Exact Solutions and Modelling of Associative Memory in Oscillatory Networks

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This paper is devoted to the study of associative memory in the networks of N coupled nonlinear oscillators interacting via complex-valued weights. Exact solutions relating to the structure of attractors have been obtained. The complete solution to the systems of two oscillators and the structural portrait of the governing dynamical system have been obtained. It is shown that homogeneous closed chains of oscillators play important role in the context of phase associative memory problems. Qualitative description of the memory in the closed chains of N oscillators is given for arbitrary N, and rigorous solutions for $N \leq 6$ are illustrated. The networks considered admit electronic, nonlinear optical and optoelectronic implementations. The background of some of them is under development.

Keywords: associative memory, dynamical systems, synchronization, nonlinear oscillators

1. Introduction.

As shown in [1,2], the associative memory model can be designed in the networks of limit-cycle nonlinear oscillators with complex matrix of connections W. In [3] the properties of this memory and also the structure of attractors in such oscillatory networks with various matrices W were studied. It was found out that in contrast with the phase oscillatory networks [4] the networks with Hermitian matrix W admit Hebbian learning if applied to the phase basis, and demonstrate the properties that make them interesting from the viewpoint of applications, in particular, rather high memory capacity: $\sim N/2$.

The present paper performs the new approach to study of associative memory in the oscillatory networks considered. Our intention is to elucidate relations between architecture of the networks and their associative memory properties. In the phase oscillatory systems [4] it was proved that the networks with tree-like architecture of connections can not have more than one stable point. Computer modelling confirms that the same fact is valid for the networks considered here. Therefore, to design matrix W providing a rich set of attractors, it is necessary to close the connections between oscillators into circles. The most simple of such architectures is a homogeneous closed chain, which has been studied in detail, n.3. The new approach is to represent the matrix W of oscillatory network with Hebbian connections as a sum of matrices of closed chains and special unclosed architectures, and describe the memory in the whole network via the memory in the separate items.

In addition to the main considerations relating to the chains, the results on two-oscillatory networks are performed in n.4: exact solution and dynamics in regimes near synchronization.

2. General

The system [1,2] governing the dynamics of N coupled limit cycle oscillators is studied from the viewpoint of associative memory modelling:

$$\dot{z_j} = (1 + i\omega_j - |z_j|^2)z_j + \sum_{k=1}^{N} W_{jk}(z_k - z_j), \quad j = 1, \ldots, N. \tag{1}$$

Here the complex variable $z_j(t) = r_j(t)e^{i\theta_j(t)}$ defines the state of j-th oscillator (r_j and θ_j are respectively the amplitude and phase of its oscillations), ω_j is its natural frequency. The first term in the right-hand side of (1) defines the intrinsic dynamics of free isolated oscillator, while the second one, responsible for interaction, is specified by Hermitian matrix of connections W.

At arbitrary parameters $\{\{\omega_j\}, W\}$ the system (1) may demonstrate the variety of complicated dynamical regimes including dynamical chaos. In n.4.2. such regimes are illustrated in the systems of two oscillators. To design the associative memory [1,2,3], we use synchronization regime in (1). The problem of associative memory can be considered as an inverse problem for dynamical system , i.e., the problem of calculation of the matrix W and frequencies ω_j providing the prescribed set of attractors. It should be reminded that, due to the invariance of the solutions of (1) relative to a constant shift of the phases θ_j, the attractors are not isolated points, but phase-locked states in (1). Below such attractors are also referred as points or memories (so, θ_1 is always assumed to be equal to zero). Note that the sum $\sum_j \omega_j$ is assumed to be zero that can always be achieved by proper rescaling of the dynamical system. Moreover, in the major part of the paper we assume $\omega_j = 0$ for all j. The systems (1) satisfying this restriction are basic from the viewpoint of design of associative memory in general case, when arbitrary Hermitian matrices W are considered.

The memories with constant amplitudes for all indices j, which are named phase memories, play the important role in the associative memory design. Hebb rule applied to the special phase basis defined below permits to obtain associative memory with memory capacity approximately $N/2$, if N is a prime number.

Consider N phase points in N-dimensional complex space C^N:

$$z_k = (r, re^{i\psi_k}, re^{2i\psi_k}, \ldots, re^{(N-1)i\psi_k}), \quad \psi_k = 2\pi k/N, \quad k = 0, \ldots, (N-1). \tag{2}$$

This set of points gives phase basis in C^N.

If we apply Hebb rule to the phase basis, i.e., take the matrix W as a weighted sum of outer-products of the basic vectors, then only cyclic matrices can be obtained. If we assume N to be a prime number, then, considering only non-diagonal part of the matrix W, one can represent it as a sum of matrices corresponding to a set of closed chains. The analysis shows that the associative memory in Hebbian systems produced from the phase basis is closely related to the memories in the corresponding closed chains. Therefore, the study of the structure of attractors in homogeneous closed chains is very useful for solution of the general problem of associative memory design as well as interesting by itself.

Noteworthy is that if N is not a prime number, then the system (1) is degenerated. In particular, in addition to isolated point attractors the attractors of greater dimensions exist. In this case the matrix W can be represented as a sum of matrices corresponding to homogeneous closed chains of sizes equal to the divisors of N, and specific unclosed architectures.

3. Homogeneous Closed Chains

Homogeneous closed chain governed by eq.(1) is determined by one complex number $(b + ic) = ue^{i\beta}$, which is the weight between two successive oscillators (ω_j are assumed to be zero for all j). The weight in the backward direction is determined by the complex conjugate. This architecture of connections is fundamental in the problem of oscillatory associative memory design. The homogeneous closed chains have been studied both analytically and numerically. Here the results of this study are briefly described.

Note that all the points of the phase basis are equilibrium points of the closed chains. It is convenient to denote an equilibrium point by k from (2). The squared amplitude r^2 can be easily calculated: $r^2 = 1 + 2(b\,cos(\beta + \psi) - b)$.

To determine stability of each point in the chain weight $(b + ic)$, it is sufficient to analyze the roots of the characteristic polynomial $D(\lambda)$ of Jacobin calculated in the corresponding point. At $N = 3, 4, 6$ these roots can be calculated analytically, so the exact description of associative memory in such chains is available. **Fig.**1a,b,d,e,f shows the solution.

It can be seen that synchronization occurs at any weight $(b+ic)$, **Fig.**1e,f (since $\omega_j \equiv 0$). At $u > 1$ the curves separating the domains of stability for different k are practically linear. Naturally, the domains are symmetrical relatively to the x-axis.

At $N = 5$, **Fig.**1c, the irreducible cofactor of degree four is present in $D(\lambda)$, and the roots of $D(\lambda)$ can not be calculated analytically. So, in this case the asymptotic result has been calculated. Here the criterion of Raus-Hurvitz describing stability of polynomials has been applied. According to this criterion, the signs of the functions depending on coefficients of $D(\lambda)$ were analyzed at u tending to infinity, and the values of β, satisfying the criterion were calculated. These values of β deliver the slopes of the lines separating different domains. The shifts of these lines relative to origin of coordinates were calculated numerically. The shift of the lines corresponding to overlap of $k = 0, 1$ and $k = 0, 4$ is non-zero, but very small. The other shifts in **Fig.**1c are practically zero. Apparently, the shifts of the lines separating domains of stability are practically zero at prime numbers exceeding $N = 5$.

It can be seen from **Fig.**1 that with the increase in N the domains of stability become more narrow and the overlap becomes larger. For example, the angles between the straight lines bounding the domains for $k = 0$ are $120°, 90°, 72°, 60°$ for $N = 3, 4, 5, 6$, respectively. The results of computer modelling confirm that for $N \geq 7$ the behavior is similar.

In the chains of three oscillators only two stable points can exist. Moreover, the domains of simultaneous stability for $k = 0, 1$ and $k = 0, 2$ are rather narrow infinite strips. In contrast, the domain for $k = 1, 2$ is large enough. So, the points of the phase basis as well as the pairs, triples etc. of these points are not equivalent in the chains. The same is valid for all N. The homogeneous closed chains for $N = 3$ are fully connected networks, so the rigorous results obtained are very useful for understanding of the dynamics of the general system (1) at $N = 3$.

Fig.1e,f shows the exact solutions for $N = 4, 6$ in the vicinity of zero. Computer modelling confirms that for larger N such behavior is typical.

At $N = 6$ the bounded domains of simultaneous stability of three points are present, **Fig.**1f, but at $u > 1$ only straight lines determine associative memory in these closed chains as well.

This is very likely that in homogeneous closed chains only the points of phase basis can be stable. Neither extra phase points, nor amplitude memories (i.e., the points with different amplitudes r_j at different j) can exist. This hypothesis has been confirmed by computer modelling of the system (1).

4. Two Coupled Oscillators

4.1. Complete Solution for the System of Two Coupled Oscillators

Equilibrium and stable points of the system (1) for arbitrary weight $W_{12} = b + ic = ue^{i\beta}$ and $\omega_1 = -\omega_2$ have been completely analyzed in the case of two oscillators, $N = 2$. It was found out that depending on

Fig.1. The domains of stability for the points of the phase basis in homogeneous. closed chains (k indicates the point; b and c are the real and imaginary parts of the weight determining a chain);
(a,b,c,e) - maximum two points can be stable at $N = 3, 4, 5$;
(d,f) - maximum three points can be stable at $N = 6$. The boundaries separating the regions of stability for different k are practically linear at $u > 1$

parameters, two or four equilibrium points can exist, and only one of them can be stable. The conditions of stability for the only stable point coincide with the conditions of its existence, and they look as follows:

1) $u \geq (\omega - c)^2$;

2) $(1-b) > 0 \quad$ OR $\quad (1-b)^2 + h^2 < u$,

here $h = \omega - c$. Note that at $\omega = 0$ these conditions are satisfied, so the system of two oscillators with zero frequencies always has a stable point.

If these conditions are satisfied, then the equilibrium point with coordinates $\{r, re^{i\theta_2}\}$, where $r = (1-b+(u-h^2)^{1/2})^{1/2}$, $\theta_2 = -i\,ln(b-ic)/(ih+(u-h^2)$ is stable. It should be noted that only phase stable points can exist in the systems of two oscillators, whereas in general systems of three oscillators non-phase points can also be stable.

4.2. Structural Portrait of Two-Oscillator Dynamical System

The parametric space of the dynamical system represents three-dimensional domain in R^3, that can be specified by parametric set $\{\kappa, \omega, \beta\}$, where $\omega = \omega_1 = -\omega_2$, $\kappa W_{12} = \kappa e^{i\beta}$, $-\infty < \omega < \infty$, $0 < \kappa < \infty$, $-\pi/2 \leq \beta \leq \pi/2$. The parameter κ, defining the absolute value of interaction strength in oscillatory system, is relevant to be extracted from matrix of connections W due to purely phase character of interaction in the system (without any influence on the amplitudes of oscillations).

Structural portrait is the decomposition of the whole parametric domain into subdomains corresponding to essentially different types of the dynamical behavior. The boundaries between the subdomains are bifurcation surfaces. There exist three subdomains for two-oscillator system:

S - the domain of synchronization,

D - the domain of so-called "amplitude death",

O - the domain of multi-frequency oscillations.

The dynamical system (1) can be represented in the form

$$\dot{z} = (D(z) + \kappa W)z, \tag{3}$$

where $z = (z_1, z_2, \ldots z_N)^\top$ is a vector of state, $D(z) = diag(D_1(z), \ldots D_N(z))$ is the diagonal matrix, $D_j(z) = 1 + i\omega_j - |z|^2$, κW is matrix of connections.

As one can see from (3), the whole set of equilibria consists of two subsets:

$z = 0 \quad$ and $\quad (D(z) + \kappa W)z = 0, z \neq 0$.

In the domain D the point $z = 0$ is the single stable equilibrium of the dynamical system. It is useful to know the boundaries of D, because this domain is obviously not suitable for associative memory modelling.

The domain S is just of interest: a set of isolated stable fixed points can exist there. Complicated equilibria of various kinds can also exist in S under various degenerations.

The domain O is not a "working" domain in the case of modelling of associative memory networks with relaxational dynamics. However, it could be quite suitable in modelling of the networks with more complicated dynamics (for example, the networks for dynamical process recognition).

For two-oscillator dynamical system the inequalities, specifying the domains S, D and O are calculated analytically. Besides, the expressions for fixed points, located in S, and the character of their stability (the Jacobian eigenvalues) also have been calculated exactly. Thus, we have the following results.

1. The domain S.

a) $\omega > 0$. $\quad \kappa \geq f_1(\omega, \beta) \quad$ at $\quad 0 < \omega \leq \omega_*$; $\quad \kappa \geq f_2(\omega, \beta) \quad$ at $\quad \omega \geq \omega_*$

b) $\omega < 0$. $\quad \kappa \geq f_3(\omega, \beta) \quad$ at $\quad 0 < |\omega| \leq \omega_{**}$; $\quad \kappa \geq f_4(\omega, \beta) \quad$ at $|\omega| \geq \omega_{**}$

2. The domain D.

a) $\omega > 0$. $\quad f_5(\omega, \beta) \leq \kappa < f_2(\omega, \beta) \quad$ at $\quad \omega \geq \omega_*$;

b) $\omega < 0$. $\quad f_5(\omega, \beta) \leq \kappa < f_4(\omega, \beta) \quad$ at $\quad |\omega| \geq \omega_{**}$;

3. The domain O.

a) $\omega > 0$. $\kappa < f_1(\omega,\beta)$ at $0 < \omega \leq \omega_*$; $\kappa < f_5(\omega,\beta)$ at $\omega \geq \omega_*$
b) $\omega < 0$. $\kappa < f_3(\omega,\beta)$ at $0 < |\omega| \leq \omega_{**}$; $\kappa < f_5(\omega,\beta)$ at $|\omega| \geq \omega_{**}$
Here it is denoted:

$$f_1(\omega,\beta) = \frac{\omega}{1+sin\beta}, \quad f_2(\omega,\beta) = \frac{\omega^2+1}{2(\omega sin\beta + cos\beta)},$$

$$f_3(\omega,\beta) = \frac{\omega}{1-sin\beta}, \quad f_4(\omega,\beta) = f_2(-|\omega|,\beta),$$

$$\omega_* = \frac{1+sin\beta}{cos\beta}, \quad \omega_{**} = \frac{1-sin\beta}{cos\beta}.$$

Two examples of structural portrait projection are shown in **Fig.**2. The first one corresponds to coupling via $W_{12} = \kappa$ (all the curves of the projection are marked by zero superscript). This projection, symmetrical with respect to axis κ, is quite similar to structural portrait of N-oscillator system with $W_{jk} = N^{-1}(1-\delta_{jk})$, obtained in [5]. However, there is the difference: in the case of N-oscillator system the narrow layered domains in the vicinity of curves f_1^0, f_3^0 exist. They corresponds to great collection of complicated multi-frequency oscillatory regimes, including chaos. Similar domains are absent in the case of two-oscillator system.

The exact analysis shows that there exist from one to four fixed points in the domain S. Only single of them is always stable (stable node), the others are saddles.

The collection of different dynamical regimes provided by two-oscillator dynamical system in parametric domain O is presented in Figs.3,4.

Two-oscillator dynamical system for complex-valued variables z_1, z_2 is equivalent to four-dimensional dynamical system for real-valued variables $x_j = Re(z_j),\ y_j = Im(z_j),\ j = 1,2$. In **Figs.**3,4 all six two-dimensional projections of phase trajectory of the dynamical system in four-dimensional phase space are presented. The following notations are used: $x_1 = x,\ y_1 = y,\ x_2 = z,\ y_2 = u$. In addition the examples of time-behavior of one of the variables (usually $x(t)$ or $u(t)$) are shown in the lower long window. The values of parameters $\omega_1, \omega_2, \kappa, \beta$ are given in the captions to **Figs.**3,4.

5. On Structural Stability of the Oscillatory Dynamical System

Strict results obtained for closed chains of small number of oscillators permit to clarify the character of structural stability (robustness) of oscillatory dynamical system under perturbations of matrix W.

In the case of networks of three oscillators with Hebbian matrix it is possible to obtain complete exact answer on structural stability basing on the results for the chains of three oscillators. Indeed, all Hebbian matrices of connections W^H, corresponding both to storing of arbitrary single memory vector and to storing of any combination of pair of memory vectors, always represent the matrices of homogeneous closed chains W^c, which are defined by single parameter - the value of chain coupling $a = b + ic$: $W^c = W^c(a)$.

Namely, we have the following relations between W^H and the corresponding W^c.

1) Three-oscillator network with memory containing a single memory vector:
$U^{(0)} = \rho^{(0)}V^{(0)}, V^{(0)} = (1,1,1) \Longrightarrow W^H = \kappa W^c(1)$
$U^{(1)} = \rho^{(1)}V^{(1)}, V^{(1)} = (1, e^{i2\pi/3}, e^{-i2\pi/3}) \Longrightarrow W^H = \kappa W^c(e^{-i2\pi/3})$
$U^{(2)} = \rho^{(1)}V^{(2)}, V^{(2)} = (1, e^{-i2\pi/3}, e^{i2\pi/3}) \Longrightarrow W^H = \kappa W^c(e^{i2\pi/3})$

2) Three oscillator network with memory containing two memory vectors:
$U^{(0)} = \rho^{(0)}V^{(0)}$ and $U^{(1)} = \rho^{(0)}V^{(1)} \Longrightarrow W^H = \kappa W^c(e^{-i2\pi/6})$
$U^{(0)} = \rho^{(0)}V^{(0)}$ and $U^{(2)} = \rho^{(0)}V^{(2)} \Longrightarrow W^H = \kappa W^c(e^{i2\pi/6})$
$U^{(1)} = \rho^{(1)}V^{(1)}$ and $U^{(2)} = \rho^{(1)}V^{(2)} \Longrightarrow W^H = \kappa W^c(-1)$

Fig.2. Two projections of structural portrait of two-oscillator system
a) $\beta = 0$.
The domain S^0 is situated over the curves f_3^0 and f_4^0 at $\omega < 0$, and over f_1^0 and f_2^0 at $\omega > 0$;
The domain D^0 - over f_5^0 and under f_4^0 at $\omega < 0$ and over f_5^0 and under f_2^0 at $\omega > 0$;
The domain O^0 - under f_3^0 and f_5^0 at $\omega < 0$ and under f_1^0 and f_5^0 at $\omega > 0$.
b) $\beta = \pi/6$
The domain S is situated over the curves f_3 and f_4 at $\omega < 0$ and over f_1 and f_2 at $\omega > 0$;
The domain D - over f_5 and under f_4 at $\omega < 0$ and over f_5 and under f_2 at $\omega > 0$;
The domain O - under f_5 and f_3 at $\omega < 0$ and under f_1 and f_5 at $\omega > 0$.

These relations just permit to extract the exhaustive information on structural stability from the picture of stability of the fixed points of the chain under variation of element a of matrix $W^c(a)$ (see the upper left picture in **Fig.**1). We obtain the following results.

1. In the case of memory, containing a single memory vector, the domain of stability of this vector in the complex a-plane really defines all admissible perturbations of a, preserving its stability. So, this domain can be considered as robustness domain. For example, in the case of storing of $U^{(0)}$ the robustness domain represents the symmetrical sector of angle $2\beta = 2\pi/3$ around the negative part of the real axis. The robustness domains in the case of storing $U^{(1)}$ and $U^{(2)}$ are also clearly seen.

2. The robustness degree is much lower in the case of storing two memory vectors. Especially it concerns the case of storing the pairs $U^{(0)}, U^{(1)}$ and $U^{(0)}, U^{(2)}$: the robustness domains are narrow strips around the rays, defined by the angles $\pm 2\pi/6$. In the case of storing of a pair $U^{(1)}, U^{(2)}$, the robustness domain is slightly greater than the sector of angle $2\pi/3$ around the negative part of the real axis.

The admissible perturbations $\Delta\beta$ of element a can be easily calculated. In the case of storing pairs $U^{(0)}, U^{(1)}$ and $U^{(0)}, U^{(2)}$, $\Delta\beta$ is proportional to κ^{-1}; in the case of storing a pair $U^{(1)}, U^{(2)}$, $\Delta\beta \geq 2\pi/6$.

These results for three-oscillator networks with Hebbian connections reflect some general features of structural stability of oscillatory networks - quite low robustness in some situations. However,as,shown [3], the class of networks of high memory capacity is delivered by Hebbian oscillaory networks with prime N. For $N > 6$ the situation when the robustness domains are just narrow strips around the definite straight lines are quite exceptional. Moreover, they can be pointed out exactly.

6. Conclusion

The central issue of the present study is the elucidation of relations between phase associative memory in oscillatory networks and associative memory in homogeneous closed chains of oscillators. Such chains of oscillators may be considered as "basic building blocks" for oscillatory networks (1) with Hermitian matrix of connections W. Moreover, the study of associative memory in the closed chains is also useful by itself, since the results of computer modelling confirm that only the points of phase basis can be stable in the chains, and no extra attractors exist.

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Fig.3. The collection of dynamical regimes displayed by two-oscillator dynamical system.

(a) - $\omega_1 = .8; \omega_2 = -.36; \kappa = .5; \beta = 0$

(b) - $\omega_1 = 1.; \omega_2 = -1.99; \kappa = .999495; \beta = 0$

(c) - $\omega_1 = 1.2; \omega_2 = -1.099; \kappa = .76; \beta = .5236$

(d) - $\omega_1 = .95; \omega_2 = 0; \kappa = .46; \beta = 0$

(e) - $\omega_1 = 1.; \omega_2 = -1.999; \kappa = .53561; \beta = 1.0472$

(f) - $\omega_1 = 8.; \omega_2 = 0; \kappa = 1; \beta = 0$

Fig. 4. The collection of dynamical regimes displayed by two-oscillator dynamical system.
(a) - $\omega_1 = .8; \omega_2 = -.4; \kappa = .5; \beta = 0$
(b) - $\omega_1 = 1.5; \omega_2 = -1.499; \kappa = .999665; \beta = .5236$
(c) - $\omega_1 = 2; \omega_2 = 0; \kappa = .8; \beta = 0$
(d) - $\omega_1 = .8; \omega_2 = -.44; \kappa = .5; \beta = 0$ (
e) - $\omega_1 = -.5; \omega_2 = 8; \kappa = 1; \beta = .25$
(f) - $\omega_1 = 3.5; \omega_2 = -3.499; \kappa = 1.87536; \beta = 1.0472$