at $\Delta \neq 0$ the polarization state tends to the polarization eigenstate with the smaller value of absorption. At $\Delta = 0$ the rotation of polarization plane with constant speed δ without change of polarization ellipse form takes place under $\bar{\sigma}_t z / \mu$ increasing. Note, that in the case the plane (Δ, δ) is the parametric space of dynamic system (3.5.9). In the case of inhomogeneous medium all possible types of behavior of function $Y = Y(\Delta\tau(z)/2\mu)$ at various relations between the sign of $\Delta(z)$ and the sign of initial value $Y_0 = V_0 / I_0$ are presented in the Fig. 3.5.1 (the solutions of the second equation of the system (3.5.9) at $|\Delta| = 0.1$) (Kuzmina, 1986 b)). Two types of function $p^2(\Delta\tau(z)/\mu)$ are depicted in Fig. 3.5.2: $I: V_0 > 0,\ \Delta < 0;\quad V_0 < 0,\ \Delta > 0;\ II: V_0 > 0,\ \Delta > 0;\quad V_0 < 0,\ \Delta < 0;$ Although both the monotonic and the non-monotonic type of behaviour with the distance are possible, the limit value is $p = 1$ (full polarization). Thus, the optically active medium acts as a polarizer. In the case of chiral medium the any type of radiation, propagating through the medium, is finally transformed into fully circularly polarized radiation.



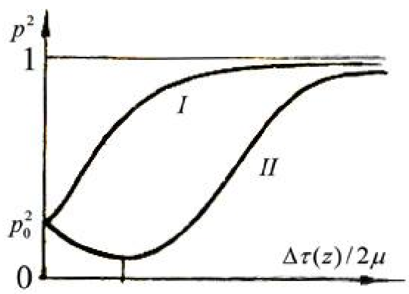


Fig. 3.5.1. The versions of function Y ($|\Delta| = 0.1$). Fig. 3.5.2 Two types of function p^2 ($|\Delta| = 0.1$).

It is worth to note the feature of the parametrical domain (Δ, δ) of dynamical system (3.5.9). The simple analysis shows, that the areas $G^{+} = \{ \Delta, \delta \mid \Delta > 0,\ \delta \neq 0\}$ and $G^{-} = \{ \Delta, \delta \mid \Delta < 0,\ \delta \neq 0\}$ are the areas of structural stability (robustness) of the system. The line $\Delta = 0$ is a bifurcation curve (it separates the parametric space into two areas of structural stability, and the dynamical system itself becomes a conservative system on the line). The line $\delta = 0$ is not a bifurcation curve, but the system (3.5.9) becomes a noncoarse dynamical system in the line (that is, the line $\delta = 0$ represents a dangerous boundary in the parametric domain (Kuzmina 1986 b); Bautin, Leontovich, 1976).

Now it is worth to briefly summarize the main features of coherently scattered radiation in the slabs of optically active media, that have been elucidated in the sections 3.1 – 3.5.

1. The four-component vector transport equation (3.1.5) can be used in radiation transport problems only in the case of weakly optically anisotropic media.

2. Geometrically isotropic optically active media are characterized by the special type of scattering phase matrices of the medium unit volume, defined by (3.1.11). There is a normalization condition on the phase matrix element $\Gamma_{14}(z;\gamma)$,

$$\Delta_s \equiv \frac{1}{2}\int_{-1}^{1} \hat{\Gamma}_{14}(z;\, \gamma)\, d\gamma\,,$$

permitting to find out whether the medium is optically isotropic (at $\Delta_s = 0$), or optically anisotropic (at $\Delta_s \neq 0$).

3. For transport problems in optically active media it is necessary to use transport equation with matrix differential operator of extinction, $\hat{\sigma}$, defined by the formula (3.1.8).

4. For transport problems with mono-directed external radiation beam, incident to the slab boundary of optically anisotropic medium, it is necessary to calculate the birefringent radiation in the slab.

3.6 Radiation transfer problems for slabs of chiral media with reflecting boundaries

The multi-scattered radiation transport problem in a slab $0 \leq z \leq H$ is the solution of the boundary value problem for transport equation (3.2.1) – (3.2.3), where the extinction operator $\hat{\sigma}(z)$ is defined by (3.2.6), and the integral operator $\hat{P}\mathbf{I}$ is specified by the expressions (3.1.10), (3.1.11). For analytical calculations the boundary conditions at the slab boundaries are convenient to be putted zero, supposing that the additional internal radiation sources to be localized at the slab boundaries $z = 0$ and $z = H$:

$$\mu\frac{\partial \mathbf{I}(z,\mathbf{s})}{\partial z}+\hat{\sigma}(z)\mathbf{I}=(\hat{P}\,\mathbf{I})(z,\mathbf{s})+\tilde{\mathbf{F}}(z,\mathbf{s}), \tag{3.6.1}$$

$$\mathbf{I}^{+}(0,\mathbf{s})=0, \tag{3.6.2}$$

$$\mathbf{I}^{-}(H,\mathbf{s})=0, \tag{3.6.3}$$

$$\tilde{\mathbf{F}}(z,\mathbf{s})=\mathbf{F}(z,\mathbf{s})+\mu\mathbf{f}_0^{+}(\mathbf{s})\tilde{\delta}(z)+|\mu|\,\mathbf{f}_H^{-}(\mathbf{s})\tilde{\delta}(z-H). \tag{3.6.4}$$

The solution of the problem (3.6.1)-(3.6.4) for coherently scattered radiation, $\mathbf{I}_c(z,\mathbf{s})$,

$$\mu\frac{\partial \mathbf{I}_c(z,\mathbf{s})}{\partial z}+\hat{\sigma}(z)\mathbf{I}_c\equiv\hat{D}\mathbf{I}_c=\mathbf{F}(z,\mathbf{s}), \tag{3.6.5}$$

$$\mathbf{I}_c^{+}(0,\mathbf{s})=0, \tag{3.6.6}$$

$$\mathbf{I}_c^{-}(H,\mathbf{s})=0, \tag{3.6.7}$$

can be written in the form

$$\mathbf{I}_c(z,\mathbf{s})=\hat{A}^{+}\tilde{\mathbf{F}}+\hat{A}^{-}\tilde{\mathbf{F}}, \tag{3.6.8}$$

where

$$(\hat{A}^{+}\tilde{\mathbf{F}})(z,\mathbf{s})=\begin{cases}\mu^{-1}\int\limits_0^z\hat{G}(\frac{z}{\mu},\frac{z'}{\mu})\tilde{\mathbf{F}}^{+}(z',\mathbf{s})dz', & \mu\ge 0\\ 0, & \mu<0\end{cases} \tag{3.6.9}$$

$$(\hat{A}^{-}\tilde{\mathbf{F}})(z,\mathbf{s})=\begin{cases}0, & \mu\ge 0\\ |\mu|^{-1}\int\limits_z^H\hat{G}(\frac{z'}{|\mu|},\frac{z}{|\mu|})\tilde{\mathbf{F}}^{-}(z',\mathbf{s})dz', & \mu<0\end{cases}. \tag{3.6.10}$$

The Green function $\hat{G}(z/\mu,z'/\mu)=\hat{G}(x-x')$ in (3.6.9), (3.6.10), is known (see (3.4.6)) and can be written as

$$\hat{G}(x-x')=e^{-(x-x')}\begin{bmatrix} ch\,\Delta)(x-x') & 0 & 0 & -sh\Delta(x-x')\\ 0 & \cos\delta(x-x') & -\sin\delta(x-x') & 0\\ 0 & \sin\delta(x-x') & \cos\delta(x-x') & 0\\ -sh\Delta(x-x') & 0 & 0 & ch\Delta(x-x')\end{bmatrix}, \tag{3.6.11}$$

where $x=\overline{\sigma}z/\mu$.

The solution of transport problem for non-scattered radiation for the slab $0\le z\le H$ with reflecting boundary $z=H$ is often of special interest. The boundary value problem can be written as

$$\hat{D}\mathbf{I}_c = \mathbf{F}(z,\mathbf{s}), \tag{3.6.12}$$

$$\mathbf{I}_c^+(0,\mathbf{s}) = 0, \tag{3.6.13}$$

$$\mathbf{I}_c^-(H,\mathbf{s}) = (\hat{R}\mathbf{I}_c^+)(H,\mathbf{s}), \tag{3.6.14}$$

where

$$(\hat{R}\mathbf{I}^+)(H,\mathbf{s}) = \int\limits_{\Omega^+} \hat{R}(\mathbf{s},\mathbf{s}')\mathbf{I}^+(H,\mathbf{s}')d\mathbf{s}', \tag{3.6.15}$$

Due to the linearity of the boundary value problem (3.6.12) - (3.6.15) it is possible to find out the relation between the solution $\mathbf{I}_c(z,\mathbf{s})$ of the problem for the slab with nonreflecting boundaries and the corresponding solution $\tilde{\mathbf{I}}_c(z,\mathbf{s})$ of the same problem with reflecting boundary $z = H$:

$$\tilde{\mathbf{I}}_c(z,\mathbf{s}) = \mathbf{I}_c(z,\mathbf{s}) + (\hat{A}^-\hat{R}\hat{A}^+\mathbf{F})(z,\mathbf{s}). \tag{3.6.16}$$

It is natural to expect, that the relation between the solution $\mathbf{I}(z,\mathbf{s})$ of problem (3.6.1)-(3.6.4) for multiply scattered radiation for slab with non-reflecting boundaries and the solution $\tilde{\mathbf{I}}(z,\mathbf{s}) \equiv \mathbf{I}_R(z,\mathbf{s})$ of the same problem for the slab with reflecting boundary $z = H$ can be found out. The relation can be written in the form (Kuzmina, 1986 b)):

$$\tilde{\mathbf{I}}_R(z,\mathbf{s}) = \mathbf{I}(z,\mathbf{s}) + \bar{\hat{S}}\sum_{k=1}^{\infty}(\hat{R}\bar{\hat{S}})^k\hat{R}(\bar{\hat{S}}\tilde{\mathbf{F}})(z,\mathbf{s}) \equiv \hat{S}\tilde{\mathbf{F}} + \bar{\hat{S}}[\hat{E} - \hat{R}\bar{\hat{S}}]^{-1}\hat{R}\bar{\hat{S}}\tilde{\mathbf{F}}, \tag{3.6.17}$$

where

$$\begin{aligned} &\mathbf{I}(z,\mathbf{s}) = \hat{S}\tilde{\mathbf{F}} \equiv \hat{D}^{-1}(\tilde{\mathbf{F}} + \hat{P}\mathbf{I}), \quad \bar{\hat{S}}\tilde{\mathbf{F}} = \hat{S}[\mathbf{f}_0\tilde{\delta}(z) + \mathbf{f}_H\tilde{\delta}(z-H)], \\ &\hat{E} = diag\{1,\ 1,\ 1,\ 1\}. \end{aligned} \tag{3.6.18}$$

It should be noted, that the relation between the solution $\tilde{\mathbf{I}}(z,\mathbf{s}) \equiv \mathbf{I}_R(z,\mathbf{s})$ of multi-scattered radiation transport problem for slab with reflecting boundary and the solution $\mathbf{I}(z,\mathbf{s})$ of the same problem with non-reflecting boundary does not depend on concrete form of Green function, governing the behaviour of non-scattered radiation in the slab. Similar relations are valid for transport problems for slabs of optically isotropic medium in analogous situations (Germogenova, 1985).

4. The estimation of medium weak anisotropy influence by a perturbation method

4.1 The reduction of transport problem for anisotropic medium to a recurrently solvable system of problems for isotropic media

We return to the problem of multi-scattered radiation transport in a slab of geometrically isotropic optically active medium, that can be formulated as a boundary value problem to the VRTE

$$\mu \frac{\partial \mathbf{I}(\tau,\mathbf{s})}{\partial z} + \hat{\sigma}(\tau)\mathbf{I} = (\hat{P}\mathbf{I})(\tau,\mathbf{s}) + \mathbf{F}(\tau,\mathbf{s}), \tag{4.1.1}$$

$$\mathbf{I}^{+}(0,\mathbf{s}) = \mathbf{f}_0^{+}(\mathbf{s}), \tag{4.1.2}$$

$$\mathbf{I}^{-}(\tau_H,\mathbf{s}) = \mathbf{f}_H^{-}(\mathbf{s}), \tag{4.1.3}$$

where (see (3.1.8*), (3.1.14))

$$\hat{\sigma}(\tau) = \bar{\sigma}_t \begin{bmatrix} 1 & 0 & 0 & \Delta(\tau) \\ 0 & 1 & \delta(\tau) & 0 \\ 0 & -\delta(\tau) & 1 & 0 \\ \Delta(\tau) & 0 & 0 & 1 \end{bmatrix}, \tag{4.1.4}$$

$\hat{P}\mathbf{I}$ is the integral operator of scattering specified by the expressions (3.1.10), (3.1.11), and

$$\bar{\sigma}_t = 0.5(\sigma_t^{+} + \sigma_t^{-}), \quad \tau \equiv \tau(0,z) = \int_0^z \bar{\sigma}_t(x)dx. \tag{4.1.5}$$

The functions $\Delta_s \equiv \Delta\sigma_s = \sigma_s^{+} - \sigma_s^{-}$, $\Delta_a \equiv \Delta\sigma_a = \sigma_a^{+} - \sigma_a^{-}$, and $\Delta n = n^{+} - n^{-}$ represent macro-characteristics of optically active medium. The optically anisotropic medium may be considered as a weakly anisotropic one if Δ_s, Δ_a and Δn (as the functions of variable z) are uniformly small for all $z \in [0,H]$. In the case a small parameter can be introduced and a perturbation method may be developed. Introduce the values

$$\bar{\Delta}_s = \sup_{z\in[0,H]} |\Delta_s(z)|, \quad \bar{\Delta}_a = \sup_{z\in[0,H]} |\Delta_a(z)|,$$

$$\bar{\delta} = \frac{\omega}{c} \sup_{z\in[0,H]} \frac{|n^{+}(z) - n^{-}(z)|}{\bar{\sigma}_t(z)}, \tag{4.1.6}$$

where the symbol *sup* (supremum) denotes the least upper bound of the function (see, for example (Rudin, 1976)). Then the value

$$\varepsilon = \max(\bar{\Delta}_s,\ \bar{\Delta}_a,\ \bar{\delta}) \tag{4.1.7}$$

can figure as the mentioned small parameter.

In terms of ε the operators the transport equation (4.1.1) can be represented in the following forms. The extinction operator can be written as

$$\hat{\sigma}(\tau) = diag(1,\,1,\,1,\,1) + \varepsilon \begin{bmatrix} 0 & 0 & 0 & \tilde{\Delta}(\tau) \\ 0 & 0 & \tilde{\delta}(\tau) & 0 \\ 0 & -\tilde{\delta}(\tau) & 0 & 0 \\ \tilde{\Delta}(\tau) & 0 & 0 & 0 \end{bmatrix}, \tag{4.1.8}$$

where

$$\tilde{\Delta} = \varepsilon^{-1}\Delta(\tau), \quad \bar{\delta} = \varepsilon^{-1}\delta(\tau), \ |\tilde{\Delta}(\tau)| \le 1, \ |\tilde{\delta}(\tau)| \le 1, \ \tau \in [0, \tau_H]. \tag{4.1.9}$$

Based on physical cosiderations (Zege, Chaikovskaya, 1984), the phase matrix $\hat{\Gamma}(\tau;\gamma)$, defining the integral operator of scattering accordingly to (3.1.10) and (3.1.11), can be also presented in the form of the sum

$$\hat{\Gamma}(\tau;\gamma) = \hat{\Gamma}^0(\tau;\gamma) + \varepsilon\,\hat{\hat{\Gamma}}(\tau;\gamma), \tag{4.1.10}$$

where

$$\hat{\Gamma}^0(\tau;\gamma) = \bar{\lambda}(\tau)\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & c_1 \\ 0 & 0 & -c_1 & a_4 \end{bmatrix}, \quad \hat{\hat{\Gamma}}(\tau;\gamma) = \begin{bmatrix} 0 & 0 & \tilde{c}_2 & \tilde{b}_2 \\ 0 & 0 & \tilde{c}_3 & \tilde{b}_3 \\ -\tilde{c}_2 & -\tilde{c}_3 & 0 & 0 \\ \tilde{b}_2 & \tilde{b}_3 & 0 & 0 \end{bmatrix} \tag{4.1.11}$$

the elements of $\hat{\hat{\Gamma}}(\tau;\gamma)$ being of the $O(1)$ order.

By using the decompositions (4.1.8) and (4.1.10) one may present the solution of the boundary value problem (4.1.1)-(4.1.3) in the form of expansion into a series on ε powers:

$$\mathbf{I}(\tau,\mathbf{s}) = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{I}^{(n)}(\tau,\mathbf{s}), \tag{4.1.12}$$

$$\mu\frac{\partial \mathbf{I}^{(0)}(\tau,\mathbf{s})}{\partial z} + \mathbf{I}^{(0)} = (\hat{P}^{(0)}\mathbf{I}^{(0)})(\tau,\mathbf{s}) + \mathbf{F}(\tau,\mathbf{s}), \tag{4.1.13}$$

$$\mathbf{I}^{(0)+}(0,\mathbf{s}) = \mathbf{f}_0^+(\mathbf{s}), \tag{4.1.14}$$

$$\mathbf{I}^{(0)-}(\tau_H,\mathbf{s}) = \mathbf{f}_H^-(\mathbf{s}), \tag{4.1.15}$$

and

$$\mu\frac{\partial \mathbf{I}^{(n)}(\tau,\mathbf{s})}{\partial z} + \mathbf{I}^{(n)} = (\hat{P}^{(0)}\mathbf{I}^{(n)})(\tau,\mathbf{s}) + (\hat{P} - \hat{\sigma})\mathbf{I}^{(n-1)}, \tag{4.1.16}$$

$$\mathbf{I}^{(n)+}(0,\mathbf{s}) = 0, \quad \mathbf{I}^{(n)-}(\tau_H,\mathbf{s}) = 0, \quad n = 1, 2, \ldots \tag{4.1.17}$$

The convergence of the perturbation method has been analyzed for the case of homogeneous optically active medium ($\Delta(\tau) = const,\ \delta(\tau) = const$) and external beam of non-polarized radiation, incident to the slab boundary $z = 0$,

$$\mathbf{F} \equiv 0, \ \ \mathbf{f}^+(0,\mathbf{s}) = \mathbf{I}^{inc}(\mathbf{s})[1,\ 0,\ 0,\ 0]^T, \quad \mathbf{f}\ \ (\tau_H,\mathbf{s}) = 0.$$

For the proof of the perturbation method convergence it was necessary to use the integral vector transport equation instead of integro-differential transport equation (4.1.1) (Kuzmina, 1991). It is worth mentioning that the properties of the scalar integral transport equation and the integral characteristic equation, defining asymptotic characteristics of deep radiative regimes, were extensively analytically studied in (Maslennikov, 1968, 1969). The effective methods of analytical and computational studies of the scalar integral characteristic equation were further proposed in the publications (Rogovtsov et al., 2009, 2016; Rogovtsov, 2015 a), b)). In particular, the application of general invariance principles to various scalar radiative transfer problems allowed to carry out a number of analytical results (such as analytical representation of "surface» and "volume" Green functions, plane and spherical albedos and others) (Rogovtsov et al., 2016).

Finally, the estimate of the total Stokes vector perturbation has been derived in the form

$$|\mathbf{I}(\tau,\mathbf{s})-\mathbf{I}^{(0)}(\tau,\mathbf{s})|\leq C\varepsilon[1,\ 1,1,\ 1]^T,\ \ (\tau,\mathbf{s})\in D=[0,\tau_H]\times\{\Omega\setminus\{\mathbf{s}|\ \mu=0\}\}. \qquad (4.1.18)$$

(The Cartesian product of the sets is denoted by the symbol $\times$ in Eq. (4.1.18)).

The constant C in Eq. (4.1.18) depends on the essential parameters of the transport problem. In the case of non-conservatively scattering medium ($\bar{\lambda}(\tau)<1$) the constant C can be estimated in terms of the following parameters

$$\begin{aligned}
&\bar{\lambda}=\sup_{\tau\in[0,\tau_H]}\bar{\lambda}(\tau), \qquad I_{\max}^{inc}=\sup_{\mathbf{s}\in\Omega} I^{inc}(\mathbf{s})\\
&|\,\Gamma\,|=\sup_{\tau\in[0,\tau_H]}\sup_{m,n}\|\Gamma_{mn}^{0}\|_{L_2[-1,1]},\\
&|\,\tilde{\Gamma}\,|=\sup_{\tau\in[0,\tau_H]}\sup_{m,n}\|\tilde{\Gamma}_{mn}\|_{L_2[-1,1]}.
\end{aligned} \qquad (4.1.19)$$

As far as the estimated by the perturbation method Stokes vector

$$\mathbf{I}(\tau,\mathbf{s})=\mathbf{I}^{(0)}(\tau,\mathbf{s})|+C\varepsilon[1,\ 1,1,1]^T, \qquad (4.1.20)$$

has to satisfy the inequality $I^2-Q^2-U^2-V^2\geq 0$, the following constraint on the small parameter has been obtained (Kuzmina, 1991):

$$\varepsilon<\varepsilon_0\frac{(1-\bar{\lambda})(1-p_{\max})I_{\min|}^{(0)}}{|\Gamma|I_{\max}^{(inc)}}, \qquad (4.1.21)$$

where

$$\begin{aligned}
&I_{min}^{(0)}=\inf_{(\tau,\mathbf{s})\in D} I^{(0)}(\tau,\mathbf{s}),\\
&p_{\max}=\sup_{(\tau,\mathbf{s})\in D}\ (I^{(0)})^{-1}[(Q^{(0)})^2+(U^{(0)})^2+(V^{(0)})^2]^{1/2}, \qquad (4.1.22)
\end{aligned}$$

ε_0 being some constant, not depending on the parameters of transport problem for slab of optically active medium, and the symbol *inf* (infimum) denotes the greatest lower bound of the function (Rudin, 1976). The constraint (4.1.21) for ε should imply that there are some transport problems for slabs of optically anisotropic media in which the perturbation of the solution due to medium anisotropy might be not small.

4.2 The estimation of weak medium anisotropy influence

The uniform estimate for total perturbation of multiple scattered radiation transport solution for a slab of weakly anisotropic optically active medium as compared to corresponding problem for optically isotropic mediun (with mean optical characteristics) by a perturbation method has been obtained. However, some remarks on possibility of estimation of weak medium anisotropy influence using the perturbation method series should be made.

The first note concerns the fact that the transport problem is defined in a non-compact domain D of variables $(\tau, \mathbf{s})$: $(\tau, \mathbf{s})\in D$, $D=[0, \tau_H]\times\{\Omega\setminus\{\mathbf{s}\,|\,\mu=0\}\}$. Usually in such a case perturbation method expansions converges non-uniformly in D, especially in the situations when the equations for perturbed problem and for unperturbed one differ qualitatively. It is just our case. Indeed, as the results of qualitative analysis of the two-dimensional dynamical system (3.5.9) showed, the point $\Delta = 0, \delta = 0$ in the parametric domain of the system (the point just corresponds to optically isotropic medium and $\varepsilon = 0$ in the perturbation method expansion) is a bifurcation point of the dynamical system (see section 3.5). However, for the transport problem, formulated in terms of Stokes vector, the polarization characteristics of radiation field represent the interest only in the regions where the radiation intensity does not vanish. So, the Stokes parameters are just the adequate characteristics in the sense, and the global deviation of the transport problem solution in the slab of optically active medium from that one in the slab of corresponding isotropic medium still can be estimated.

The second note concerns the constraint (4.1.21) for ε, obtained in the process of the perturbation method convergence proof. The constraint should imply that the radiation transport problems for optically active media actually should be better treated independently, as a special class of transport problems.

4.3 An example: the estimation of polarization characteristics perturbation in a slab of isotropic medium with non-block-diagonal scattering phase matrix

Optically isotropic medium composed of chaotically distributed non-spherical scatterers can be characterized by non-block-diagonal phase matrix of the type (3.1.11). Sometimes it can be of interest what is the effect of the phase matrix non-block-diagonality on polarization characteristics of multiply scattered radiation in isotropic medium (in comparison with the same transport problem for the medium specified by the block-diagonal phase matrix). A qualitative answer can be obtained via application of the perturbation method.

Consider the radiation transport problem for the slab of $0 \le z \le H$ of optically isotropic mediim with non-reflecting boundaries, defined by the equations (see (3.2.1)-(3.2.5):

$$\mu \frac{\partial \mathbf{I}(z,\mathbf{s})}{\partial z} + \hat{\sigma}(z)\mathbf{I} = (\hat{P}\,\mathbf{I})(z,\mathbf{s}) + \mathbf{F}(z,\mathbf{s}), \tag{4.3.1}$$

$$\mathbf{I}^{+}(0,\mathbf{s}) = \mathbf{f}_0^{+}(\mathbf{s}), \tag{4.3.2}$$

$$\mathbf{I}^{-}(H,\mathbf{s}) = \mathbf{f}_H^{-}(\mathbf{s}). \tag{4.3.3}$$

where $\hat{\sigma}_t$ is the scalar operator, $\hat{\sigma}_t = \sigma_t(z)\cdot diag\{1,\ 1,\ 1,\ 1\}$. Let the integral operator of scattering $\hat{P}$ (see (3.1.10) is defined by the phase matrix

$$\hat{\Gamma}(z;\gamma) = \hat{\Gamma}^{(0)}(z;\gamma) + \varepsilon\,\hat{\Gamma}^{(1)}(z;\gamma), \tag{4.3.4}$$

where

$$\hat{\Gamma}^{(0)} = \overline{\sigma}_s \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & c_1 \\ 0 & 0 & -c_1 & a_4 \end{bmatrix}, \quad \hat{\Gamma}^{(1)} = \overline{\sigma}_s \begin{bmatrix} 0 & 0 & c_2 & b_2 \\ 0 & 0 & c_3 & b_3 \\ -c_2 & -c_3 & 0 & 0 \\ b_2 & b_3 & 0 & 0 \end{bmatrix} \tag{4.3.5}$$

and $\varepsilon \ll 1$ (see (4.1.10)-(4.1.11)). As it was marked in the section 3.1, the condition

$$\frac{1}{2}\int_{-1}^{1} \Gamma_{14}(\gamma)d\gamma \equiv \frac{1}{2}\int_{-1}^{1} b_2(\gamma)d\gamma = 0 \tag{4.3.6}$$

should be fulfilled in the case of optically isotropic medium. The “influence” of small non-block-diagonality of the phase matrix on the transport problem solution for a slab of optically isotropic medium can be estimated by the perturbation method (Kuzmina , 1987).

It is convenient to schematically present the solution of multi-scattered radiation transfer problem (4.3.1)-(4.3.5) in the compact form (the decomposition on successive orders of scattering)

$$\mathbf{I}(z,\mathbf{s}) \equiv \hat{S}\tilde{\mathbf{F}} = \sum_{k=0}^{\infty}(\hat{A}\hat{P})^k \hat{A}\,\tilde{\mathbf{F}}(z,\mathbf{s}), \tag{4.3.7}$$

where $\tilde{\mathbf{F}}$ is defined by the formula

$$\tilde{\mathbf{F}}(z,\mathbf{s}) = \mathbf{F}(z,\mathbf{s}) + \mu\mathbf{f}_0^{+}(\mathbf{s})\tilde{\delta}(z) + |\mu|\,\mathbf{f}_H^{-}(\mathbf{s})\tilde{\delta}(z-H), \tag{4.3.8}$$

(see (3.4.3)). The operator $\hat{A}$ in Eq. (4.3.7) is the operator of attenuation (extinction) of non-scattered radiation, defined by the expression

$$\hat{A}\tilde{\mathbf{F}}(z,\mathbf{s}) = \hat{A}^{+}\tilde{\mathbf{F}} + \hat{A}^{-}\tilde{\mathbf{F}}, \tag{4.3.9}$$

where $\hat{A}^{+}\tilde{\mathbf{F}}$, $\hat{A}^{-}\tilde{\mathbf{F}}$ are defined accordingly to (3.6.9) and (3.6.10). For optically anisotropic medium the operator $\hat{A}$ is the matrix operator whereas for optically isotropic medium it is the scalar operator, defined by the scalar Green function

$$\hat{G}(z/\mu, z'/\mu) = e^{-\frac{\overline{\tau}(z,z')}{\mu}}\, diag\{\,1, 1, 1, 1\,\}, \quad \overline{\tau}(z,z') = \int_z^{z'} \sigma_t(x)dx \tag{4.3.10}$$

Therefore, as far as radiation attenuation in optically isotropic medium does not change the polarization characteristics of radiation ($\hat{A}\hat{P}=\hat{P}\hat{A}$), the series (4.3.7) can be rewritten in the form

$$\mathbf{I}^{(isotr)}(z,\mathbf{s}) \equiv \hat{S}^{(isotr)}\tilde{\mathbf{F}} = \sum_{k=0}^{\infty}\mathbf{I}^{(k)} = \sum_{k=0}^{\infty}\hat{A}^{k+1}\hat{P}^{k}\,\tilde{\mathbf{F}}(z,\mathbf{s}). \tag{4.3.11}$$

As one can see, the *k*-the term of the expansion (4.3.11) may be presented in the form

$$\mathbf{I}^{(k)}(z,\mathbf{s}) = \hat{A}\hat{\hat{P}}\mathbf{I}^{(k-1)} + \hat{A}S^{(isotr)}\hat{\hat{P}}\mathbf{I}^{(k-1)}, \tag{4.3.12}$$

where $\hat{\hat{P}}$ is the integral operator of scattering, defined by the phase matrix $\hat{\hat{\Gamma}} = \hat{\Gamma}^{(0)}\hat{\Gamma}^{(1)}$ (obviously, the matrix $\hat{\hat{\Gamma}}$ possesses the algebraic structure similar to that of the matrix $\hat{\Gamma}^{(1)}$).

For the further analysis of the terms $\mathbf{I}^{(k)}(z,\mathbf{s})$ of the series (4.3.11) (expressed through $\hat{P}^{k}\,\mathbf{I}$) it is necessary to exploit the representation of $\mathbf{I}(z,\mathbf{s})$ in the form of Fourier series decomposition over the system of generalized spherical functions $\{\hat{Y}^{l}_{ms}\}$. If the Stokes parameters are used as the vector $\mathbf{I}(z,\mathbf{s})$ components ($\mathbf{I}(z,\mathbf{s})=(I,Q,U,V)^{T}$), the Fourier series can be written in the form (Kuzmina, 1978);

$$\mathbf{I}(z;\mu,\varphi)=\sum_{l=0}^{\infty}\sum_{s=-l}^{l}\hat{Y}^{l}_{s}\,\mathbf{I}^{l}_{s}(z;\mu,\varphi), \tag{4.3.13}$$

where

$$\hat{Y}^{l}_{s}=\begin{bmatrix} Y^{l}_{0s} & 0 & 0 & 0\\ 0 & \frac{1}{2}(Y^{l}_{2s}+Y^{l}_{-2s}) & \frac{i}{2}(Y^{l}_{2s}-Y^{l}_{-2s}) & 0\\ 0 & \frac{i}{2}(Y^{l}_{2s}-Y^{l}_{-2s}) & \frac{1}{2}(Y^{l}_{2s}+Y^{l}_{-2s}) & 0\\ 0 & 0 & 0 & Y^{l}_{0s}\end{bmatrix},\quad \mathbf{I}^{l}_{s}=\begin{bmatrix} I^{l}_{0s}\\ \frac{1}{2}(Q^{l}_{2s}+Q^{l}_{-2s})-\frac{i}{2}(U^{l}_{2s}-U^{l}_{-2s})\\ \frac{1}{2}(U^{l}_{2s}+U^{l}_{-2s})+\frac{i}{2}(Q^{l}_{2s}-Q^{l}_{-2s})\\ V^{l}_{0s}\end{bmatrix} \tag{4.3.14}$$

The functions $(\hat{P}^{k}\mathbf{I})(z;\mu,\varphi)$, entering into the series (4.3.11), can be written in terms of series over $\{\hat{Y}^{l}_{ms}\}$ as

$$(\hat{P}^{k}\mathbf{I})(z;\mu,\varphi)=\sum_{l=0}^{\infty}\sum_{s=-l}^{l}\hat{Y}^{l}_{s}(\hat{\Gamma}^{(l)}_{Y})^{k}\,\mathbf{I}^{l}_{s}(\tau,\mathbf{s}), \tag{4.3.15}$$

where $\hat{\Gamma}^{(l)}_{Y}$ are the matrices, containing proper combinations of the decomposition coefficients on $\{\hat{Y}^{l}_{ms}\}$ of the phase matrix $\hat{\Gamma}$ (Kuzmina, 1987).

The estimation of the solution perturbation caused by small non-block-diagonality of the phase matrix can be rather easily obtained for the simplest transport problem - the axially symmetric transport problem for slab, illuminated by external mono-directed linearly polarized radiation beam, normally incident to the boundary $z=0$:

$$\sigma_t = \sigma_t^{(0)}(z),\ \mathbf{F}(z,\mathbf{s}) = 0,\ \mathbf{f}_H(\mathbf{s}) = 0,\ \mathbf{f}_0(\mathbf{s}) \equiv \mathbf{f}_0^* = (I_0,\ Q_0,\ 0,\ 0)^T\ \delta(z) \qquad (4.3.16)$$

Via using the expression (4.3.12) and the decomposition (4.3.15), the following relations can be obtained:

$$\hat{\hat{P}}\tilde{\mathbf{F}} = \hat{\hat{P}}\mathbf{f}_0^*\delta(z) = \mathbf{I}^* \equiv (0,\ 0,\ U^*,\ V^*)^T;\quad \hat{\hat{P}}\tilde{\mathbf{F}} = \bar{\mathbf{I}}^*;\quad \hat{P}^{(isotr)}\mathbf{I}^* = \mathbf{I}^0 \equiv (I^*,\ Q^*,\ 0,\ 0)^T, \quad (4.3.17)$$

Using (4.3.7), one can finally obtain, that the Stokes vector $\mathbf{I}^{(1)}(z;\mu,\varphi)$ is of the form

$$\mathbf{I}^{(1)}(z;\mu,\varphi) = (0,\quad 0,\quad U_1(z,\mu,\varphi),\ \ V_1(z,\mu,\varphi))^T, \qquad (4.3.18)$$

where the functions $U_1(z,\mu,\varphi)$ and $V_1(z,\mu,\varphi)$ in (4.3.18) do not contain the zero $\mu-$harmonics. It is the consequence of the fact that due the medium isotropy ($\Delta_s = 0$, see (3.1.13)) we have $\hat{\hat{\Gamma}}^{(0)} = \hat{0}$. Further qualitative analysis based on the decomposition (4.3.15) allows to elucidate the general properties of the remaining functions $\mathbf{I}^{(k)}(z,\mathbf{s}),\ k>1$: the functions $\mathbf{I}^{(2k-1)}(z,\mathbf{s})$, defining the contributions $O(\varepsilon^{2k-1})$ into the total solution, have the form

$$\mathbf{I}^{(2k-1)}(z;\mu,\varphi) = (0,\quad 0,\quad U_{2k-1}(z,\mu,\varphi),\ \ V_{2k-1}(z,\mu,\varphi))^T, \qquad (4.3.19)$$

whereas the functions $\mathbf{I}^{(2k)}(z,\mathbf{s})$, defining the contributions $O(\varepsilon^{2k})$ have the form

$$\mathbf{I}^{(2k)}(z;\mu,\varphi) = (I_{2k}(z,\mu,\varphi),\ \ Q_{2k}(z,\mu,\varphi),\quad 0,\quad 0)^T. \qquad (4.3.20)$$

The Fourier analysis allows to extract an essential qualitative information on the radiation transport problem solution for the slab of optically isotropic medium specified by non-block-diagonal phase matrix, defined by (4.3.4)-(4.3.5). Namely, we have the following result: a) the multiply scattered light in the slab is weakly elliptically polarized; b) the deviations of both the radiation intensity and the linear polarization degree from the corresponding characteristics of unperturbed transport problem (specified with block-diagonal phase matrix $\hat{\Gamma}^{(0)}$) are of $O(\varepsilon^2)$ (Kuzmina, 1978).

5. An outline of some results on radiation transfer problems in anisotropic media of another types

5.1 Anisotropic media in the Earth atmosphere remote sensing problems

It is now well recognized that cirrus clouds have a major influence on the Earth-ocean-atmosphere energy balance. The macroscopic optical properties of a disperse media consisting of scattering particles, randomly oriented in the space, is ultimately defined by particle microscopic characteristics (particle size, shape and the refractive index) and the distribution function on the particle orientations. If the orientations of non-spherical particles of disperse medium are not totally random, the medium is proved to be optically anisotropic. The ice crystal clouds (cirrus and cirrostratus) provide the examples of optically anisotropic media, demonstrating the well-known atmospheric optical phenomenon of halo. The crystals responsible for halo may be horizontally oriented flat, hexagonal plates or oriented column-shaped crystals**.** The ice crystals can be suspended near the ground, in which case they are referred to as diamond dust. When the dust anisotropic medium is formed by column-shaped crystals the known phenomenon of light pillars can be observed.

A) Ice crystal clouds

Ice cloud disperse optically anisotropic media, formed by spatially oriented suspended tiny ice crystals, belong to the general class of essentially optically anisotropic media, in which medium optical anisotropy is accompanied by its geometrical anisotropy. The well-known atmospheric optical phenomenon of halo is just created by light reflection from these anisotropic media. Another familiar phenomenon is light pillars that is produced by light reflection from anisotropic media formed by column-shaped ice crystals (see Fig. 5.1.1 - 5.1.2). Modeling of polarized radiative transfer in the anisotropic media requires construction of the matrix extinction operator and the scattering phase matrix of the vector transport equation, governing radiation transport in the anisotropic medium. Various models of disperse anisotropic media were designed and the operators of the VRTE were constructed. In particular, disperse medium models composed of chiral particles were created (Ablitt et al., 2006; Liu et al., 2013). The radiative transfer in a chiral anisotropic medium was studied via Monte Carlo simulations, and the effects of medium chirality were elucidated (Ablitt et al., 2006). The optical properties of scattering anisotropic medium models formed by ice crystals of cirrus clouds can be obtained based on geometrical and physical optics approaches (Borovoi et al., 2000, 2006, 2007, 2010; Borovoi, 2005, 2006, 2013).

Modeling of radiation transport processes in cirrus clouds is of importance for the Earth atmosphere remote sensing problems (Takano et al., 1989; 1993; Mishchenko et al., 2000; Liou, 2002; Mishchenko et al., 2002; Liou et al., 2011). The attempt of radiative transfer problem analysis for optically anisotropic medium, formed by horizontally oriented ice cloud crystals, has been performed in (Takano et al., 1993), based on vector transport equation with scalar extinction operator. As it was discussed in (Mishchenko, 1994), the approach could provide a significant error in radiative transfer problem solutions. To estimate the error it would be desirable to find the exactly solvable problem for the anisotropic medium for comparison the exact and the approximate results. Such comparison was previously fulfilled for another type of anisotropic medium model (composed of perfectly aligned prolate and oblate spheroids), and a significant discrepancy was demonstrated (Tsang et al., 1991; Ishimaru et al., 1984).

The Monte-Carlo simulations of radiation transfer in crystal cloud optically anisotropic media models have been performed (Grishin et al., 2004; Prigarin et al., 2005; Mishchenko et al., 2005). The medium models have been designed, and the functions entering into the vector transport equation have been calculated (Volkovitski et al., 1984; Takano, Liou, 1989; Borovoi et al., 2000]; Grishin et al., 2004; Kokhanovsky, 2005; 2006).

The Monte Carlo simulations of halos in crystal cloud models of optically anisotropic media have been performed. The computer simulation results demonstrated that the anisotropy of cloud medium can strongly affect the cloud optical properties. In particular, both halo patterns and angular distributions of the upward and downward radiation are strongly dependent on optical anisotropy characteristics. Besides it was found, that the cloud optical anisotropy can result not only from the shape and spatial orientation of cloud particles, but, in addition, it can be a consequence of non-Poisson spatial particle distribution (Prigarin et al., 2005 a), b); Prigarin et al., 2007; 2008).

It should be added, that although many important studies of ice crystal cloud media, composed of non-randomly distributed particles, have been undertaken (Liou, 2002; Kokhanovsky, 2003, 2004, 2005 a), b); Prigarin et al., 2005; 2007; 2008), further investigations of radiative transfer problems for optically anisotropic cloud media are still required.

Fig. 5.1.1 The example of ice halo

[www.ice-halo.net]

Fig. 5.1.2 The example of light pillars

[Sterlitamak, Russia, 19.12.2015, S.Lifanov]

B) Densely packed disperse media

Radiation transport in dense scattering media is of interest both from the viewpoint of the Earth remote sensing problems and from the viewpoint of a variety of other applications, including non-invasive medical investigation of biological tissues. Calculation of extinction matrix and scattering phase matrix for densely packed media composed of non-spherical wavelength-sized scatterers demands taking into account all the details of strongly inhomogeneous scattered radiation field in the vicinity of any scatterer (Borovoi et al., 1983, 2005, 2013).

The necessity of studying of multi-scattered radiation transport processes in the Earth ice-snow cover follows from the fact that both ice and snow covers belong to the class of strongly reflecting Earth surfaces (the reflectance of pure snow cover can achieve 90% in the visible wavelength band) (Kokhanovsky, 1998; Farrell et al., 2005; Kokhanovsky, 2011). The importance of ice-snow cover monitoring is related to climatology problems: as it is established experimentally, the Earth surface, covered by ice and snow, is shrinking rather quickly over the past 25 years (Munneke, 2009) (see Fig. 5.1.3).

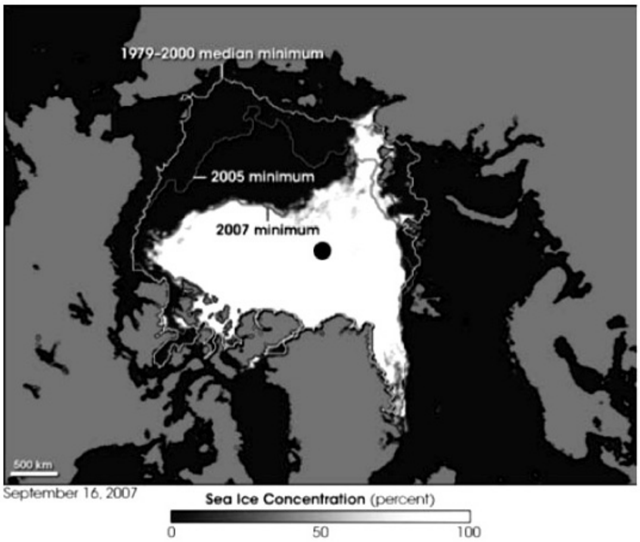


Fig. 5.1.3 The minimum sea-ice extent and concentration in the Arctic Ocean. http;//www.ncidc.org, National Snow and Ice Data Center, USA. [Munneke, 2009].

In remote sensing problems both snow and clouds can be treated as disperse media consisting of mutually independent ice crystals. The snow layers can be also modelled as a random disperse media with densely packed particles of non-spherical shape (Kokhanovsky, 1998). Sometimes the snow layer can be also modelled as an ice cloud consisting of fractal particles (in the visible wavelength band) (Kokhanovsky, 2003; Liou et al., 2011). Snow particle size, pollutant concentrations, and the snow layer thickness represent the essential model parameters. The analytical approximation to radiation transport processes in snow cover layers has been developed (Kokhanovsky, 2011). It provided the possibility to compare the calculated snow cover characteristics (such as snow particle size, pollutant concentration, snow cover albedo) with the results of satellite measurements (Kokhanovsky, 2005 a); 2011).

The results of accurate computer study of multiple electromagnetic radiation scattering by densely packed disperse medium models can be found in (Tse et al, 2007; Tsang et al., 2007; Tseng, 2008; Okada, Kokhanovsky, 2009; Randrianalisoa, 2010; Dlugach et al., 2011).

In another approach a two-layer model of radiation transfer in the atmosphere-snow system was designed, in which the lower layer was modelled as a disperse medium consisting of hexagonal ice crystals (Munneke, 2009). It has been previously hypothesized (Wiscombe, Warren, 1980) that scattering and absorptive properties of any ice crystal model can be approximated by the appropriate disperse medium model composed of spherical particles, as long as the volume-to-surface radio is conserved. Accurate computation of radiative transfer problem in the two-layer snow-atmosphere model has been performed. The medium model was further extended to that consisting of both snow and cloud layers. The influence of cloud layer presence on snow surface albedo was demonstrated. Thus, it could be estimated as an additional evidence that clouds have a considerable impact on the radiation balance of the atmosphere-snow system (Munneke, 2009). In a whole, clouds increase the broadband clear-sky albedo of the snow cover. The concurrent observations were compared with model calculations, providing good results.

The major results, obtained for multiply cattered radiation transport problems in snow cover, were found under the assumption that the effective continuous medium, corresponding to coherently scattered radiation propagation, is an optically isotropic medium. However, in the case of densely packed ensembles of scatterers in a number of situations the effective medium may turned out to be optically anisotropic. There exist recent papers devoted to the studying of optical characteristics of these effective media (see, for example, (Alonova et al, 2013)). The model of disperse medium of densely packed spheres, used in the problem of active microwave remote sensing of terrestrial snow, was treated in (Tsang et al, 2011), and a significant value of cross polarization was obtained. The results are consistent with the experimental observations.

C) The extinction and scattering phase matrices for models of disperse optically anisotropic media

A great variery of disperse optically anisotropic medium models (formed by ensembles of non-spherical particles with random and preferred types of particle orientations) have been designed with the aim of accurate calculation of extinction operators and scattering phase matrices of the VRTE governing the radiative transfer in anisotropic media. It allowed to study the dependence of the medium scattering macro-characteristics on the parameters of medium microstructure (Mishchenko et al., 1992; Alexandrov, et al., 1993; Mishchenko, 1994; Bolgov et al., 1998; Roux et al., 2001; Mishchenko et al., 2007; 2016; Xie et al., 2011; 2012; Shefer, 2013; 2016; Gao et al., 2012; 2013; Liu et al., 2013; Yang et al., 2013; Marinyuk, 2015). For some models the medium backscattering efficiencies have been also estimated. For example, it was done for a model of polydisperse medium consisting of disordered randomly distributed infinite Mie cylinders with different refractive indices. Under medium illumination perpendicularly to the cylinder axes the albedo problem for homogeneous half-space was analyzed. The coherent backscattering factors for several two-dimensional medium models were found as well (Mishchenko et al, 1992). In the paper (Gao et al., 2012) the medium phase matrices were calculated (by the Discrete Dipole Approximation method) and the backscattering efficiencies were estimated for the disperse medium composed of small layered plates. Several ice cloud models consisting of smooth, roughened, homogeneous and inhomogeneous hexagonal ice crystals with various aspect ratios were designed and studied with the aim of application to the satellite-based retrieval of ice cloud properties (Xie et al., 2012). The extinction matrices were calculated for the medium model composed of plates (both infinite-radius plates and finite-size particles) (Gao et al., 2013). A medium model composed of chiral particles was designed, a helical liquid crystal model of a capsule shape being used for modeling of single medium scatterer (Liu et al., 2013), The distribution on particle orientations of twist type was constructed. The matrix extinction operator of the VRTE (with 16 elements), providing the medium ability of differentiating left and right circularly polarized light, was obtained.

5.2 Magneto-gyrotropic media

The magnetoactive plasma – cold rarefied plasma in a permanent magnetic field $\mathbf{H}^0$ – represents an example of non-absorbing optically anisotropic (gyrotropic) medium with elliptical birefringence (Zheleznyzkov, 1977, 1996; Gnedin et al., 1979). In appropriate parametric domain the magnetoactive plasma can possess strong optical anisotropy. To analyze the anisotropy it is convenient to present the components of dielectric permittivity tensor $\hat{\varepsilon}$ in terms of parameters u and v, where $u = \omega_H^2 / \omega^2$, $v = \omega_L^2 / \omega^2$, ω_H – the cyclotron frequency for electron, ω_L – the plasma frequency. If one uses the coordinate system where the wave vector $\mathbf{k}$ is directed along the z axis, and $\mathbf{H}^0$ is located in the plane

(x, y), α being the angle between $\mathbf{H}^0$ and z axis (that is, $\mathbf{H}^0 = (0, H_y^0, H_z^0)$), then the tensor $\hat{\varepsilon}$ can be written in the form:

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon_0 - g\sqrt{u} & ig\cos\alpha & -ig\sin\alpha \\ -ig\cos\alpha & \varepsilon_0 - g\sqrt{u}\cos^2\alpha & 0 \\ ig\sin\alpha & g\sqrt{u}\cos\alpha\sin\alpha & \varepsilon_0 - g\sqrt{u}\sin^2\alpha \end{bmatrix}, \qquad (5.2.1)$$

where

$$\varepsilon_0 = \varepsilon^{(isotr)} = 1 - v, \quad g = \frac{v\sqrt{u}}{1-u}.$$

The tensor $\hat{\varepsilon}$ completely defines the refractive index squares n_o^2, n_e^2 of the normal wave modes in the medium, and their polarization states can be found through calculation of the eigenvalues and eigenvectors of the two-dimensional projection $\hat{\varepsilon}_{\perp}$ of the tensor (4.2.1) (Zheleznyzkov, 1977, 1996):

$$\hat{\varepsilon}_{\perp} = \begin{bmatrix} \varepsilon_0 - g\sqrt{u} & ig\cos\alpha \\ -ig\cos\alpha & \varepsilon_0 - g\sqrt{u}\cos^2\alpha \end{bmatrix}. \qquad (5.2.2)$$

Not writing down the explicit formulas for n_o^2, n_e^2 and the polarization states (Zheleznyzkov, 1996), we can mark here some limit cases and qualitative consequences. In general case (at $\alpha \neq 0$, $\alpha \neq \pi/2$) the normal waves are elliptically polarized and their polarization states are almost orthogonal. The n_o^2, n_e^2 are expressed in the form of complicated functions of u, v, ω, which are simplified in the cases of lengthwise ($\alpha = 0$) and transverse ($\alpha = \pi/2$) directions of propagation. The polarizations of the normal waves are reduced to circular (for lengthwise propagation) and to linear (for transverse propagation). It also should be noted that in general case the polarization states of the normal waves are not orthogonal, and in the situation the radiation intensity does not equal to the sum of intensities of the normal waves. The fact should be taken into account in transport problems for strongly optically anisotropic media.

5.3 Optically active media occurred in bio-medical field of research

Biological tissues belong to optically inhomogeneous absorbing media with the refracting indices greater than the refractive index of the air. They can be divided into two main classes – strongly scattering (turbid) and weakly scattering (transparent). The analysis of polarization characteristics of multiply scattered radiation in biological media is one the most important instruments for estimation the features of the internal media structure (via solution of the inverse problems of radiation transport). A wide variety of biological tissues belong to optically anisotropic media, demonstrating the birefringence of various types. For example, optical anisotropy of bio-tissues can be a consequence of the refractive index difference of the base matter and the collargen fibers. Chiral molecules are typically enclosed in bio-tissues, and so circular birefringence and optical activity are two common phenomena for radiation transfer in the media. In a whole, the bio-tissues usually belong to four large classes of optically anisotropic media: optically isotropic media, uniaxial crystals, biaxial crystals and optically active (chiral) media. The measurement of bio-tissue refractive indices is one of the actual problems of bio-tissue optics. On the other hand, the results of analysis of multiply scattered radiation transport through optically inhomogeneous bio-tissues provide a valuable information about the features of their internal structure. So, the design of adequate mathematical models of disperse bio-tissues is of importance. The four-component vector transport equation with matrix differential operator is necessary for modeling the radiation transport processes in optically anisotropic bio-tissue media.

In traditional polarimetry the multiply scattered light depolarization measurements are widely used for determining the concentrations of optically active molecules (such as glucose) in the scattering medium. The accurate modelling of polarized light propagation in turbid media, serving as templates for the biological tissues, and the comparison of the results of modelling with the measured data often demonstrates good agreement (Maruo et al, 2003; Larin et al, 2002). An example of application of Monte Carlo modeling of multiply scattered polarized light transport in linearly birefringent and optically active media, figuring as the models of biological tissues, was provided in (Wood et al., 2007). Measurements were also made using a Stokes polarimeter that detected the scattered light in different geometries. The comparison of the results of Monte Carlo simulations with the measurements showed a close agreement between both the results.

A closely related area of research concerns the application of radiation transport theory approaches to the problems of non-invasive medical diagnostics of non-heterogeneities in biological tissues. In the papers (Bass et al., 2009; Bass et al., 2010) the method of non-heterogeneities retrieval was based on the solution of direct problem – the obtaining of multiply scattered radiation field in the 3D spatial regions of optically isotropic turbid media (in arbitrary 3D region in (x, y, z)-geometry and axially symmetrical cylindrical region) under the illumination by an anisotropic (collimated) laser radiation source. The deterministic method for calculating the multiply scattered radiation field in the 3D regions was applied (instead of Monte Carlo simulations). The simulation of ultra-short light pulse propagation in turbid media was additionally realized (Bass et al., 2010), and the parallel computational algorithms were applied. The methods developed in (Bass et al., 2009; Bass et al., 2010) might be easily generalized to the corresponding problems for optically active media.

5.4 Multilayered anisotropic media

Multilayered plane structures consisting of various optically anisotropic materials have become increasingly widely used in modern optical systems such as narrow-band birefringent filters and many other semiconductor devices. In practice the optically anisotropic media are made of thin films, composites, artificial materials. The characteristics of these devices and their design are usually based on the detailed understanding of electromagnetic radiation propagation through these anisotropic layered media. The development of general theory of electromagnetic radiation propagation in birefringent layered media began since 1970-ths (see, for instance (Yeh, 1979; 1980)). Analytical approaches to studying of radiative processes in multilayer optically anisotropic structures were also developed (Stammes et al., 2001; Farrell et al., 2005; Kiasat et al., 2011).

Such new phenomena as the exchange Bragg scattering, optical surface waves, oscillatory evanescent waves were found in these media. Since the birefringent multilayer waveguides are of great importance in the integrated optics, the phenomena of radiation reflectance and transmittance in layered anisotropic media were extensively studied. The reflectance and transmittance coefficients for multilayered birefringent media can be expressed in terms of the overall transfer matrix components. The behavior of the evanescent and the guided waves give rise to interesting features of radiation transport in the birefringent layered structures. In contrast to optically isotropic media, where the evanescent waves have a pure imaginary propagation constant, in birefringent layered media the evanescent wave can decay exponentially with an oscillatory intensity distribution. Another interesting feature of periodic plane structures concerns the possibility of resonant radiation interaction with the medium (when radiation wave length is approximately equal to the layered structure period).

Multilayered structures, designed based on the porous silicon (PSi), play currently an important role in various applications. These include microcavities, photonic crystals, waveguide structures, photodetectors, sensors, etc. Besides, optically active materials can be designed based on porous silicon structures (for instance, via infiltration appropriate electro- or thermo-optic media into the pores). Therefore, the porous silicon structures are turned out to be excellent candidates for tunable optical interconnects and switches. Novel layered anisotropic structures are also applied in material science, electro-analytical chemistry, biological interfaces, tissue engineering, physics, and optics.

Two-dimensional periodic optically anisotropic structures are known as photonic crystals (PCs). These periodic structures have been currently extensively studied due to their wide abilities to control the light flows. Light transfer inside PCs can be analyzed via modelling the processes governed by the Maxwell equations. A variety of photonic devices (such as polarization-independent waveguides, wavelength demultiplexers, beam deflectors, and routers) can de designed by utilizing the interesting PC features (Kiasat et al., 2011; Giden et al., 2014).

Usually the PC lattices possess translational, rotational or mirror symmetry. However, under some conditions the PC cell may convert into a chiral medium. Photonic quasi-crystals are also represent significant interest and have been intensively studied, the PCs with chiral optical properties being of special interest. A variety of new photonic devices is expected to be created (beam routers, splitters, deflectors) based on understanding the features of the PC optical anisotropy.

5.5 Liquid crystals and optical fibers

As well known, in a normal liquid the molecule arrangement is equally disordered in all directions. Liquid crystals are anisotropic: the molecules have some degree of alignment, and the liquid crystal properties depend on the direction. In the nematic phase, the molecules are not layered and are free to rotate or slide. In the smectic phase the molecules maintain the general order of the nematic phase, but in addition aligned into layers. In the cholesteric phase, the molecules are directionally oriented and stacked in a helical pattern, each layer being rotated at a slight angle (see Fig. 5.5.1). Because of their anisotropic structures, liquid crystals exhibit unusual optical and electrical properties that are exploited in a great variety of applications. With the rapid development of nanosciences, and the synthesis of many new anisotropic nanoparticles, the number of various types of liquid crystals is quickly increasing. Theoretical study of fluid crystal microstructure is a quite complicated task because of their high density, many-particle correlations and anisotropy of particle interactions (de Gennes, 1974; Chandrasekhar, 1977; Yariv, Yeh, 1984). Chiral liquid crystal molecules usually give rise to chiral mesophases. A number of unusual interference effects can be observed in the chiral mesophases, which are very interesting for applications. For example, chiral liquid crystals can be used as tunable filters in electrooptical devices (for hyperspectral imaging).

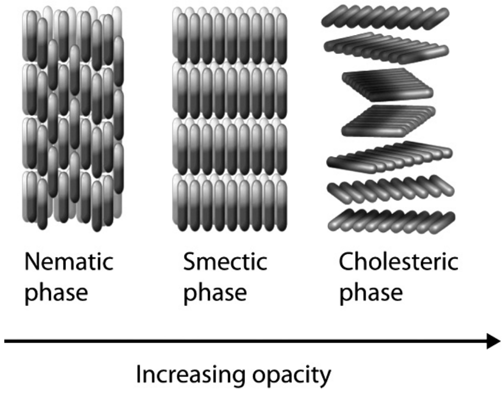


Fig. 5.5.1. The molecule arrangement in nematic, smectic and cholesteric liquid crystals.

Liquid crystals can demonstrate phase transitions of second order, spontaneous symmetry breaking, strong fluctuations, discontinuity (Arsenova, 2009).

Liquid crystal technology is exploited in many areas of science and engineering, as well as in device technology. Promising applications of this special kind of material are possible, providing new effective solutions to a great variety of problems.

Optical fibers are flexible, transparent fibers made by drawing glass of a diameter slightly thicker than that of a human hair. They are used as a means to transmit light between the two ends of the fiber and demonstrate successful utilization in the field of fiber-optic communications (since signals travel along them with extremely small dissipation). In addition, the fibers are also characterized by providing an electromagnetic interference. Specially designed fibers are successfully used in a variety of applications (including creation of fiber sensors). Fibers are also actively used in remote sensing. The fiber optics is the actively developed field of applied science and engineering. Optical fibers are widely used as sensors in measurements of intensity, phase, polarization, wavelength.

An optical fiber can be modelled as a cylindrical dielectric waveguide that transmits light along its axis with the help of the process of total internal reflection. The fiber consists of a core surrounded by a cladding layer. To confine the optical signal in the core, the refractive index of the core must be greater than that of the cladding. The boundary between the core and cladding may either be abrupt or gradual. In addition to internal diffuse light scattering, attenuation also occur due to selective absorption of specific wavelengths. The design of optical fibers requires the selection of materials based on knowledge of its properties and limitations.

Deeper understanding of light propagation in fiber-based materials is possible via modelling multiply scattered light transport problems in infinitely long cylindrical fibers, the structural properties of the fiber being taken into account. Numerical solution of radiative transfer equation by Monte Carlo method was so far used. In the way the relations between light diffusion and fiber structure characteristics were partially elucidated (Linder, 2014) The propagation of short pulses in birefringent single-mode fibers was studied as well (Menyuk, 1988).

The radiation transport problems for optically anisotropic media of fiber-like geometry might be considered as a new class of radiation transport problems where the work is still at the very beginning.

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References

Ablitt B.P., Hopcraft K.I., Turpin K.D., Chang P.C.Y., Walker C.G. & Jakeman E., (2006). Imaging and multiple scattering through media containing optically active particles, DOI: 10.1088/0959-7174/9/4/308.

Alexandrov M.D., Rogozkin D.B, and Remizovich V.S, (1993). Multiple light scattering in a two-dimensional medium with large scatterers, J. Opt. Soc. Am. A **10**, 2602–2610.

Alonova] M.V., Angelsky O.V., Ermolenko S.B., Zimnyakov D.A, Isaeva E.A., Sina J.S., Skurlov I.D., Tverdova A.A., Ushakova O.V., 2013: Optical properties of densely packed dispersive systems: Effective medium approximation, Vestnik SGP, **3,** 72(in Russian).

Apresyan L.A, Kravtsov Yu.A., (1996). *Radiation transfer. Statistical and wave aspects*. Basel, Gordon and Breach 1996. (Original Russian edition: Nauka, Moscow, 1979.)

Arsenova E.A., 2009. *Correlation functions and the features of transfer and scattering of waves in liquid crystals*, Doct. Thesis, S.-Petersburg,(in Russian).

Astrov D.N., (1960). The magnetoelectric effect in antiferromagnetics, *Zh. Eksp. Teor. Fiuz*., **38**, 984-985 (in Russian).

Azzam R.M. and Bashara N.M., (1989). *Ellipsometry and polarized light*, North Holland PC, NY.

Azzam R.M., (1978). Propagation of partially polarized light through anisotropic media with without depolarization: a differential 4 4· matrix calculus, *J. Opt. Soc. Am*. **68**, 1756 -1767.

Barabanenkov Yu.N., (1975). Multiple scattering of waves by the ensembles of particles and the theory of radiation transport., *Sov. Phys – Uspekhi*, **18**, 673-689 (in Russian).

Barabanenkov Yu. N., (1973). Wave corrections to the transfer equation for "back" scattering. *Radiophys Quantum Electron,* **16,** 65–71 (in Russian).

Barabanenkov Yu.N, Kravtsov Yu. A, Ozrin V.D, Saichev A.I, (1991). Enhanced backscattering in optics. *Prog Opt* **,** **29,** 65–197.

Barabanenkov Yu.N., Zurk L.M., Barabanenkov M.Yu., (1995). Poynting's theorem and electromagnetic wave multiple scattering in dense media near resonance: modified radiative transfer equation*, J. Electromag. Waves and Appl*., **9**, 1393-1420.

Bass L.P., Nikolaeva O.V., Kuznetsov V.S., Bykov A.V., A.V.Priezzhev A.V., Dergachev A.A., 2009: Modeling of optical radiation propagation in bio-tissue phantom with using of the supercomputer MBC1000, *Mathem. Modelirovanie*, **21**, 3 – 14 (in Russian).

Bass L.P., Nikolaeva O.V., Kuznetsov V.S., Bykov A.V., A.V.Priezzhev A.V., (2010). Parallel algorithms for simulation of ultrashort pulse propagation in turbid media, IL *NUOVO CIMENTO*, **33** C, n. 1.

Bautin N.N., Leontovich E.L., (1976). *Methods of qualitative analysis of dynamical systems in the plane*, M., Nauka, (in Russian).

Bolgov D.I., Remizovich V.S., and Rogozkin D.B., (1998). Multiple scattering of light in a 2-D medium with large-scale inhomogeneities: an exactly solvable model and approximate methods of calculation, *Laser Phys.* **8**, 462–470 .

Born M, Wolf E., (1975). *Principles of optics, Fifth Edition,* Pergamon.

Borovoi A.G.. (1966) a). Iteration method in multiple scattering. *Izv. Vyssh. Ucheb. Zaved., Fiz*., No. **2**, 175-177.

Borovoi A.G.. (1966) b). Iteration method in multiple scattering: radiative transfer equation. *Izv. Vyssh. Ucheb. Zaved., Fizika*, No. **6**, 50-54.

Borovoi A.G., (1967) a). Multiple scattering of short waves by a system of correlated particles. I. Averaged field, *Izv. Vyssh, Ucheb, Zaved. Fizika*, n. **4**, 97-101.

Borovoi A.G., (1967) b). Multiple scattering of short waves by a system of correlated particles. II. Kinetic equation, *Izv. Vyssh, Ucheb, Zaved. Fizika*, n. **5**, 7-11.

Borovoi A.G., (1983). Light propagation in media with closely packed particles. *Optics and Spectroscopy* 54, 449-450, 1983

Borovoi A.G., Grishin I.A., Oppel U.G., (2000). Mueller matrix for oriented hexagonal ice crystals of cirrus clouds. In: Eleventh Internationa l Workshop On Multiple Scattering LIDAR Experiments (MUSCLE 11), November 1 - 3, 2000, Williamsburg , Virginia , USA, 2000.

Borovoi A., Grishin I., Naats E., Oppel U., (2002). Light backscattering by hexagonal ice crystals, *Journal of Quantitative Spectroscopy and Radiative Transfer*, **72 (4),** 403-417.

Borovoi A.G., (2005). Multiple scattering of optical waves in media containing discrete scatterers, *Doct. Thesis*., Tomsk.

Borovoi A.G., (2006). Multiple scattering of short waves by uncorrelated and correlated scatterers, *Light Scattering Reviews*, **1**, 181-252.

Borovoi A., Kustova N., (2006). Statistical approach to light scattering by convex ice crystals, *Opt. Lett*. **31**, 1747-1749.

Borovoi A.G., Burnashov A.V., Cheng A.Y.S., (2007). Light scattering by horizontally oriented ice crystal plates, *Journal of Quantitative Spectroscopy and Radiative Transfer*. **106 (1),** 11-20.

Borovoi A.G., Kustova N.V., (2010). Light scattering by large faceted particles, *Polarimetric, Detection, and Remote Sensing* , Springer, Dordrecht, The Netherlands.

Borovoi A.G., (2013). Light scattering by large particles: physical optics and the shadow-forming field, *Light Scattering Reviews* **8**, 115-138.

Brosseau C., (1995). Evolution of the Stokes parameters in optically anisotropic media, *Opt. Lett.* **20**, 1221−1223.

Cairns B., Waquet F., Knobelspiesse K., Chowdhary J., and Deuze J.- L., (2010). Polarimetric remote sensing of aerosols over land surfaces, in *Satellite Aerosol Rmote Sensing Over Land,* eds A. A. Kokhanovsky and G. de Leeuw (Chichester: pringer-Praxis), 295–325.

Chandrasekhar S., (1960). *Radiative transfer*. Oxford: Oxford University Press.

Chandrasekhar S., (1977). *Liquid Crystals*, Cambridge, Cambridge Univ.Press.

Cheng T.H., Gu X.F., Xie D.H., Li Z.Q., Yu T., and Chen X. F., (2011). Simultaneous retrieval of aerosol optical properties over the Pearl River Delta, China using multi-angular, multi-spectral, and polarized measurements. *Remote Sens. Env*. 115, 1643–1652, doi: 10.1016/j.rse.2011.02.020

Dlugach J.M., Mishchenko M.I., Liu L., Mackowski D.V., (2011). Numerically exact computer simulations of light scattering by densely packed, random particulate media, **112,** Is. 13, 2068–2078.

Dolginov A.Z, Gnedin Yu. N, Silant'ev N.A., (1970) *J Quant Spectrosc Radiat Transfer,* **10**, 707.

Dolginov A.Z, Gnedin Yu.N, Silant'ev N.A., (1975). Photon polarization and frequency change in multiple scattering. *J Quant Spectrosc Radiat. Transfer,* **10,** 707–754.

Dolginov A.Z., Gnedin Yu.N., and Silant'ev N.A., (1995). *Propagation and Polarization of Radiation in Cosmic Media* (Gordon and Breach, Basel). (Original Russian edition: Nauka, Moscow, 1979.)

Dubovik O., Herman M., Holdak A., Lapyonok T., Tanré D., Deuzé J. L., et al., (2011). Statistically optimized inversion algorithm for enhanced retrieval of aerosol properties from spectral multi-angle polarimetric satellite observations. *Atmos. Meas. Tech*. **4**, 975–1018; doi: 10.5194.

Dullemond K., Peeters K., (1991 - 2010). *Introduction to Tensor Calculus*, Copyright 1991-2010, English translation 2008 – 2010; www.ita.uni-heidelberg.de/~dullemond/lectures/tensor/tensor/tensor.pdf.

Dzyaloshinskii L.E., (1960). On the magnetoelectrical effect in antiferromagnetics, *Soviet Phys. JETP,* **10**, 628-669 (in Russian).

Farrell R., Rouseff A.D., McCally R.L., (2005). Propagation of polarized light through two- and three-layer anisotropic stacks, *J. Opt. Soc. Am*. A, **22**, 1981-1992.

Faure R., Kaufmann A.M., Denis-Papin M., (1964). *Mathematiques Nouvelles*, Dunod, Paris.

Fedorov F.I., (1976). *Theory of the gyrotropy,* Minsk, Nauka i Technika, (in Russian).

Fedorov F.I., Philippov V.V., (1976). *Reflection and refraction of light by transparent crystals,* Minsk, Nauka i Tekhnika, (in Russian).

Foldy LL., (1945). The multiple scattering of waves. *Phys Rev.* **67**: 107–119.

Gao M., You Y., Yang P., Kattawar G.W., (2012). Backscattering properties of small layered plates: a model for iridosomes , *OPTICS EXPRESS,* **20**, no. 22.

Gao M, Yang P, Kattawar G.W., (2013). Polarized extinction properties of plates with large aspect ratios. *J. Quant. Spectrosc. Radiat. Transfer* **131**, 72–81.

de Gennes P.G., (1974). *The Physics of Liquid Crystals*, Oxford, Clarendon Press.

Germogenova T.A., (1985). On the inverse problems of atmosphere optics, Sov. Dokl. **285**, №5, (in Russian).

Germogenova T.A., Konovalov N.V., Kuzmina M.G., (1989) The mathematical foundations of polarized radiation transport theory (strict results). In the issue *Invariance Principle and Its Applications*, Proceedings of the Symposium, Oct. 26-30, 1981, Buarakan., Erevan, Armenia; 271-284.

Ghosh N., Wood M.F.G., Vitkin I.A., (2008). Mueller matrix decomposition for extraction of individual polarization parameters from complex turbid media exhibiting multiple scattering, optical activity, and linear birefringence, *J. Biomed. Opt*. **13**(4), 044036.

Giden I.H., Turduev M., Kurt H., (2014). Reduced symmetry and analogy to chirality in periodic dielectric media, *Opt.Soc J. Europ. Opt. Soc, Public.*, **9**, 14045i.

Ginzburg V.L., A.A.Rukhadze A.A., (1975). *Waves in magneto-active plasma*. Nauka, Moscow, 1975 (in Russian)

Grishin I.A., (2004). *Light Scattering on Ice Crystals Typical for Cirrus*. Doctor Thesis. (150p.), (in Russian).

Hasekamp, O.P., Litvinov, P., and Butz, A., (2011). Aerosol properties over the ocean from PARASOL multiangle photopolarimetric measurements. *J. Geophys. Res*. **116**, D14204; doi: 10.1029/2010JD015469.

Hovenier J.W, editor, (1996). Light scattering by non-spherical particles. *J Quant Spectrosc Radiat Transf* ; **55,** 535–694.

Hovenier J.W, van der Mee C, Domke H., (2004). *Transfer of polarized light in planetary atmospheres*. Dordrecht: Kluwer.

Van de Hulst H.C., (1957). *Light scattering by small particles*. New York: Wiley.

Van de Hulst H.C, (1980). *Multiple light scattering*, Academic Press, New York.

Ishimaru A., (1978). *Wave Propagation and Scattering in Random Media*, v. 1 and 2, N Y, Acad. Prèss, (574 p).

Ishimaru A., Lesselier D., Yeh C., (1984). Multiple scattering calculations for nonspherical particles based on the vector radiative transfer theory, Radio Sci. **19**, 1356-1366.

Katsev I.L., Prikhach A.S., Zege E.P., Ivanov A.P., and Kokhanovsky A.A., (2009). Iterative procedure for retrieval of spectral aerosol optical thickness and surface reflectance from satellite data using fast radiative transfer code and ts application to MERIS measurements," in *Satellite Aerosol Remote Sensing ver Land,* eds A. A. Kokhanovsky and G. de Leeuw (Berlin: Springer-Praxis), pp.101–134.

Kiasat Y., Szabo Z., Chen X., Li E., (2011). Light interaction with multilayer arbitrary anisotropic structure: an explicit analytical solution and application for subwavelength imaging, *JQSAB*.

Knobelspiesse K., Cairns B., Redemann J., Bergstrom R.W., and Stohl A., (2011). Simultaneous retrieval of aerosol and cloud properties during the MILAGRO field campaign. *Atmos. Chem. Phys*. **11**, 6245–6263; doi: 10.5194/acp-11-6245-2011

Kokhanovsky A.A., (1998). On light scattering in random media with large densely packed particles, *J. Geophys. Res.* D, **103**, 6089 -6096.

Kokhanovsky A.A., (1999) a). Radiative transfer in chiral random media, Phys. Rev. E, **60**, no. 4, 4899-4907.

Kokhanovsky A.A., (1999) b). *Light Scattering Media Optics: Problems and Solutions*, Wiley-Praxis, Chichester.

Kokhanovsky A.A., (2000). The tensor radiative transfer equation, *J. Phys. A*: Math. Gen. **33**, 4121–4128.

Kokhanovsky A.A., (2003). Optical properties of irregularly shaped particles, *J. Phys*., D**36**, 915-923.

Kokhanovsky A.A., (2004). Optical properties of terrestrial clouds, *Earth Science Reviews*, **64**, 189-241.

Kokhanovsky A.A., Zege E.P., (2004). Scattering optics of snow, *Applied Optics*, **43**, 1589-1602.

Kokhanovsky A.A., (2005) a). Reflection of light from particulate media with irregularly shaped particles, *J. Quant. Spectr. Rad. Transfer*, **96**, 1-10.

Kokhanovsky A.A., (2005) b). Phase matrix of ice crystals in noctilucent clouds, *Proc. SPIE*, **5829**, 44-52.

Kokhanovsky A.A., (2006). *Cloud Optics*, Dordrecht: Springer, 2006.

Kokhanovsky, A.A., Deuzé, J.L., Diner, D.J., Dubovik, O., Ducos, Emde, C.,et al., (2010). The intercomparison of major aerosol retrieval algorithms using simulated intensity and polarization characteristics of reflected light. *Atmos. Meas.Tech.* **3**, 909–932, 2010. doi: 10.5194/amt-3-909-2010.

Kokhanovsky A.A., (2011). *Solar radiation transport in clouds and snow cover and its application to the problems satellite Earth Remote sensing*, Doct.Thesis , St. Petersburg.

Kokhanovsky A.A., (2015). The modern aerosol retrieval algorithms based on the simultaneous measurements of the intensity and polarization of reflected solar light: a review, *Frontiers in Environmental Science,* v.**3**.

Kong J.A., (1974). Optics of bianisotropic media, *J. of the Opt. Soc. of America*, **64,** iss. 10, 1304-1308.

Kong J.A., (1990). *Electromagnetic waves theory*, (Second edition) Wiley Interscience Publising. John Wiley&Sons, Inc., New York.

Kravtsov Yu.A., Bieg B., and Bliokh K.Yu., (2007). Stokes-vector evolution in a weakly anisotropic inhomogeneous medium, *arxiv.org/pdf/0705.4450*

Kravtsov Yu.A., Bieg B., (2010). Propagation of electromagnetic waves in wearly anisotropic media: Theory and applications/ *Optica Applicata*, v. **XL**, No. 4.

Kravtsov Yu.A., Orlov Yu.I., (1990). *Geometrical optics of inhomogeneous media,* Springer Verlag, Berlin, Heidelberg.

Kurt H., Turduev M., and Giden I.H., (2012). Crescent shaped dielectric periodic structure for light manipulation," *Opt. Express* **20**, 7184–7194.

Kuzmina M.G., (1976). Polarized radiation transport equation in anisotropic media, Preprint KIAM-**68**, (in Russian).

Kuzmina M.G., (1978). General functional properties of polarized radiation transport equation, Sov. Docl., v. **238**, 314-317, 1978 (in Russian).

Kuzmina M.G., (1986) a). To the formulation of polarized radiation transfer problems for slabs of optically active media, Preprint KIAM -**110**, (in Russian).

Kuzmina M.G., (1986) b). Polarized radiation transport in slabs of optically active media, Preprint KIAM -**123**, (in Russian).

Kuzmina M.G., (1987). The perturbation method in transport problems for optically active media, Preprint KIAM-**9** , 1987 (in Russian).

Kuzmina M.G., (1989). The perturbation method in radiation transfer problems for slabs of optically active media, Sov. Dokl. **308**, 335-341.

Kuzmina M.G., (1991). A perturbation method and Stokes parameters estimates in polarized radiation transfer problems in the slabs of optically active media, *TTSP*, **20**(1), . 69-81.

Landau L.D., Lifshitz E.M., (1960). *Electrodynamics of continuous media*, Addison –Wesley, Reading, Mass.

Larin K.V, Motamedi M., Eledrisi M.S, and Esenaliev R.O, (2002). Noninvasive blood glucose monitoring with optical coherence tomography, *Diabetes Care* **25**, 2263–2267.

Lax M., (1951). Multiple scattering of waves*, Rev. Mod. Phys.* **23,** 287–310.

Linder T., (2014). Light Scattering in Fiber-based Materials. *A foundation for characterization of structural properties,* Doct. Thesis, Dept. of Computer Science, Electrical and Space Engineering Lule˚a University of Technology Lule˚a, Sweden.

Liou K.N., Takano Y., Yang P., (2011). Light absorption and scattering by aggregates: Application to black carbon and snow grains, *JQSRT*, **112**, .1581-1594.

Liou K.N., (2002). *An introduction to atmospheric radiation,* 2*nd* ed.,. San Diego, USA, Academic Press.

Liou K.N., (1992). *Radiation and cloud processes in the atmosphere: theory, observation, and modeling*. New York: Oxford University Press.

Liu J., Kattawar G.W., (2013). Detection of dinoflagellates by the light scattering properties of the chiral structure of their chromosomes**,** *J. Quant. Spectrosc. Radiat. Transfer* **131**, 24–33.

Maslennikov M.V., (1968, 1969). The Milne problem with anisotropic scattering, Proc. Steklov Inst. of Math., v. **97**, 1968 (in Russian); Amer. Math. Soc., Providence, Rhode Island, 1969.

Menyuk C.R., (1988). Stability of solitons in birefringent optical fibers. II. Arbitrary amplitudes**,** *J. of the Opt. Soc. of America* B, **5**, n. 2, 392-402.

Marinyuk V.V., Dlugach J.M., and Yanovitskij E.G., (1992). Multiple light scattering by polydispersions of randomly distributed, perfectly aligned Mie cylinders illuminated perpendicularly to their axes, *J. Quant. Spectrosc. Radiat. Transfer* **47**, 401-410.

Marshak A., Davis A.B,, editors, (2005). *3D radiative transfer in cloudy atmospheres*. Berlin: Springer.

Maruo K., Tsurugi M., Chin J., Ota T., Arimoto H, Yamada Y., Tamura M., Ishii M., and Ozaki Y., (2003). Noninvasive blood glucose assay using a newly developed near-infrared system, *IEEE J. Sel. Top.Quantum Electron.* **9**, 322–330 .

Mishchenko M.I., (1994). Transfer of polarized infrared radiation in optically anisotropic media: application to horizontally oriented ice crystals: comment, *J. Opt. Soc. Am*. A, **11**, n.4.

Mishchenko M.I., Hovenier J.W., Travis L.D. (Eds), (2000). Light Scattering by Nonspherical Particles. Theory, Measurements, and Applications, Academic Press

Mishchenko M.I., (2002) Vector radiative transfer equation for arbitrarily shaped and arbitrarily oriented particles: a microphysical derivation from statistical electromagnetics, *Appl Opt*. **41**, 7114–34.

Mishchenko M.I., (2003). Microphysical approach to polarized radiative transfer: extension to the case of an external observation point. *Appl Opt* , **42,** 4963–4967.

Mishchenko M.I., (2014). *Electromagnetic scattering by particles and particle groups: an introduction*. Cambridge, UK: Cambridge University Press.

Mishchenko M.I., Travis L.D., and Lacis A.A., (2002). *Scattering, Absorption and Emission of Light by Small Particles*, Cambridge University Press.

Mishchenko M.I Travis L.D., and Lacis A.A., (2006). *Multiple scattering of light by particles: radiative transfer and coherent backscattering.* Cambridge, UK, Cambridge University Press.

Mishchenko M.I., (2011). Directional radiometry and radiative transfer: a new paradigm. *J. Quant Spectrosc Radiat Transf***,** **112**, 2079–2094.

Mishchenko M.I., Tishkovets V.P., Travis L.D., et al., (2011). Electromagnetic scattering by a morphologically complex object: fundamental concepts and common misconceptions. *J. Quant Spectrosc Radiat. Transf.*, **112,** 671–92.

Mishchenko M.I., (2008) a). Multiple scattering by particles embedded in an absorbing medium. 1. Foldy–Lax equations, order-of-scattering expansion, and coherent field*.* *Opt Express*, **16,** 2288–2301.

Mishchenko M.I., (2008) b). Multiple scattering by particles embedded in an absorbing medium. 2. Radiative transfer equation. *J.Quant. Spectrosc. Radiat. Transf* . **109,** 2386–2390.

Mishchenko M.I., Liu L., Mackowski D.V., Cairns B., Videen G., (2007). Multiple scattering by random particulate media: exact 3D results, *Opt. Express* **15**, 2822–2836 .

Mishchenko M.I., Dlugach J.M., and Yanovitskij E.G., (1992). Multiple light scattering by polydispersions of randomly distributed, perfectly aligned Mie cylinders illuminated perpendicularly to their axes. *J. Quant. Spectrosc. Radiat. Transfer* **47**, 401-410**.**

Mishchenko M.I., (1994). Asymmetry parameters of the phase function for densely packed scattering grains**,** *JQSRT* **52,** 95-110.

Mishchenko M.I., (2010). The Poynting-Stokes tensor and radiative transfer in discrete random media: the microphysical paradigm. *Opt. Express* **18**, 19770-19791.

Mishchenko M.I., Dlugach J.M., Yurkin M.A., Bi L., Cairns B., Liu L., Panetta R.L., Travis L.D., Yang P., and Zakharova N.T., (2016) a). First-principles modeling of electromagnetic scattering by discrete and discretely heterogeneous random media. *Phys. Rep*. **632**, 1–75.

Mishchenko M.I., Dlugach J.M., and Zakharova N.T., (2016) b). Demonstration of numerical equivalence of ensemble and spectral averaging in electromagnetic scattering by random particulate media. *J. Opt. Soc. Am. A* **33**, 618-624.

Mishchenko M.I., (2014). Light propagation in a two-dimensional medium with large inhomogeneities, *J. Opt. Soc. Am.* A **32**, 1330–1336 .

Munneke P.K, (2009). *Snow, ice and solar radiation*, Institute of Marine and Atm. Research Utrecht (IMAU); Dept. of Physics and Astronomy, Faculty of Sci., Utrecht University.

Newton R.G., (1982). *Scattering theory of waves and particles*. New York, Springer -Verlag, New York, (2nd edition).

Nikolaeva O.V., Bass L.P., Germogenova T.A., Kuznetsov V.S., (2007). Algorithms to calculation of radiative fields from localized sources via the Code Raduga-5.1(P*), Transport Theory and Statistical Physics*, v. **36**, iss. 4 – 6, 439–474.

Okada Y., Kokhanovsky A.A., (2009). Light scattering and absorption by densely packed groups of spherical particles, *JQSRT,* **110**, 902–917.

Prigarin S.M., Boovoi A.G, Buscaglioni P., Cohen A., Grishin I.A, Oppel U.G., Zhuravleva T.B., (2005) a). Monte Carlo simulation of radiation transfer in optically anisotropic clouds*, Proc. SPIE*, **5829**, 88 – 94.

Prigarin S.M, Oppel U.G, (2005) b). A hypothesis of 'fractal' optical anisotropy in clouds and Monte Carlo simulation of relative radiation effects, *Proc. SPIE*, **5829**, 102 – 108.

Prigarin S.M., Borovoi A.G., Grishin I.A., U.G. Oppel U.G., (2007). Monte Carlo simulation of radiation transfer in optically anisotropic crystal clouds, Atmospheric and Oceanic Optics, **20**, n.3, 183 – 188.

Prigarin S.M., Borovoi A.G., Grishin I.A., Oppel U.G., (2008). Monte Carlo simulation of halos in crystal clouds, *XV International Symposium "Atmospheric and Ocean Optics. Atmospheric Physics", June 22-28, 2008, Krasnoyarsk. Abstracts.* p.109.

Randrianalisoa J., Baillis D., (2010). Radiative properties of densely packed spheres in semitransparent media: A new geometric optics approach, *JQSRT* **111**, is.10, 1372 –1388.

Rogovtsov N.N., Borovik F.N., (2009). The characteristic equation of radiative transfer theory. In: Kokhanovsky A.A., editor. *Light Scattering Reviews*, **4**, 47–429, Chichester, UK; Springer-Praxis Publishing.

Rogovtsov N.N., (2015) a). Constructive theory of scalar characteristic equations of the theory of radiation transport: I Basic assertions of theory and conditions for the applicability of truncation method. *Differential Equations*, **51**, 268–281.

Rogovtsov N.N., (2015) b). Constructive theory of scalar characteristic equations of the theory of radiation transport: II Algorithms for finding solutions and their analytic representations. *Differential Equations*, **51**, 661–273.

Rogovtsov N.N, Borovik F.N., (2016). Application of general invariance relations reduction method to solution of radiation transfer problems. *Journal of Quantitative Spectroscopy & Radiative Transfer,* **183,** 128-153.

Roux L., Mareschal P., Vukadinovic N., Thibaud J.-B., and . Greffet J.-J., (2001). Scattering by a slab containing randomly located cylinders: comparison between radiative transfer and electromagnetic simulation. *J. Opt. Soc. Am*. A **18**, 374–384.

Rozenberg G.V., (1955). *Usp. Fiz. Nauk* v. **61**, p. 77.

Rudin W., (1976). *Principles of Mathematical Analysis*, (Third ed.), McGraw Hill.

Rytov S.M., Kravtsov Yu.A., Tatarsky V.I., (1978). *Introduction to Statistical Radiophysics*: *Random Fields,* Fizmat, Moscow (in Russian).

Shefer O., (2013). Numerical study of extinction of visible and infrared radiation transformed by preferentially oriented plate crystals. *J. Quant. Spectrosc. Radiat. Transfer* **117**, 104–113.

Shefer O., (2016). Extinction of radiant energy by large atmospheric crystals with different shapes. *J. Quant. Spectrosc. Radiat. Transfer* **178**, 350–360.

Stamnes J., Sithambaranathan G.S., (2001). Reflection and refraction of an arbitrary electromagnetic wave at a plane interface separating anisotropic and a biaxial medium, *J. Opt. Soc. Am*. A **22**, 3119-3129.

Takano Y., Liou K.N., (1989). Solar radiative transfer in cirrus clouds. Part II: Theory and computations of multiple scattering in a anisotropic medium, *J. of Atm. Sci*., **46**, n. 3.

Takano Y., Liou K.L., (1993). Transfer of polarized infrared radiation in optically anisotropic media: application to horizontally oriented ice crystals," *J. Opt. Soc. Am*. A **10**, 1243–1256.

Tishkovets V., Mishchenko M.I., (2004). Coherent backscattering of light by a layer of discrete random media, *JQSRT* **86**, 161.

Tsang L., Pan J., Liang D., Li Z., (2011). Modeling Active Microwave Remote Sensing of Snow Using Dense Media Radiative Transfer (DMRT) Theory with Muftiple Scattering Effects, *IEEE Transactions on Geoscience and Remote Sensing_* **45** , n. 4.

Tsang L., Ding K.-H, (1991). Polametric signatures of a layer of random nonspherical discrete scatterers overlying a homogeneous half-space based on first- and second-order vector radiative transfer theory, *IEEE Trans. Geosci. Remote Sens*., **29**, 242–253.

Tsang L., Kong J.A., (2001). *Scattering of electromagnetic waves*, John Wiley & Sons, Inc.

Tsang L., Pan J., Liang D., Li Z., Cline D.W., Tan Y., (2007). Modeling Active Microwave Remote Sensing of Snow Using Dense Media Radiative Transfer (DMRT) Theory With Multiple-Scattering Effects. *IEEE Transactions on Geosci*. **45**, iss. 4.

Tse K.K., Tsang L., Chan C.H., Ding K.H., and Leung K.W., (2007). Multiple scattering of waves by dense random distributions of sticky particles for applications in microwave scattering by terrestrial snow, *Radio Sci.* **42**, RS5001.

Tseng S., (2008). Optical characteristics of a cluster of closely-packed dielectric spheres, *Opt. Commun*. **281**, 1986–1990.

Volkovitski O.A., Pavlova L.N., Petrushin A.G., (1984). *Optical Properties of Crystal Clouds*, Leningrad, Gidrometeoizdat, (in Russian).

Watson K.M., (1969) Multiple scattering of electromagnetic waves in an underdense plasma. *J. Math Phys.*, **10**, 688–702.

Watson K.M., (1953). Multiple scattering and the many-body problem – applications to photomeson production in complex nuclei. *Phys Rev***. 89,** 575–87.

Wiscombe W.J., Warren S.G., (1980). A model for the spectral albedo of snow. I: Pure snow, *J. Atmos. Sci*., **37**, 2712–2733.

Wood M.F.G., Guo X., Vitkin I.A., (2007). Polarized light propagation in multiply scattering media exhibiting both linear birefringence and optical activity: Monte Carlo model and experimental methodology, *Journal of Biomedical Optics,* **121**.

Xie Y., Yang P., Kattawar G.W., Baum B.A., and Hu Y., (2011). Simulation of the optical properties of plate aggregates for application to the remote sensing of cirrus clouds**,** *Applied Optics*, **50**, 1065–1081.

Yariv A., Yeh P., (1984). *Optical Waves in Crystals*, New York: Wiley.

Yeh P., (1979). Electromagnetic propagation in birefringent layered media, *J. Opt. Soc. Am*. **69**, 742–755,.

Yeh P., (1980). Optics of Anisotropic layered Media: A New 4 X 4 Matrix Algebra, *Surface Science*, .**96**, 41-53.

Zege E.P. L.I.Chaikovskaya L.I., (1984). *Optics and Spectroskopy*, **5**, p. 1060.

Zege E.P, Ivanov A.P, Katsev I.L., (1991). *Image transfer through a scattering medium*. Berlin,   Springer.

Zheleznyakov V.V., (1977). *Electromagnetic waves in cosmic plasma*, Nauka, Moscow.

Zheleznyakov V.V., (1996). *Radiation in Astrophysical Plasma*, Kluwer.