Recurrent Associative Memory Network of Nonlinear Coupled Oscillators

Margarita Kuzmina[1], Edward Manykin[2] and Irina Surina[2]

[1] - Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia
[2] - RRC Kurchatov Institute, Moscow, Russia

Abstract. The recurrent associative memory networks with complex-valued Hebbian matrices of connections are designed from interacting limit-cycle oscillators. These oscillatory networks have peculiarities and advantages as compared to Hopfield neural network model. In particular, the class of networks with high memory characteristics (the capacity close to 1, low extraneous memory) exists. At zero values of oscillator frequencies the designed networks are closely related to the known "clock" neural networks (networks from complex-valued neurons). Pattern recognition of colored images and recognition of objects with complicated topological structure look quite natural in the context of such models. Exact solutions have been obtained for a few types of the networks considered, in particular, for homogeneous closes chains.

1. Introduction.

During more than twenty years the systems of coupled oscillators are used for modelling of various phenomena in physics, chemistry, biology. Recently the studies on artificial neural-like oscillatory networks were started. In the first series of publications, using phase approximation of oscillatory dynamics (so-called phase model), the phenomenon of clasterization of oscillatory population into the state of synchronization in the vicinity of phase transition was exploited [1]. Later the attempt to design oscillatory networks of associative memory resembling Hopfield networks was made with the use of Ginzburg-Landau oscillatory system and its phase approximation [2]. However, in this study the design has not been completed with a clear construction of the weight matrix. Another attempt based on the special case of phase model, was made recently [3], but this approach leads to some difficulties caused by the specific properties of the model chosen.

In [4-6], the recurrent oscillatory networks of associative memory with complex-valued generalization of Hebbian matrix of connections were designed using amplitude-phase dynamical model. Although the networks consist of oscillators, the relaxational dynamics in synchronization regime is used in the design of the associative memory.

The attractive feature of the designed networks is that they admit electronic and optical implementations based on optoelectronic and nonlinear optics principles. The background of the implementations is currently under development.

2. Dynamical Equations. The Problem of Design of Associative Memory Network.

The system [4-6] governing the dynamics of N coupled limit cycle oscillators is studied from the viewpoint of associative memory modelling:

$$\dot{z_j} = (1 + i\omega_j - |z_j|^2)z_j + \kappa \sum_{k=1}^{N} W_{jk}(z_k - z_j), \qquad j = 1, ..., N. \tag{1}$$

Here the complex variable $z_j(t) = r_j(t)exp(i\theta_j(t))$ defines the state of j-th oscillator (r_j and θ_j are respectively the amplitude and phase of oscillators), ω_j is its natural frequency. The first term in the right-hand side of (1) defines the intrinsic dynamics of free isolated oscillator, while the second one, responsible for interaction, is specified by the matrix of connections κW. Non-negative parameter κ defines the absolute value of interaction strength in oscillatory system and the matrix $W = [W_{jk}]$, which in the case of symmetrical interaction is Hermitian one, specifies the weights of connections. As a weight matrix, W satisfies the natural restrictions: $W_{jj} = 0, \quad |W_{jk}| \leq 1, \quad \sum_{k=1}^{N} |W_{jk}| \leq 1$.

The system (1) can be rewritten in the vector form:

$$\dot{z} = (D(z) + \kappa W)z, \tag{2}$$

where the column-vector $z = (z_1, \ldots, z_N)^\top$ is a state vector of the oscillatory system and $D(z)$ is the diagonal matrix,

$$D(z) = diag(D_1(z), \ldots, D_N(z)), \quad D_j(z) = 1 + i\omega_j - |z_j|^2 - \kappa \sum_{k=1}^{N} W_{jk}.$$

At arbitrary set $\{\{\omega_j\}, \kappa, W\}$ of the parameters the system (1) demonstrates the variety of complicated dynamical regimes including dynamical chaos. The regime of synchronization, used in associative memory modelling, is quite simple from the viewpoint of nonlinear dynamics: if $\sum_j \omega_j = 0$ (this restriction on the frequencies can be always satisfied by proper rescaling of the dynamical system), this is relaxation to stable equilibria.

The problem of design of associative memory network can be formulated as combination of two (independent) subproblems for the governing dynamical system:

1) the inverse problem for the dynamical system, i.e., design of the system possessing the prescribed set of stable equilibria with large enough basins of attraction;

2) a kind of control problem for the designed dynamical system, i.e., the choice of an adequate learning algorithm.

In the present study, the results on subproblem 1) for the system (1) in the regime of synchronization are performed. Moreover, only *phase memory* (with attractors located in the points of the phase space of (1) that have equal moduli for all coordinates) is suggested. Computer modelling of non-phase associative memory shows that in many cases non-phase memories can be obtained from phase ones by relatively slight distortions, therefore the results on phase memory are useful for solution of the general problem.

First the results of the analysis of equilibria of oscillatory networks with small number of oscillators and of special interconnection architectures are performed.

3. The System of Two Coupled Oscillators. Structural Portrait.

Since the system (1) can be represented in the form (2), it has always the equilibrium $z = 0$. So, in general case the whole set of equilibria of (1) consists of two subsets: $z = 0$ and $z \neq 0$ & $(D(z) + \kappa W)z = 0$.

In the case of the system of two coupled oscillators the parametrical space of the dynamical system is (ω, κ, β), where $\omega = \omega_1 = -\omega_2$, $W_{12} = exp(i\beta)$, $-\infty < \omega < \infty$, $0 < \kappa < \infty$, $-\pi/2 < \beta \leq \pi/2$.

Let $\mathcal{D}$ denotes the subdomain of the parametrical space where the equilibrium $z = 0$ is stable (this domain is usually regarded as "amplitude death" [7]) and $\mathcal{S}$ - the subdomain where stable equilibria $(D(z) + \kappa W)z = 0$ are located ($\mathcal{S}$ is the synchronization domain). The following results have been obtained analytically for two-oscillatory system: (both the expressions for equilibria and the eigenvalues of Jacobian defining the character of stability of the equilibria have been calculated in the explicit form):

- there exist from one to four fixed points of (1) in different domains of parametrical space (the corresponding subdomains have been specified);

- only one fixed point U^1 is stable ($U^1 \in \mathcal{S}$, U^1 is a stable node; the other fixed points are saddles);

- the structural portrait of the system (1) has been obtained in the whole parametrical space; all the boundaries have been calculated in the explicit analytical form.

As an example we can give the projection of the structural portrait into the quadrant $(\omega, \kappa, 0), \omega \geq 0$. This projection resembles the structural portrait of the system (1) with $W_{jk} = N^{-1}(1 - \delta_{jk})$ obtained in [7]:

$$\mathcal{D} = \{1 \leq \kappa \leq \frac{\omega^2 + 1}{2\omega}\}, \qquad \mathcal{S} = \{\kappa \geq \omega \ \ at \ \ \omega \leq 1 \ \& \ \kappa \geq \frac{\omega^2 + 1}{2\omega} \ \ at \ \ \omega > 1\},$$

The domain of unsteady dynamics is $\{\kappa < \omega \ \ at \ \ \omega \leq 1 \ \& \ \kappa < 1 \ \ at \ \ \omega \geq 1\}$.

4. Exact Solutions for Homogeneous Closed Chains.

Phase memory has been studied both analytically and with the use of modelling for a few instructive types of matrices W, in particular, for closed chains of oscillators with constant weight $(b + ic)$ and zero frequencies. It can be seen that N phase points P_k, $k = 0, \ldots, N-1$, with coordinates $(r, r \cdot exp(i\psi), r \cdot exp(2i\psi), \ldots, r \cdot exp((N-1)i\psi))$, where ψ is $2\pi k/N$, $r = (1 + 2(b \cdot cos(\psi) - b - c \cdot sin(\psi)))^{1/2}$ are equilibrium points for such closed chains. Their stability has been studied by direct calculation of the corresponding spectra of Jacobian matrices. For $N = 3, 4, 6$ the complete solutions have been obtained. The results can be displayed in the plane c, b. For $N = 3$ P_0 is stable if $b > ((1+3c^2)^{1/2} - 1)/3$, P_1 – if $b < c\sqrt{3} + 1/6 - (4c^2/3 + 1/36)^{1/2}$, P_2 – if $b < -c\sqrt{3} + 1/6 - (4c^2/3 + 1/36)^{1/2}$. Thus, in two narrow unbounded strips two pairs of points, i.e., P_0,P_1 and P_0, P_2 are stable, and in the expanding region one pair, i.e., P_1,P_2, is stable. For $N = 4$ the plane (c,b) is divided into the regions in the following way: P_0 is stable if $b > (-1 + (1+4c^2)^{1/2})/2$, P_1 – if $(3c^2 - b^2 + 2bc - c > 0)$ & $c < 0$, P_2 – if $b < (1 - (1+20c^2)^{1/2})/10$, P_3 - if $(3c^2 - b^2 - 2bc + c > 0)$ & $c > 0$. Fig. 1 shows the regions of stability for P_k, $k = 0, \dots, 5$, in the chains from six oscillators in two scales. Here the digits mean the values of k determining ψ. As one can see, in three small bounded regions three points are simultaneously stable. It looks likely that no extra (non-phase) stable points exist in homogeneous closed chains. For $N < 25$ this hypothesis has been confirmed using computer modelling. For $N \geq 7$ the overlap of the stability regions increases and more points can be stable simultaneously. Some strict solutions have been also obtained for cyclic matrices W of low orders ($N < 7$) other than the closed chains

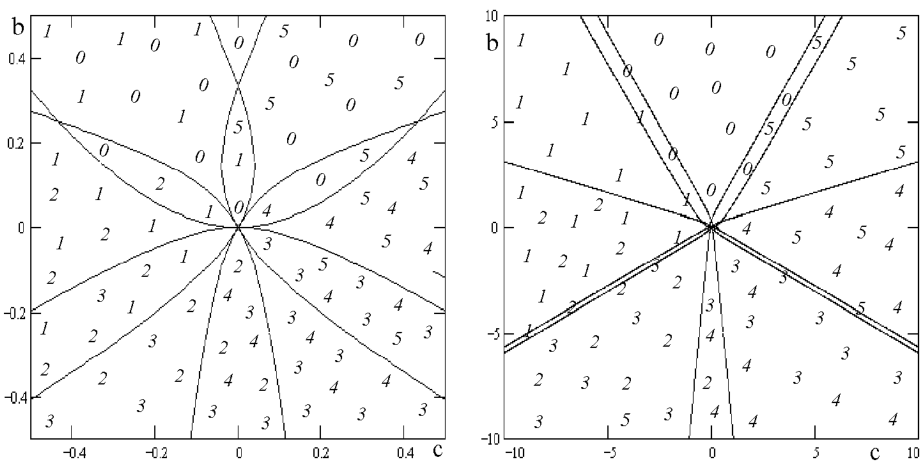

Fig. 1. Phase memories in the chains of six oscillators. Two scales.

5. Oscillatory Networks with Hebbian Matrices of Connections. Related Phasor Networks.

Dynamical system (1) with $\omega_j \equiv 0$ represents the important special case of oscillatory system which can be regarded as phasor networks and can be viewed as natural generalization of the known "clock" neural networks.

The equilibria of oscillatory networks and corresponding phasor networks proved to be closely related. The following proposition reflects the relation.

- Let $\mathcal{N}(\{\omega_j\}, \kappa, W)$ be an oscillatory network with arbitrary frequencies ω_j

satisfying the condition $\sum_j \omega_j = 0$.

Let the corresponding phasor network $\mathcal{N}(\{0\}, \kappa, W)$ possesses the collection of M memory vectors $\{U^1, \ldots U^M\}$. Define $\tilde{\kappa} > \kappa$ satisfying the condition: $\gamma \equiv \Omega/\tilde{\kappa} \ll 1$, where $\Omega = max_j|\omega_j|$.

Then oscillatory network $\mathcal{N}(\{\omega_j\}, \tilde{\kappa}, W)$ has memory vectors $\tilde{U}^1, \ldots, \tilde{U}^M$, which represent perturbations of the corresponding $U^1, \ldots, U^M$.

The proof of this proposition has been obtained using the perturbation method on small parameter γ. It was confirmed by computer studies of phase portraits of the dynamical system (1) at small N.

The most essential feature of oscillatory networks is that the memory vectors to store cannot be chosen arbitrarily: they are completely defined be special symmetrical set of orthogonal vectors in N-dimensional complex space C^N — "phase" basis $\mathcal{B}_\mathcal{N}$:

$$\mathcal{B}_\mathcal{N} = \{ \ V^m \ \mid (V^s)^+ V^m = N\delta_{sm} \ \ m, s = 1, \ldots, N.\}$$

(Here V^m is a column-vector $(V_1^m, ...V_N^m)^\top$ and $(V^m)^+$ is the corresponding conjugated row-vector: $(V^m)^+ = (\bar{V}_1^m, ..., \bar{V}_N^m))$.

The phase basis is defined by single generating vector $V^0 = (1, ...1)^\top$ and the single parameter $\varphi = 2\pi/N$. All other vectors are of $\mathcal{B}_\mathcal{N}$ can be calculated with the help of recurrent process.

The basis $\mathcal{B}_\mathcal{N}$ is the eigenbasis of any Hermitian weight matrix W of size $N \times N$. At the same time the matrix W^H of rank M,

$$W^H = \sum\nolimits_{m=1}^M V^m (V^m)^+, \qquad M = rankW, \tag{3}$$

is the matrix of the projection operator into M-dimensional subspace of C^N spanned on $V^1, \ldots, V^M$.

The following results are valid for phasor networks with W^H as the matrix of connections.

- 1. Let N be a prime number.

Define basis $\mathcal{B}_\mathcal{N}$ and choose any subset of $M \leq N$ vectors from this basis $\{V^1, \ldots V^M\}$. Construct W^H in accordance with (3).

Then the phasor network has memory vectors $U^1, \ldots U^M, U^m = cV^m$, where $c = 1$ if $V^0 \in \{V^1, \ldots, V^M\}$ and $c = (1+\kappa)^{1/2}$ if $V^0 \notin \{V^1, \ldots, V^M\}$.

All memory vectors $U_1, \ldots U^M$ have equal basins of attraction.

The sizes of the basins can be controlled if weighted Hebbian matrix $\tilde{W}^H = \sum_{m=1}^M \lambda^m V^m (V^m)^+$ is used.

- 2. Let the number of oscillators N be not prime.

The main feature of the network memory in this case is that the memory is not completely controllable in distinction to the previous case. Namely, only special odd numbers M of vectors from the basis $\mathcal{B}_\mathcal{N}$ can be imposed into the network memory. If M is different from the mentioned special numbers, the recalling process is impossible at all: the dynamical system (1) has continual set of degenerated equilibria.

It should be noted also that all matrices W^H are irreducible if N is prime and are reducible ones otherwise.

Conclusions.

The special class of recurrent oscillatory and the corresponding phasor networks of high performance is designed. It is characterized by completely controllable memory of high storage capacity: up to $N-1$ memory vectors defined by some specific set ("phase" basis) can be loaded into the memory of the network consisting of N processing units. The weight matrix is designed in complex-valued Hebbian form. Extraneous memory exists, but it can be easily separated due to its non-phase character. The results of complete strict analysis of closed homogeneous oscillatory chains are presented. Oscillatory networks are promising from many viewpoints, in particular, in view of possible of nonlinear optical implementations.

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