ASSOCIATIVE MEMORY OSCILLATORY NETWORKS WITH HEBBIAN AND PSEUDO-INVERSE MATRICES OF CONNECTIONS

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The systems of symmetrically coupled limit cycle oscillators admit the design of recurrent associative memory networks. Complex-valued matrices of connections are proved to be the proper extension for specification the modifiable interconnection architecture of the networks. The expressions for weighted Hebbian and pseudo-inverse matrices of connections in terms of the set of memory vectors to store are obtained.

It is shown that in the parametric domain, corresponding to strong oscillatory interaction, the set of slightly perturbed eigenvectors of the matrix of connection forms the set of the network memory vectors. Their locations can be calculated by perturbation method on appropriate small parameter. The retrieval characteristics of these oscillatory networks coincide with those of "clock" neural networks (the networks of magnetic spins on a plane).

Keywords: associative memory, coupled oscillators, Hebbian learning

1 Introduction

Large systems of coupled oscillators [1-3] in the regime of synchronization (phase locking) have an ability to memorize information. So the problem of design of recurrent associative memory oscillatory networks, similar to corresponding problem for neural networks, arises. The design includes the specification of matrix of network connections in terms of prescribed set of memory vectors to store and also the proper choice of other modifiable parameters of governing dynamical system to provide the effective retrieval characteristics of the network.

It should be noted that one of the most attractive features of oscillatory systems is undoubtedly wide field of possible physical implementations. In particular, oscillatory models promise direct and by this reason much more effective implementations as compared to optoelectronical ones. For instance, the potentialities of implementations based on photon-echo effect [12] are in fact essentially greater than those already achieved in the known schemes.

As for theoretical study of oscillatory systems from the viewpoint of associative memory modeling there is a number of various ways of associative memory modeling based on systems of coupled oscillators. One of them is connected with the existence of two subpopulations of oscillatory system in the vicinity of phase transition into synchronized state - the subpopulations of oscillators in synchronized and nonsynchronized states. This approach has been developed in a number of works (see, for instance, [4]).

The modeling of *recurrent* associative memory oscillatory network on the base of oscillatory systems in the state of *synchronization* is still at the very beginning [5-7]. Up to now only the special kind of oscillatory system with the simplest kind of interaction - limit-cycle oscillators with linear interactions of pairs - have been studied in various aspects so far. Such the special type of oscillatory systems is proved to be closely related to the systems of magnetic spins on the plane ("clock" spin glasses, or phasor systems). The associative memory "clock" neural networks with Hebbian matrix of connections have been studied and their retrieval characteristics has been obtained [8].

It worth mentioning also that an attempt has been done to study phasor networks with asymmetrical complex-valued matrix of connections and non-zero thresholds [10]. This model is the natural generalization of the phasor network model which had been studied in [9]. Further study of this model would be quite desirable.

The presented work contains novel results on design of recurrent associative memory oscillatory networks with Hebbian and pseudo-inverse matrices of connections.

2 The Dynamical Equations of the Model of Phase Oscillators.

We consider the system of N limit-cycle oscillators on a plane with symmetrical nonhomogeneous coupling, the state of each being defined as a point $z_j = r_j exp(i\theta_j)$ of complex plane. In appropriate parametric domain the dynamical system governing the dynamics of oscillatory system

$$\dot{z_j} = (1 + i\omega_j - |z_j|^2)z_j + K \sum_{k=1}^{N} W_{jk}(z_k - z_j), \quad j = 1, ..., N. \tag{1}$$

can be reduced to "phase" dynamical system

$$\dot{\theta}_j = \omega_j + K \sum_{k=1}^{N} \mathcal{W}_{jk} sin(\theta_k - \theta_j + \beta_{jk}), \quad j = 1, ..., N. \tag{2}$$

where $\omega_j, \quad j = 1, ...N$, are the natural frequencies on the cycles and complex-valued Hermitian $N \times N$ matrix $W = [W_{jk}] = [\mathcal{W}_{jk} exp(i\beta_{jk})], \quad W = \bar{W}^\top \equiv W^+$ specifies the weights of connections of oscillators in the network, the real value K defines the absolute value and the sign of interaction strengths in the system [13]. The dynamical system (2) defines the model of system of "phase oscillators". Obviously, the state vector of the network of phase model is $z = (z_1, ..., z_N)^\top, \quad z_j = exp(i\theta_j)$. Note that the matrix W in (1) should not have zero diagonal unlike to the case of neural networks. This property is just the consequence of special form of operator of interaction of amplitude-phase dynamical system (1).

Any Hermitian matrix W can be represented in a form

$$W = N^{-1} \sum_{m=1}^{M} \lambda^m V^m (V^m)^+, \quad M = rankW, \tag{3}$$

where $\{V^m\}$ is the set of mutually orthogonal eigenvectors of W corresponding to the set of its nonzero eigenvalues [13]:

$$WV^m = \lambda^m V^m, \quad (V^s)^+ V^m = N\delta_{ms}, \quad m = 1, ...N. \tag{4}$$

where δ_{ms} denotes the Kronecker symbol. With the help of expansion (3) the dynamical system (2) can be rewritten in the form

$$\dot{\theta}_j = \omega_j + (K/N) \sum_{m=1}^{M} \sum_{k=1}^{N} \lambda^m sin([\theta_k - \beta_k^m] - [\theta_j - \beta_j^m]). \tag{5}$$

One more form of the system (2) can be obtained if one uses the expansion of state vector z in eigenbasis $\{V^m\}$ of matrix W

$$z = \sum_{m=1}^{M} Z^m V^m, \quad Z^m = N^{-1}(V^m)^+ z = N^{-1} \sum_{j=1}^{N} exp(i[\theta_j - \beta_j^m]) = R^m exp(i\psi^m), \tag{6}$$

The variables Z^m, the inner products of current state vector z and the basis vectors V^m, are the macrovariables (in the case of high dimension N of the dynamical system). They are just the "overlaps" which are usually used in asymptotical analysis of retrieval characteristics of associative memory neural networks.

Being rewritten in terms of macrovariables $Z_m R^m exp(i\psi^m)$, the system (2) has the form of N independent equations [13]:

$$\dot{\theta}_j = \omega_j + K \sum_{m=1}^{M} \lambda^m R^m sin(\psi^m + \beta_j^m - \theta_j), \tag{7}$$

System (7) provides the self-consistent field approach description of oscillatory network. In particular, it is very convenient for the analysis of fixed points of the phase dynamical system.

3 The "Hebbian" Solution to Associative Memory Network Design Problem

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As it is very well known from the theory of associative memory neural networks, the matrix of connections in the form of sum of outer products on the orthogonal set of memory vectors just provides the simplest, natural solution to network design problem. The outer-product matrices themselves are usually regarded as "Hebbian" because of the relation to Hebbian learning algorithm. We show that the "Hebbian" solution to the associative memory design problem also exists for the networks of "phase oscillators". More exactly, we have the following results.

I. The case $\omega_j = 0$ (phasor networks).

Let the set $\{\mathcal{V}^m\}$, $\mathcal{V}_j^m = exp(i\beta_j^m)$, of $M (M \leq N)$ linearly independent vectors to store be given. Then the "weighted Hebbian" solution to the problem can be realized in the following steps:

- Find the orthogonal system of vectors $\{V^m\}$, corresponding to the set $\{\mathcal{V}^m\}$.

• Define the matrix W by formulas (3),(4), where λ^m are some real positive values subject to natural condition $\sum_{m=1}^{M}\lambda^m = 1$. The formula (3) for W can be also rewritten in the more familiar form

$$W = V\Lambda V^+, \quad \Lambda = diag(\lambda^1, ...\lambda^M), \tag{8}$$

where $N \times M$ matrix $V = [V_j^m]$ has the vectors V^m as its columns. The choice of coinciding $\{\lambda^m\}$, which is used in the neural networks problems, is known to lead to drastically great extraneous memory (what is quite natural from the viewpoint of linear algebra). So the choice of noncoinciding $\{\lambda^m\}$ seems to be preferable [11].

II. The case $\omega_j \neq 0, \quad \sum_{j=1}^{N}\omega_j = 0.$

It can be shown that at arbitrary ω_j in the case of "strong" interaction the oscillatory network with Hebbian matrix has the set of memory vectors $\tilde{V}^m, m = 1, ...M$. Each $\tilde{V}^m$ is located at small vicinity of corresponding V^m, being its slight perturbation. The set $\{\tilde{V}^m\}$ can be estimated by perturbation method on small parameter $\gamma = \Omega/K$, where $\Omega = max_j|\omega_j|$. It provides the following result:

$$\tilde{V}_j^m = V_j^m + \epsilon(\lambda^m)^{-1}\omega_j + O(\epsilon^2), m = 1, ...M, j = 1, ...N. \tag{9}$$

These facts permit to conclude that the retrieval characteristics of oscillatory network in the region of strong interaction coincide with those for "clock" neural networks obtained in [8] (storage capacity $\alpha \sim 0.037$, the limit value of "overlap" ~ 0.9). The characteristics of the oscillatory network at arbitrary admissible γ differ essentially from those of corresponding phasor network: under gradual increasing of γ the fixed points of oscillatory dynamical system disappear step by step in saddle-node bifurcations.

4 The Oscillatory Networks with Pseudo-inverse Matrix of Connections

Just as in the case of neural networks the problem of imposing to the memory of the set of independent, but not orthogonal memory vectors can be solved by calculation of pseudo-inverse W. Let $\{U^m\}$ be such a set of vectors and $U = [U_j^m]$ be the $N \times M$ matrix having the vectors U^m as its columns. The pseudo-inverse matrix W is usually defined as the solution to the matrix equation $WU = U$. In the case of complex-valued W and U the solution can be written as

$$W = U(U^+U)^{-1}U^+, \quad N > M.$$

Here the Hermitian $M \times M$ matrix $C = U^+U, \quad (C_{kl} = (U^l)^+U^k)$ has a sense of correlation matrix of the set $\{U^m\}$. If one uses for C the expansion (3) in the basis of its eigenvectors Q^m:

$$C \equiv U^+U = N^{-1}\sum_{m=1}^{M}\nu^m Q^m(Q^m)^+, \quad CQ^m = \nu^m Q^m,$$

the explicit form for W can be found:

$$W = N^{-1}\sum_{m=1}^{M}(\nu^m)^{-1}\mathcal{U}^m(\mathcal{U}^m)^+, \quad \mathcal{U}_j^m = \sum_{k=1}^{M}U_j^k Q_k^m. \tag{10}$$

If $\{U^m\}$ is the system of mutually orthogonal vectors, one can easily get from (10) that the pseudo-inverse matrix is reduced to Hebbian one.

Further, it is useful to note that the matrix C is directly related to Gram matrix G of a set vectors $\{U^m\} : C = G^+$. It permits to verify the fact that the design of the network with pseudo-inverse matrix is equivalent to the design of the network with Hebbian matrix after the preliminary Gram - Schmidt orthogonalization of the given set $\{U^m\}$. The last way seems to be preferable in view of the advantages of Hebbian matrix - the relation to local Hebbian learning algorithm and the additivity with respect to number of vectors to store.

5 Summary

- The formula for weighted Hebbian matrix of connections for recurrent associative memory oscillatory network is obtained. It is shown that in the region of strong oscillatory interaction ($\gamma = \Omega/K$ is small) the retrieval characteristics of such a network coincide with those of corresponding phasor network.

- The explicit form of expression for pseudo-inverse matrix of connections in terms of network memory vectors to store is obtained.

Acknowledgment
The research was supported by Russian fund of fundamental investigations, grant number 94-01-00406.

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