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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Families of Periodic Solutions and Invariant Tori of Hamiltonian System

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Александр Дмитриевич Брюно

Семейства периодических решений и инвариантных торов системы Гамильтона.
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Вблизи неподвижного решения, вблизи периодического решения и вблизи инвариантного тора аналитической системы Гамильтона рассматривается нормальная форма функции Гамильтона. Обычно нормализующее преобразование расходится в полной окрестности каждого указанного исходного объекта, но существуют сходящиеся преобразования, которые нормализуют функцию Гамильтона лишь на некоторых множествах, примыкающих к исходному объекту. Эти множества аналитичны, включают все формальные семейства периодических решений, а при некотором условии на малые знаменатели они включают некоторые формальные семейства инвариантных торов с подобными базисами частот. Поэтому в случае общего положения вещественная система Гамильтона с n степенями свободы имеет: а) однопараметрические семейства периодических решений, б) однопараметрические семейства n-мерных неприводимых инвариантных торов и в) $(l+1)$-параметрические семейства $k(<n)$-мерных неприводимых инвариантных торов с ровно $2l$ собственными значениями, имеющими нулевые вещественные части, и все их мнимые части соизмеримы с частотами.

Ключевые слова: система Гамильтона, стационарное решение, периодическое решение, инвариантный тор, нормальная форма.

Alexander Dmitrievich Bruno

Families of Periodic Solutions and Invariant Tori of Hamiltonian System.

Near a stationary solution, near a periodic solution and near an invariant torus of an analytic Hamiltonian system we consider the normal form of its Hamiltonian function. Usually, the normalizing transformation diverges in the whole neighborhood of each mentioned initial object, but there exist convergent transformations, which normalize the Hamilton function only in some sets adjoining the initial object. The sets are analytic, include all formal families of periodic solutions and under a condition on small divisors, they include some formal families of invariant tori with similar bases of frequencies. So generically the real Hamiltonian system with n degrees of freedom has: (a) one-parameter families of periodic solutions, (b) one-parameter families of n-dimensional invariant irreducible tori and (c) $(l+1)$-parameter families of $k(<n)$ dimensional irreducible invariant tori with exactly $2l$ eigenvalues having zero real parts, and for all of them imaginary parts are commensurable with frequencies.

Key words: Hamiltonian system, stationary solution, periodic solution, invariant torus, normal form.

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1. Introduction

We consider the real analytic autonomous Hamiltonian system with a finite number of degrees of freedom and without parameters near its stationary solution (Section 2), near its periodic solution (Section 3) and near its invariant torus (Section 4).

Our aim is to study families of periodic solutions and families of invariant tori adjoining the mentioned above initial objects.

For that, we reduce the Hamiltonian system to its normal form. Using it, we select such formal sets, for which there exist convergent normalizing transformations and so the sets are analytic. They are the set $\tilde{\mathcal{A}}$, if small divisors are absent, and the set $\mathcal{B}$, if small divisors present and satisfy a condition. The set $\tilde{\mathcal{A}}$ contains all families of periodic solutions.

It appeared that in general case a periodic solution lies at an one-parameter family of periodic solutions with near periods and invariant irreducible torus of the maximal possible dimension also lies at an one-parameter family of such tori with similar bases of frequencies, but formal families of invariant tori of non-maximal dimension are analytic, only if their eigenvalues satisfy an additional condition: all eigenvalues with zero real part have imaginary parts commensurable with set of frequencies.

- Huygens in XVII century began to study periodic solutions.
- Kolmogorov [Kolmogorov, 1954] began to study analytic invariant tori.
- His successors Arnold, Moser and others developed a theory named KAM.

Here another approach from the [Bruno, 1989, Part II] is used. In the present work there are three new methods:

- a division of a formal set $\mathcal{A}^k$ into components $\mathcal{A}^k_I$;
- an introduction of the reduced normal form, not depending on time;
- writing the sets $\mathcal{A}^k_I$ and $\mathcal{B}^k$ for the reduced normal form.

2. A neighbourhood of a stationary solution

2.1. Normal form. Consider the Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}, \quad j = 1, \ldots, n \tag{2.1}$$

with n degrees of freedom in a neighborhood of the stationary point

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) = 0, \quad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_n) = 0. \tag{2.2}$$

If the Hamiltonian $\gamma(\boldsymbol{\xi}, \boldsymbol{\eta})$ is analytic at this point, then it can be expanded in the power series

$$\gamma(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}, \tag{2.3}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $\boldsymbol{\xi}^{\mathbf{p}} = \xi_1^{p_1} \xi_2^{p_2} \cdots \xi_n^{p_n}$, $\gamma_{\mathbf{pq}}$ are constant coefficients.

Since point (2.2) is stationary, expansion (2.3) begins with quadratic terms. These terms are associated with the linear part of system (2.1). The eigenvalues of the matrix of the linear system are organized in pairs

$$\lambda_{j+n} = -\lambda_j\,, \quad j = 1,\ldots,n\,.$$

Let $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)$. The canonical changes of coordinates

$$\boldsymbol{\xi},\boldsymbol{\eta} \longrightarrow \mathbf{x},\mathbf{y} \tag{2.4}$$

preserve the Hamiltonian property of the system.

Theorem 2.1 ([Bruno, 1972, § 12])**.** *There exists a canonical invertible formal transformation* (2.4) *that reduces initial system* (2.1) *to the normal form*

$$\dot{x}_j = \frac{\partial g}{\partial y_j}\,, \quad \dot{y}_j = -\frac{\partial g}{\partial x_j}\,, \quad j = 1,\ldots,n\,, \tag{2.5}$$

where the series

$$g(\mathbf{x},\mathbf{y}) = \sum g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} \tag{2.6}$$

contains only resonant terms with

$$\langle \mathbf{p} - \mathbf{q},\boldsymbol{\lambda}\rangle = 0. \tag{2.7}$$

Here $\langle \mathbf{p},\boldsymbol{\lambda}\rangle = p_1\lambda_1 + \cdots + p_n\lambda_n$ is the scalar product.

For the real original system (2.1), the coefficients $g_{\mathbf{pq}}$ of the complex normal form (2.6) satisfy special real type relations, and under the standard canonical linear change of coordinates $(\mathbf{x},\mathbf{y}) \to (\mathbf{X},\mathbf{Y})$, system (2.5) takes the real form [Bruno, 1994, Ch. I].

Let $I = \{i_1,\ldots,i_m\}$ be a set of increasing natural indices $i \leqslant n$. Here $1 \leqslant m \leqslant n$. Consider the coordinate subspace

$$K_I = \{\mathbf{x},\mathbf{y} : x_j = y_j = 0 \text{ for all } j \notin I\}\,.$$

If $I = \{1,\ldots,n\}$, then K_I is the space $\mathbb{C}^{2n}$ with coordinates $\mathbf{x},\mathbf{y}$, which we denote as K_n.

Let us mention four numerical characteristics of a coordinate subspace K_I:

1) Its *half-dimension* $m_I = m$.
2) Its *multiplicity of resonances* $\varkappa_I$, as a number of linearly independent integer relations $\sum_{i\in I} p_i\lambda_i = 0$ with integer p_i.
3) Its *degree of irrationality* $\sigma_I = m_I - \varkappa_I$.
4) Its *subsets of eigenvalues* $\boldsymbol{\lambda}_I = \{\lambda_i, i \in I\}$ and $\boldsymbol{\alpha} = \{\alpha_i, i \in I\}$.

Below we will consider the case where all eigenvalues $\lambda_1,\ldots,\lambda_n$ are pure imaginary: $\lambda_j = \mathrm{i}\alpha_j$, $\alpha_j \in \mathbb{R}$, and all $\lambda_j \neq 0$.

2.2. Convergency of the normalizing transformation.
Condition ω

Let $\omega_k = \min|\langle \mathbf{p},\boldsymbol{\lambda}\rangle|$ over $\langle \mathbf{p},\boldsymbol{\lambda}\rangle \neq 0$, $||\mathbf{p}|| < 2^k$, $\mathbf{p} \in \mathbb{Z}^n$, where $||\mathbf{p}|| = \sum |p_j|$. Then

$$-\sum_{k=1}^{\infty} \frac{\log \omega_k}{2^k} < \infty.$$

It is very weak numerical restriction on the eigenvalues $\boldsymbol{\lambda}$. It satisfies for almost all sets $\boldsymbol{\lambda}$. In particularly, it satisfies if all λ_j are pairwise commensurable, then the multiplicity of resonances $\varkappa_n$ for the whole space K_n is equal to $n-1$. Condition ω is a restriction on ‘‘small divisors’’ arising in the normalizing transformation.

Condition A
There exists such a power series $a(\mathbf{x},\mathbf{y})$, that in the normal form (2.5)

$$\frac{\partial g}{\partial y_j} = \lambda_j x_j a, \quad \frac{\partial g}{\partial x_j} = \lambda_j y_j a, \quad j = 1, \ldots, n.$$

It is a very hard restriction on the right parts of the normal form (2.5). It satisfies very seldom.

Theorem 2.2 ([Bruno, 1971]). *If eigenvalues $\boldsymbol{\lambda}$ satisfy Condition ω and the normal form* (2.5) *satisfies Condition* **A**, *then the normalizing transformation converges in a neighborhood of the stationary point* $\mathbf{x} = \mathbf{y} = 0$.

According to [Bruno, 1972, § 12] Condition **A** is equivalent to the condition that the Hamiltonian $g(\mathbf{x},\mathbf{y})$ of the normal form (2.5) is a power series of one variable $\sum\limits_{j=1}^{n} \lambda_j x_j y_j$.

2.3. Set $\mathcal{A}$. Suppose that the functions $f_1(\mathbf{x},\mathbf{y}), \ldots, f_r(\mathbf{x},\mathbf{y})$ are analytic and vanish at the point $\mathbf{x} = \mathbf{y} = 0$. Then the system of equations

$$f_j(\mathbf{x},\mathbf{y}) = 0, \quad j = 1, \ldots, r \tag{2.8}$$

defines an *analytic set $\mathcal{N}$* containing the point $\mathbf{x} = \mathbf{y} = 0$. If $f_1, \ldots, f_r$ are formal power series, we say that (2.8) defines a *formal set $\mathcal{N}$*.

Problem 2.1. Which invariant formal sets of the system (2.1) are analytic?

The fact is that it is comparatively easy to calculate formal invariant sets using the normal form (2.5)–(2.7). We need to select among them only those that are analytic for the initial system (2.1).

From the normal form (2.5)–(2.7) we form the formal set

$$\mathcal{A} = \left\{ \mathbf{x},\mathbf{y} : \frac{\partial g}{\partial y_j} = \lambda_j x_j a,\ \frac{\partial g}{\partial x_j} = \lambda_j y_j a,\ j = 1, \dots, n \right\} \tag{2.9}$$

where a is a free parameter. We can exclude it from the equations and obtain a representation of the set $\mathcal{A}$ in the form (2.8).

All solutions from the set $\operatorname{Re}\mathcal{A}$ are conditionally-periodic (including periodic and stationary solutions). In fact, the value of the parameter a is constant in each solution and we have

$$x_j = x_j^0 \exp \lambda_j a t, \quad y_j = y_j^0 \exp\left(-\lambda_j a t\right), \quad j = 1, \dots, n.$$

Definition 2.1. Let K_I be a coordinate subspace in $\mathbf{x}$,$\mathbf{y}$. The *component* $\mathcal{A}_I$ *of the set* $\mathcal{A}$ in this subspace is defined by the system of equations

$$\frac{\partial g}{\partial y_j} = \lambda_j x_j a, \quad \frac{\partial g}{\partial x_j} = \lambda_j y_j a, \quad j \in I,$$

and all x_j,$y_j \neq 0$ for $j \in I$. In particularly, there is the component $\mathcal{A}_n$ for the whole space K_n.

Theorem 2.3. *If in the normal form* (2.5) *Hamiltonian* $g(\mathbf{x},\mathbf{y})$ *is analytic, then each component* $\operatorname{Re}\mathcal{A}_I$ *is a family of irreducible invariant tori of dimension* σ_I *with frequencies* $a\boldsymbol{\alpha}_I$. *Generically the family is either one-parametric (along* a*) or it is empty. In degenerate cases it can have more than one parameter.*

A torus of dimension 1 is a periodic solution.

A coordinate subspace $K_{\tilde I}$ is called *rational* if the corresponding eigenvalues $\boldsymbol{\lambda}_{\tilde I}$ are pairwise commensurable. Let $\widetilde{\mathcal{K}}$ be the union of all rational subspaces, and $\tilde{\mathcal{A}} = \mathcal{A} \cap \widetilde{\mathcal{K}}$

Theorem 2.4 ([Bruno, 1989, Part II, § 3])**.** *There exists an analytic canonical transformation* $\boldsymbol{\xi},\boldsymbol{\eta} \to \mathbf{x},\mathbf{y}$ *which transforms initial system* (2.1) *to normal form in the set* $\tilde{\mathcal{A}}$ *and the set is analytic.*

2.4. Set $\mathcal{B}$. Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a diagonal matrix. In the set $\mathcal{A}$ we consider the $2n \times 2n$ matrix

$$B = \begin{pmatrix} \dfrac{\partial^2 g}{\partial \mathbf{y} \partial \mathbf{x}} - \Lambda a & \dfrac{\partial^2 g}{\partial \mathbf{y} \partial \mathbf{y}} \\ -\dfrac{\partial^2 g}{\partial \mathbf{x} \partial \mathbf{x}} & -\dfrac{\partial^2 g}{\partial \mathbf{x} \partial \mathbf{y}} + \Lambda a \end{pmatrix},$$

where a is the same parameter as in equations (2.9). We define the formal set $\mathcal{B}$ as that subset of the set $\mathcal{A}$ in which the matrix B is nilpotent, that is,

$$\mathcal{B} = \left\{ \mathbf{x},\mathbf{y} : \quad \mathbf{x},\mathbf{y} \in \mathcal{A}, \quad B^{2n} = 0 \right\}.$$

Theorem 2.5 ([Bruno, 1989, Part II]). *Generically* $\mathcal{B} = \mathcal{A}_n$.

Theorem 2.6 ([Bruno, 1989, Part II]). *If eigenvalues* $\boldsymbol{\lambda}$ *satisfies Condition* ω *then there exists an analytic canonical transformation* $\boldsymbol{\xi},\boldsymbol{\eta} \to \mathbf{x},\mathbf{y}$ *which transforms initial system* (2.1) *to normal form in the set* $\mathcal{B}$ *and the set is analytic.*

2.5. Example 2.1. Suppose that $\lambda_1, \dots, \lambda_n$ are linearly independent over integral numbers, that is, the equation $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$ has only the trivial solution $\mathbf{p} = 0$ for the integers $\mathbf{p}$. Then in the normal form (2.5)-(2.7),

$$g = f\left(\rho_1, \dots, \rho_n\right), \qquad \text{where } \rho_j = x_j y_j, \quad j = 1, \dots, n.$$

We therefore have

$$\mathcal{A} = \left\{ \mathbf{x}, \mathbf{y} : x_j \frac{\partial f}{\partial \rho_j} = \lambda_j x_j a, \quad y_j \frac{\partial f}{\partial \rho_j} = \lambda_j y_j a, \quad j = 1, \dots, n \right\}.$$

We consider the set $\mathcal{A}$ in the cartesian coordinates $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$. In the general case, each coordinate subspace (with respect to $\boldsymbol{\rho}$) contains one onedimensional (with respect to $\boldsymbol{\rho}$) component of the set $\mathcal{A}$ that does not lie in a smaller coordinate subspace. Consequently, the set $\mathcal{A}$ consists of $2^n - 1$ such components; for each $d \leqslant n$ there are exactly $n!/[d!(n-d)!]$ of these components situated in d-dimensional (with respect to $\boldsymbol{\rho}$) coordinate subspaces. In particular, there is one component,

$$\mathcal{A}_n = \left\{ \boldsymbol{\rho} : \frac{\partial f}{\partial \rho_j} = \lambda_j a, \quad j = 1, \dots, n \right\}$$

situated outside the coordinate subspaces. $B^2 = 0$ in this component.

In fact,

$$B = \begin{pmatrix} R - \Lambda a & 0 \\ 0 & -R + \Lambda a \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} S & S \\ -S & -S \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}$$

where the blocks are the following $n \times n$ matrices:

$$R = \left\{ \frac{\partial f}{\partial \rho_1}, \dots, \frac{\partial f}{\partial \rho_n} \right\}, \quad U = \left\{ x_1, \dots, x_n \right\}, \quad V = \left\{ y_1, \dots, y_n \right\}$$

these are diagonal, and
$$S = \left(\frac{\partial^2 f}{\partial\rho_j\partial\rho_k}\right).$$

In the set $\mathcal{A}$ we have
$$B = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} S & S \\ -S & -S \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}$$
that is, $B^2 = 0$. In the general case, the matrix B is not nilpotent in coordinate subspaces, since there for some derivatives $\partial f/\partial\rho_j \neq \lambda_j a$. Thus, $\mathcal{B} = \mathcal{A}_n$.

Moreover, the n components
$$\mathcal{A}'_j = \{\boldsymbol{\rho}: \quad \rho_i = 0, \quad i \neq j, \quad 1 \leqslant i \leqslant n\}, \quad j = 1, \ldots, n,$$
are coordinate axes with respect to $\boldsymbol{\rho}$ and exhaust all rational subspaces; taken together, they make up the set $\tilde{\mathcal{A}}$. According to Theorem 2.4, the initial system (2.1) has n analytic one-parameter families $\mathcal{A}'_j$ of periodic solutions (these are Lyapunov [Lyapunov, 1892] families).

If the eigenvalues $\boldsymbol{\lambda}$ satisfy Condition ω, then by Theorems 2.6, 2.5 the component $\mathcal{A}_n = \mathcal{B}$ is also an analytic set.

Let
$$f = g = \langle\boldsymbol{\rho}, \boldsymbol{\lambda}\rangle + \frac{1}{2}\langle\boldsymbol{\rho}, T\boldsymbol{\rho}^*\rangle + \cdots \tag{2.10}$$
where T is a symmetric matrix. In the general case, det $T \neq 0$ and the system of equations in (2.9) has a one-dimensional solution
$$\boldsymbol{\rho}^* = T^{-1}\boldsymbol{\lambda}^*(a-1) + o(a-1), \tag{2.11}$$
where * means transposition.

If the initial system (2.1) is real, the coordinates x_j, y_j are connected by the reality relation $\bar{x}_j = -\mathrm{i}y_j$. Consequently, $-\arg x_j = \arg y_j - \pi/2$, that is, $\arg(x_j y_j) = \pi/2$.

Thus, purely imaginary ρ_j with $\operatorname{Im}\rho_j \geq 0, j = 1, \ldots, n$ correspond to real values of the initial coordinates. Each set $\mathcal{A}'_j$ has a real part,
$$\operatorname{Re}\mathcal{A}'_j = \{\rho_j : \operatorname{Re}\rho_j = 0, \quad \operatorname{Im}\rho_j \geq 0\}$$
which is a real one-parameter family of periodic solutions. For the real Hamiltonian (2.10) the matrix T is also real. If the vector $T^{-1}\operatorname{Im}\boldsymbol{\lambda}^*$ has coordinates of different signs, then according to (2.11), the set $\mathcal{B}$ has only a trivial real part, $\operatorname{Re}\mathcal{B} = 0$.

If all the coordinates of $T^{-1}\boldsymbol{\alpha}^*$ are of the same sign, the real set $\operatorname{Re}\mathcal{B}$ is a one-parameter family of n-dimensional irreducible invariant tori with frequency basis $\alpha_1 a, \ldots, \alpha_n a$. As $a \to 1$ the tori of this family tend towards the fixed point $\mathbf{x} = \mathbf{y} = 0$. For a detailed analysis of various situations see in [Bruno, 1989, Part II, § 3].

3. A neighborhood of a periodic solution

3.1. Local coordinates. Let a real Hamiltonian system with $n+1$ degrees of freedom have a real 2π -periodic solution $\mathcal{M}$ and the Hamiltonian function is analytic in some neighborhood of it.

According to [Bruno, 1994, Ch. II, Sect. 2.1] near the solution $\mathcal{M}$ we can introduce such real local canonical coordinates $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, ψ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$, ρ, that the solution $\mathcal{M}$ is given by equations

$$\boldsymbol{\xi} = \boldsymbol{\eta} = 0, \quad \rho = 0, \quad \psi = \psi_0 + t$$

and the Hamiltonian has the form

$$\gamma = \Sigma\gamma_{\mathbf{pq}l}(\psi)\boldsymbol{\xi}^{\mathbf{p}}\boldsymbol{\eta}^{\mathbf{q}}\rho^l = \rho + \cdots , \tag{3.1}$$

where integers $\mathbf{p}$,$\mathbf{q} \geqslant 0$, integer $l \geqslant 0$, real analytic functions $\gamma_{\mathbf{pq}l}(\psi)$ have in ψ the period 2π and they are expanded in the Fourier series.

Then the Hamiltonian system is

$$\begin{aligned} \dot{\xi}_j &= \frac{\partial\gamma}{\partial\eta_j}, & \dot{\eta}_j &= -\frac{\partial\gamma}{\partial\xi_j}, \quad j = 1, \ldots, n, \\ \dot{\psi} &= \frac{\partial\gamma}{\partial\rho}, & \dot{\rho} &= -\frac{\partial\gamma}{\partial\psi}. \end{aligned}$$

3.2. Normal form. For $\rho = 0$ and $\psi = t$ quadratic in $\boldsymbol{\xi}$,$\boldsymbol{\eta}$ part γ_2 of the Hamiltonian (3.1) defines 2π-periodic linear in $\boldsymbol{\xi}$,$\boldsymbol{\eta}$ system

$$\dot{\xi}_j = \frac{\partial\gamma_2}{\partial\eta_j}, \quad \dot{\eta}_j = -\frac{\partial\gamma_2}{\partial\xi_j}, \quad j = 1, \ldots, n. \tag{3.2}$$

Let $\nu_1, \ldots, \nu_{2n}$ be eigenvalues of its monodromy matrix, i. e. matrix of substitution of fundamental matrix of solutions to the system (3.2) in the period 2π. Let all $|\nu_j| = 1$ and $\nu_j \neq -1$. We put

$$\alpha_j = \frac{1}{2\pi \mathrm{i}}\ln\nu_j, \quad \alpha_j \in \mathbb{R}, \quad \alpha_j \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad j = 1, \ldots, 2n.$$

Using correct numeration one obtains

$$\alpha_{j+n} = -\alpha_j, \quad j = 1, \ldots, n.$$

Let denote $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Theorem 3.1 ([Bruno, 1994, Ch. II]; [Bruno, 2020])**.** *There exists a complex formal invertible* 2π*-periodic in* ψ *and* φ *canonical transformation of coordinates in the form of Poisson series*

$$\boldsymbol{\xi},\psi,\boldsymbol{\eta},\rho \longleftrightarrow \mathbf{x},\varphi,\mathbf{y},r, \tag{3.3}$$

which reduces the Hamiltonian γ *into normal form*

$$g(\mathbf{x},\varphi,\mathbf{y},r) = r + \mathrm{i}\sum_{j=1}^{n}\alpha_j x_j y_j + \sum g_{\mathbf{pq}lm}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}r^l e^{\mathrm{i}m\varphi}, \tag{3.4}$$

where $\mathbf{x},\mathbf{y} \in \mathbb{C}^n$*,* $0 \leqslant \mathbf{p},\mathbf{q} \in \mathbb{Z}^n$*,* $l \geqslant 0$ *and* m *are integer, all terms of the second sum are resonant, i. e.*

$$\langle \mathbf{p}-\mathbf{q},\boldsymbol{\alpha}\rangle + m = 0. \tag{3.5}$$

Theorem 3.2. *There exists a canonical transformation*

$$\begin{aligned} &x_j = u_j \exp(-\mathrm{i}\beta_j\varphi), \quad y_j = v_j \exp(\mathrm{i}\beta_j\varphi), \quad j = 1,\ldots,n, \\ &r = s - \mathrm{i}\sum_{j=1}^{n}\beta_j u_j v_j \end{aligned} \tag{3.6}$$

with rational β_j*, which reduces the normal form of the Hamiltonian* (3.4)*,* (3.5) *to an autonomous power series*

$$h(\mathbf{u},\mathbf{v},s) = s + \mathrm{i}\sum_{j=1}^{n}\gamma_j u_j v_j + \sum h_{\mathbf{pq}l}\mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}}s^l, \tag{3.7}$$

where in the first sum all nonzero γ_j *are irrational numbers and in the second sum* $0 \leqslant \mathbf{p},\mathbf{q} \in \mathbb{Z}^n$*,* $0 \leqslant l \in \mathbb{Z}$*,* $h_{\mathbf{pq}l} = \mathrm{const} \in \mathbb{C}$ *and present only resonant terms with*

$$\langle \mathbf{p}-\mathbf{q},\boldsymbol{\gamma}\rangle = 0,$$

where $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_n)$.

Similar theorem is in [Bruno, 2020]. Theorem 3.2 is a particular case of Theorem 4.2 that will be given in Section 4 with the proof.

Variable s now is a formal integral of the following system

$$\dot{u}_j = \frac{\partial h}{\partial v_j}, \quad \dot{v}_j = -\frac{\partial h}{\partial u_j}, \quad j = 1,\ldots,n, \tag{3.8}$$

$$\dot{\varphi} = \frac{\partial h}{\partial s}. \tag{3.9}$$

If the initial Hamiltonian γ is real for real coordinates $\boldsymbol{\xi}$,ψ,$\boldsymbol{\eta}$,ρ, then in Theorem 3.1 variables $\mathbf{x}, \mathbf{y}$ are complex but variables ψ, ρ and φ, r are real. Here according to [Bruno, 1994, Chs. I, II] variables $\mathbf{x}$,$\mathbf{y}$ are connected with real variables $\mathbf{X} = (X_1, \dots, X_n)$, $\mathbf{Y} = (Y_1, \dots, Y_n)$ by the linear standard transformation

$$x_j = \frac{1}{\sqrt{2\mathrm{i}}} (\mathrm{i}X_j - Y_j), \quad y_j = \frac{1}{\sqrt{2\mathrm{i}}} (\mathrm{i}X_j + Y_j), \quad j = 1, \dots, n.$$

3.3. Convergency of the normalizing transformation. Condition ω^1

Let $\omega_k = \min |p_0 + \langle \mathbf{p}, \boldsymbol{\alpha} \rangle|$ over $|p_0 + \langle \mathbf{p}, \boldsymbol{\alpha} \rangle| \neq 0$, $|p_0| + ||\mathbf{p}|| < 2^k$, $p_0 \in \mathbb{Z}$, $\mathbf{p} \in \mathbb{Z}^n$. Then

$$-\sum_{k=1}^{\infty} \frac{\log \omega_k}{2^k} < \infty.$$

Condition $\mathbf{A}^1$

There exists such a Poisson series $a(\mathbf{x}, \varphi, \mathbf{y}, r)$ that in the normal form (3.4)

$$\begin{aligned} &\frac{\partial g}{\partial y_j} = \mathrm{i}\alpha_j x_j a, \quad \frac{\partial g}{\partial x_j} = \mathrm{i}\alpha_j y_j a, \; j = 1, \dots, n, \\ &\frac{\partial g}{\partial \varphi} = 0, \qquad\quad \frac{\partial g}{\partial r} = a. \end{aligned} \tag{3.10}$$

According to [Bruno, 1972, § 11] it equivalents to the condition that Hamiltonian g is a power series of one variable $r + \mathrm{i} \sum_{j=1}^{n} \alpha_j x_j y_j$.

Theorem 3.3 ([Bruno, 1972], § 11)**.** *The normalizing transformation* (3.3) *converges in a neighborhood of our periodic solution if numbers* $\boldsymbol{\alpha}$ *satisfy Condition* ω^1 *and the normal form* (3.4)*,* (3.5) *satisfies Condition* $\mathbf{A}^1$.

3.4. Set $\mathcal{A}^1$. Let now a is an arbitrary parameter. Then the system of equations (3.10) defines the set $\mathcal{A}^1$. For the reduced normal form (3.7), (3.8), (3.9), the set $\mathcal{A}^1$ is defined by the system

$$\begin{aligned} &\frac{\partial h}{\partial v_j} = \mathrm{i}\gamma_j u_j a, \quad \frac{\partial h}{\partial u_j} = \mathrm{i}\gamma_j v_j a, \quad j = 1, \dots, n, \\ &\frac{\partial h}{\partial s} = a. \end{aligned} \tag{3.11}$$

Subsystem of equations (3.11) defines the set of all periodic solutions of the subsystem (3.8). Equation (3.9) $\dot{\varphi} = \partial h / \partial s$ gives dependence of φ from t for each of these solutions.

To each set of increasing indices $I = \{i_1, \dots, i_m\}$, $1 \leqslant i_1, i_m \leqslant n$, $1 \leqslant m \leqslant n$, there correspond the *coordinate subspaces*

$$K_I^1 = \{\mathbf{x}, \mathbf{y}, r, \varphi : x_j = y_j = 0 \text{ for all } j \notin I\},$$
$$L_I^1 = \{\mathbf{u}, \mathbf{v}, s, \varphi : u_j = v_j = 0 \text{ for all } j \notin I\}.$$

Now the coordinate r or s corresponds to the frequency 1, so each subspace K_I^1 and L_I^1 corresponds to the set of frequencies $\{1, \alpha_{i_1}, \dots, \alpha_{i_m}\}$, $\{1, \gamma_{i_1}, \dots, \gamma_{i_m}\}$. As in Section 2, we define for it *multiplicity of resonances* $\varkappa_I^1$ and *degree of irrationality* $\sigma_I^1 = m + 1 - \varkappa_I^1$.

As in the transformation (3.6) all β_j are rational numbers, then these numbers coinside for K_I^1 and L_I^1.

As before, in subspaces K_I^1 and L_I^1 we define *components* $\mathcal{A}_I^1$ *of the set* $\mathcal{A}^1$, as parts of intersection of set $\mathcal{A}^1$ with the subspace, excluding points lying at the smaller coordinate subspaces K_J^1 and L_J^1 correspondingly. Sets $\mathcal{A}_I^1$ in K_I^1 and in L_I^1 are the same and connected by the transformation (3.6). If $I = \{1, \dots, n\}$, then K_I^1 and L_I^1 are the whole space $\mathbb{C}^{2(n+1)}$, which we denote $K_n^1 = L_n^1$. For components $\mathcal{A}_I^1$ Theorem 2.3 about families of invariant tori is true.

Let $\tilde{I}^1$ is a set of all indices j with rational α_j or zero γ_j. Define $\mathcal{A}_{\tilde{I}}^1 = \mathcal{A}^1 \cap L_{\tilde{I}^1}^1$. Similarly to Theorem 2.4, there exists a converge transformation, normalizing at the set $\mathcal{A}_{\tilde{I}}^1$, and the set is analytic. It contains all families of periodic solutions adjoining the initial periodic solution $\mathcal{M}$. In it equations (3.11) take the form

$$\frac{\partial h}{\partial v_j} = 0 = \frac{\partial h}{\partial u_j}, \quad j \in \widetilde{I}^1.$$

This system defines a set of stationary points, and equation $\dot{\varphi} = \dfrac{\partial h}{\partial s} = a$ give dependence of t for them. The set $\mathcal{A}_{\tilde{I}}^1$ contains the set $\mathcal{A}_0^1$, where all $u_j = v_j = 0$, $j = 1, \dots, n$, i. e. it is not empty. Hence, each periodic solution of a Hamiltonian system belongs to a family of periodic solutions (see [Bruno, 1994, Ch. II, § 2]).

3.5. Set $\mathcal{B}^1$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a diagonal matrix. At the set $\mathcal{A}^1$ in coordinates $\mathbf{u}, \mathbf{v}$, s we consider the square $(2n+1) \times (2n+1)$ matrix

$$B_1 = \begin{pmatrix} \dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{u}} - \mathrm{i}\Gamma a & \dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{v}} & \dfrac{\partial^2 h}{\partial \mathbf{v} \partial s} \\ \dfrac{\partial^2 h}{\partial s \partial \mathbf{u}} & \dfrac{\partial^2 h}{\partial s \partial \mathbf{v}} & \dfrac{\partial^2 h}{\partial s \partial s} \\ -\dfrac{\partial^2 h}{\partial \mathbf{u} \partial \mathbf{u}} & -\dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{u}} + \mathrm{i}\Gamma a & -\dfrac{\partial^2 h}{\partial \mathbf{u} \partial s} \end{pmatrix},$$

where a is the same parameter as in equations (3.10).

The *set* $\mathcal{B}^1$ is such subset of the set $\mathcal{A}^1$ on which the matrix B_1 is nilpotent, i. e. $B_1^{2n+1} = 0$.

Theorem 3.4. *Generically*

$$\mathcal{B}^1 = \mathcal{A}^1_n$$

and Re $\mathcal{B}^1$ *is a one-parameter (along* a*) family of invariant tori of dimension* σ^1_n*.*

Theorem 3.5 ([Bruno, 1989, Part II, § 3])**.** *Under Condition* ω^1 *there exists such analytic canonical transformation* (3.2) *that reduces the initial system to the normal form at the set* $\mathcal{B}^1$ *and the set is analytic.*

If no resonant relation $p_0 + \langle \boldsymbol{\alpha}, \mathbf{p} \rangle = 0$ exists for numbers $1, \boldsymbol{\alpha}$, where $p_0 \in \mathbb{Z}$, $\mathbf{p} \in \mathbb{Z}^n$, then generically the set Re $\mathcal{B}^1$ is a one-parameter family of irreducible invariant tori of dimension $n + 1$.

4. Neighborhood of an invariant torus

4.1. Local coordinates. Let a real analytic Hamiltonian system with $k + n$ degrees of freedom has a k-dimesional invariant torus $\mathcal{T}^k$. The torus is named *regular*, if in some canonical analytical coordinates $\boldsymbol{\xi}$, $\boldsymbol{\psi}$, $\boldsymbol{\eta}$, $\boldsymbol{\rho}$ ($\boldsymbol{\psi}, \boldsymbol{\rho} \in \mathbb{R}^k$, $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$):

1) the torus $\mathcal{T}^k$ is defined by equations

$$\boldsymbol{\xi} = \boldsymbol{\eta} = 0, \quad \boldsymbol{\rho} = 0;$$

2) coordinates $\boldsymbol{\psi}$ are cyclic (angular) $\mod 2\pi$ and in the torus $\mathcal{T}^k$ they satisfy to the system of equations

$$\dot{\psi}_i = \Omega_i = \text{const} \in \mathbb{R}, \quad i = 1, \ldots, k;$$

3) in a neighborhood of the torus $\mathcal{T}^k$ the system is Hamiltonian

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}, \quad j = 1, \ldots, n$$
$$\dot{\psi}_i = \frac{\partial \gamma}{\partial \rho_i}, \quad \dot{\rho}_i = -\frac{\partial \gamma}{\partial \psi_i}, \quad i = 1, \ldots, k$$

with analytic Hamiltonian function $\gamma(\boldsymbol{\xi}, \boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\rho})$ which is expanded into the convergent Poisson series

$$\gamma = \sum \gamma_{\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{m}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}} \boldsymbol{\rho}^{\mathbf{m}} \exp \mathrm{i} \langle \mathbf{l}, \boldsymbol{\psi} \rangle , \tag{4.1}$$

where $0 \leqslant \mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $0 \leqslant \mathbf{m} \in \mathbb{Z}^k$, $\mathbf{l} \in \mathbb{Z}^k$,

4) The variational system is reducible, i. e. the square in $\boldsymbol{\xi}$,$\boldsymbol{\eta}$ part $\gamma_2(\boldsymbol{\xi},\boldsymbol{\psi},\boldsymbol{\eta},\boldsymbol{\rho})$ of the Hamiltonian γ for $\boldsymbol{\rho} = 0$ does not depend on $\boldsymbol{\psi}$:

$$\gamma_2 = \frac{1}{2} \langle \boldsymbol{\zeta}, G\boldsymbol{\zeta} \rangle ,$$

where $\boldsymbol{\zeta} = (\boldsymbol{\xi},\boldsymbol{\eta})$ and G is a constant symmetric $2n \times 2n$ matrix. Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of the matrix $A = JG$, where $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ and E_n is the identity $n \times n$ matrix. Applying the correct numbering one has

$$\lambda_{j+n} = -\lambda_j, \quad j = 1, \ldots, n.$$

We denote $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\Omega} = (\Omega_1, \ldots, \Omega_k)$.

4.2. Normal form.

Theorem 4.1. *There exists a canonical invertible formal transformation of coordinates*

$$\boldsymbol{\xi}, \boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\rho} \longleftrightarrow \mathbf{x}, \boldsymbol{\varphi}, \mathbf{y}, \mathbf{r} \tag{4.2}$$

in the form of Poisson series of type (4.1)*, which near a regular invariant torus* $\mathcal{T}^k$ *reduces the Hamiltonian* (4.1) *to its normal form*

$$g = \sum_{j=1}^{n} \lambda_j x_j y_j + \sum_{i=1}^{k} \Omega_i r_i + \sum g_{\mathbf{plqm}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \mathbf{r}^{\mathbf{m}} \exp \mathrm{i} \langle \mathbf{l}, \boldsymbol{\varphi} \rangle \tag{4.3}$$

where the third sum contains only resonant terms, for which

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle + \mathrm{i} \langle \mathbf{l}, \boldsymbol{\Omega} \rangle = 0. \tag{4.4}$$

Its proof is similar to the proof of Theorem 3.1 in Ch. II of [Bruno, 1994]. Below in this Section we will consider the case where all eigenvalues $\boldsymbol{\lambda}$ are pure imaginary: $\lambda_j = \mathrm{i}\alpha_j$, $\alpha_j \in \mathbb{R}$, and all $\lambda_j \neq 0$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Theorem 4.2. *If equation* $\langle \mathbf{l}, \boldsymbol{\Omega} \rangle = 0$ *in* $\mathbf{l} \in \mathbb{Z}^k$ *has only zero solution* $\mathbf{l} = 0$*, then there exists a canonical transformation*

$$\begin{aligned} &x_j = u_j \exp\left(-\mathrm{i} \langle \mathbf{A}_j, \boldsymbol{\varphi} \rangle\right), \quad y_j = v_j \exp \mathrm{i} \langle \mathbf{A}_j, \boldsymbol{\varphi} \rangle , \quad j = 1, \ldots, n, \\ &\mathbf{r}^* = \mathbf{s}^* - B\mathbf{w}^*, \end{aligned} \tag{4.5}$$

where $\mathbf{w} = (u_1v_1, \dots, u_nv_n)$*,* B *is a* $k \times n$ *matrix with rational elements, and matrix* $A = \begin{pmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{pmatrix} = B^*$*, which transforms the Hamiltonian normal form* (4.3), (4.4) *to the autonomous series*

$$h(\mathbf{u}, \mathbf{v}, \mathbf{s}) = \mathrm{i}\sum_{j=1}^{n} \gamma_j u_j v_j + \sum_{i=1}^{k} \Omega_i s_i + \sum h_{\mathbf{pqm}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}} \mathbf{s}^{\mathbf{m}}, \tag{4.6}$$

which does not depend on angles $\boldsymbol{\varphi}$ *and contains only resonant terms with*

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\gamma} \rangle = 0, \tag{4.7}$$

where $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$*.*

Here asterisk * means transposition.

Proof. In assumption of Theorem equation (4.4) takes the form

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\alpha} \rangle + \langle \mathbf{l}, \boldsymbol{\Omega} \rangle = 0. \tag{4.8}$$

Let equation

$$\langle \mathbf{p}, \boldsymbol{\alpha} \rangle + \langle \mathbf{l}, \boldsymbol{\Omega} \rangle = 0 \tag{4.9}$$

have μ linearly independent integer solutions $(\mathbf{p}, \mathbf{l})$. Then integer solutions $(\mathbf{p}, \mathbf{l})$ of equation (4.9) form a lattice in $\mathbb{R}^{n+k}$ and as a consequence of linear independence $\boldsymbol{\Omega}$ we have $0 \leqslant \mu \leqslant n$. Let the set of vectors

$$(\mathbf{p}_1, \mathbf{l}_1), \dots, (\mathbf{p}_\mu, \mathbf{l}_\mu) \tag{4.10}$$

forms a basis of the lattice. Then the $\mu \times n$ matrix $C = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_\mu \end{pmatrix}$ has such μ columns that their determinant is different from zero. For simplicity of notations we assume that they are columns with numbers $1, \dots, \mu$. Now all vectors $\mathbf{p} = (p_1, \dots, p_n)$ we divide into two parts $\mathbf{p}' = (p_1, \dots, p_\mu)$ and $\mathbf{p}'' = (p_{\mu+1}, \dots, p_n)$. Similarly we divide the matrix $C = (C', C'')$, where matrix C' has dimension $\mu \times \mu$, and matrix C'' has dimension $\mu \times (n - \mu)$. For vectors of the basis (4.10), from equation (4.9) we obtain the system of equations

$$C' \boldsymbol{\alpha}'^* + C'' \boldsymbol{\alpha}''^* + L \boldsymbol{\Omega}^* = 0,$$

where L is a matrix with integral elements.

Let us solve the last system for $\boldsymbol{\alpha}'^*$:

$$\boldsymbol{\alpha}'^* = -C'^{-1}C''\boldsymbol{\alpha}''^* - C'^{-1}L\boldsymbol{\Omega}^* \stackrel{\mathrm{def}}{=} -D\boldsymbol{\alpha}''^* - \tilde{A}\boldsymbol{\Omega}^*, \tag{4.11}$$

where matrices $D = C'^{-1}C''$ and $\tilde{A} = C'^{-1}L$ have rational elements and dimensions $\mu \times (n-\mu)$ and $\mu \times k$ respectively. Here numbers $\boldsymbol{\alpha}''$ and $\boldsymbol{\Omega}$ are linearly independent over integral numbers, i. e. the equation

$$\langle \mathbf{p}'',\boldsymbol{\alpha}''\rangle + \langle \mathbf{l}, \boldsymbol{\Omega}\rangle = 0$$

in integer $(\mathbf{p}'',\mathbf{l})$ has only zero solution $\mathbf{p}'' = 0, \mathbf{l} = 0.$ Let define the $n \times k$ matrix

$$A = \begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}. \tag{4.12}$$

Taking into account (4.11), now we consider integer solutions $(\mathbf{p},\mathbf{q},\mathbf{l})$ to the equation (4.8)

$$\begin{aligned}
&\langle (\mathbf{p}-\mathbf{q})',\boldsymbol{\alpha}'\rangle + \langle (\mathbf{p}-\mathbf{q})'',\boldsymbol{\alpha}''\rangle + \langle \mathbf{l},\boldsymbol{\Omega}\rangle = \\
&\qquad = \left\langle (\mathbf{p}-\mathbf{q})', -D\boldsymbol{\alpha}''^* - \tilde{A}\boldsymbol{\Omega}^*\right\rangle + \langle (\mathbf{p}-\mathbf{q})'',\boldsymbol{\alpha}''\rangle + \langle \mathbf{l},\boldsymbol{\Omega}\rangle = \\
&\quad = \left\langle (\mathbf{p}-\mathbf{q})', -\tilde{A}\boldsymbol{\Omega}^*\right\rangle - \langle \mathbf{l},\boldsymbol{\Omega}\rangle - \langle (\mathbf{p}-\mathbf{q})', D\boldsymbol{\alpha}''^*\rangle + \langle (\mathbf{p}-\mathbf{q})'',\boldsymbol{\alpha}''\rangle = \\
&\qquad\qquad = \left\langle \mathbf{l} - \tilde{A}^*(\mathbf{p}-\mathbf{q})',\boldsymbol{\Omega}\right\rangle + \langle (\mathbf{p}-\mathbf{q})'' - D(\mathbf{p}-\mathbf{q})',\boldsymbol{\alpha}''^*\rangle = 0.
\end{aligned}$$

As $\boldsymbol{\alpha}''$ and $\boldsymbol{\Omega}$ are linearly independent over integral numbers, then the last equality gives equations

$$\begin{aligned} \mathbf{l} &= \tilde{A}^*\,(\mathbf{p}-\mathbf{q})', \\ (\mathbf{p}-\mathbf{q})'' &= D^*\,(\mathbf{p}-\mathbf{q})'. \end{aligned} \tag{4.13}$$

Now let us compute dependence from $\boldsymbol{\varphi}$ for a resonant monomial after substitution (4.5)

$$\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}\exp \mathrm{i}\,\langle \mathbf{l},\boldsymbol{\varphi}\rangle = \mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}}\exp \mathrm{i}\,(\langle \mathbf{q}-\mathbf{p}, A\boldsymbol{\varphi}^*\rangle + \langle \mathbf{l},\boldsymbol{\varphi}\rangle),$$

where matrix A was defined in (4.12). So here dependence of $\boldsymbol{\varphi}$ is

$$\left\langle (\mathbf{q}-\mathbf{p})',\tilde{A}\boldsymbol{\varphi}^*\right\rangle + \langle \mathbf{l},\boldsymbol{\varphi}\rangle = \left\langle \tilde{A}^*(\mathbf{q}-\mathbf{p})' + \mathbf{l},\boldsymbol{\varphi}\right\rangle = \langle 0,\boldsymbol{\varphi}\rangle = 0$$

according to (4.13).

Other statements of Theorem are easily verified. Proof is finished. □

Remark 4.1. In system

$$\dot{u}_j = \frac{\partial h}{\partial v_j}, \qquad \dot{v}_j = -\frac{\partial h}{\partial u_j}, \qquad j = 1,\dots,n, \tag{4.14}$$
$$\dot{\varphi}_i = \frac{\partial h}{\partial s_i}, \qquad \dot{s}_i = -\frac{\partial h}{\partial \varphi_i}, \qquad i = 1,\dots,k,$$

corresponding to the reduced normal form of Hamiltonian (4.6), all coordinates **s** are parameters. For the stationary points $(\mathbf{u}^0,\mathbf{v}^0,\mathbf{s}^0)$ of the subsystem (4.14), which satisfy the ''algebraic'' system of equations

$$\frac{\partial h}{\partial v_j} = \frac{\partial h}{\partial u_j} = 0, \quad j = 1,\dots,n,$$

equations

$$\dot{\varphi}_i = \left.\frac{\partial h}{\partial s_i}\right|_{\mathbf{u}^0,\mathbf{v}^0,\mathbf{s}^0}, \quad i = 1,\dots,k,$$

define frequencies on the corresponding invariant tori.

Remark 4.2. If frequencies $\Omega_1,\dots,\Omega_k$ are linearly independent over integral numbers, i. e. equation $\langle \mathbf{l},\boldsymbol{\Omega}\rangle = 0$ in $\mathbf{l} \in \mathbb{Z}^k$ has only zero solution $\mathbf{l} = 0$, then the initial torus $\mathcal{T}^k$ is irreducible.

4.3. Convergency of the normalizing transformation.

Condition ω^k

Let $\omega_m = \min|\langle \mathbf{p},\boldsymbol{\alpha}\rangle + \langle \mathbf{l},\boldsymbol{\Omega}\rangle| = 0$ for $\langle \mathbf{p},\boldsymbol{\alpha}\rangle + \langle \mathbf{l},\boldsymbol{\Omega}\rangle \neq 0$, $||\mathbf{p}|| + ||\mathbf{l}|| < 2^m$, $\mathbf{p} \in \mathbb{Z}^n$, $\mathbf{l} \in \mathbb{Z}^k$. Then

$$-\sum_{m=1}^{\infty} \frac{\log \omega_m}{2^m} < 0.$$

Condition $\mathbf{A}^k$

There exists such a Poisson series $a(\mathbf{x},\boldsymbol{\varphi},\mathbf{y},\mathbf{r})$ that in the normal form (4.3)

$$\frac{\partial g}{\partial y_j} = \mathrm{i}\alpha_j x_j a, \quad \frac{\partial g}{\partial x_j} = \mathrm{i}\alpha_j y_j a, \quad j = 1,\dots,n,$$
$$\frac{\partial g}{\partial \varphi_i} = 0, \qquad \frac{\partial g}{\partial r_i} = \Omega_i a, \qquad i = 1,\dots,k. \tag{4.15}$$

It equivalents to the condition that Hamiltonian g is a power series of one variable

$$\sum_{j=1}^{n} \lambda_j x_j y_j + \sum_{i=1}^{k} \Omega_i r_i.$$

Theorem 4.3. *The normalizing transformation* (4.2) *converges in a neighborhood of our invariant torus* $\mathcal{T}^k$ *if numbers* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\Omega}$ *satisfy Condition* ω^k *and the normal form* (4.3), (4.4) *satisfies Condition* $\mathbf{A}^k$.

4.4. Set $\mathcal{A}^k$. Let now a is an arbitrary parameter. Then the system of equations (4.15) defines the formal set $\mathcal{A}^k$. For the reduced normal form (4.6), (4.7) the set $\mathcal{A}^k$ is defined by the system of equations

$$\begin{aligned} &\frac{\partial h}{\partial v_j} = \mathrm{i}\gamma_j u_j a, \quad \frac{\partial h}{\partial u_j} = \mathrm{i}\gamma_j v_j a, \quad j = 1,\dots,n,\\ &\frac{\partial h}{\partial s_i} = \Omega_i a, \quad i = 1,\dots,k. \end{aligned} \tag{4.16}$$

To each set of increasing indices $I = \{i_1,\dots,,i_m\}$, $1 \leqslant i_1, i_m \leqslant n$, $1 \leqslant m \leqslant n$, there correspond the coordinate subspaces

$$\begin{aligned} K_I^k &= \{\mathbf{x},\mathbf{y},\mathbf{r},\boldsymbol{\varphi} : x_j = y_j = 0 \text{ for all } j \notin I\},\\ L_I^k &= \{\mathbf{u},\mathbf{v},\mathbf{s},\boldsymbol{\varphi} : u_j = v_j = 0 \text{ for all } j \notin I\}. \end{aligned}$$

If $I = \{1,\dots,n\}$, then $K_I^k = L_I^k = \mathbb{C}^{2(n+k)} \stackrel{\text{def}}{=} L_n^k$.

Now coordinates $\mathbf{r},\boldsymbol{\varphi},\mathbf{s}$ correspond to frequencies $\boldsymbol{\Omega}$. So the coordinate subspaces K_I^k and L_I^k correspond to the set of frequencies $\{\boldsymbol{\Omega},\alpha_{i_1},\dots,\alpha_{i_m}\}$ and $\{\boldsymbol{\Omega},\gamma_{i_1},\dots,\gamma_{i_m}\}$. For each K_I^k and L_I^k we denote $m_I = m$ and :

1) *half-dimension* $m_I^k = k + m_I$;
2) *multiplicity of resonances* $\varkappa_I^k$ as the number of linearly independent integer relations
$$\sum_{j\in I} p_j\alpha_j + \langle \mathbf{l},\boldsymbol{\Omega}\rangle = 0 \text{ and } \sum_{j\in I} p_j\gamma_j + \langle \mathbf{l},\boldsymbol{\Omega}\rangle = 0$$
with integer p_j and $\mathbf{l} \in \mathbb{Z}^k$;
3) *degree of irrationality* $\sigma_I^k = k + m_I^k - \varkappa_I^k$;
4) *subset of frequencies* $\boldsymbol{\Omega},\boldsymbol{\alpha}_I$ and $\boldsymbol{\Omega},\boldsymbol{\gamma}_I$.

Numbers $m_I^k,\varkappa_I^k,\sigma_I^k$ coincide for K_I^k and L_I^k, because in transformation (4.5) all elements of matrices A and B are rational numbers.

As before in subspaces K_I^k and L_I^k we define *components* $\mathcal{A}_I^k$ of the set $\mathcal{A}^k$ as parts of intersections of the set $\mathcal{A}^k$ with the subspaces excluding points belonging to smaller subspaces K_J^k and L_J^k. Sets $\mathcal{A}_I^k$ in K_I^k and in L_I^k are the same and connected by transformation (4.5). So, there is component $\mathcal{A}_n^k$.

Theorem 4.4. *If the normalized Hamiltonian* (4.3), (4.4) *is analytic, then each component* $\operatorname{Re}\mathcal{A}_I^k$ *is a family of irreducible invariant tori of dimension* σ_I^k *with frequencies* $\boldsymbol{\Omega}a$, $\boldsymbol{\gamma}_I a$. *In the generic case these families have* $\varkappa_I^k$ *parameters.*

4.5. Set $\mathcal{B}^k$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a diagonal matrix. In the set $\mathcal{A}^k$ in coordinates **u**,**v**,**s** we consider the $(2n + k) \times (2n + k)$ matrix

$$B_k = \begin{pmatrix} \dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{u}} - \mathrm{i}\Gamma a & \dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{v}} & \dfrac{\partial^2 h}{\partial \mathbf{v} \partial \mathbf{s}} \\ \dfrac{\partial^2 h}{\partial \mathbf{s} \partial \mathbf{u}} & \dfrac{\partial^2 h}{\partial \mathbf{s} \partial \mathbf{v}} & \dfrac{\partial^2 h}{\partial \mathbf{s} \partial \mathbf{s}} \\ -\dfrac{\partial^2 h}{\partial \mathbf{u} \partial \mathbf{u}} & -\dfrac{\partial^2 h}{\partial \mathbf{u} \partial \mathbf{v}} + \mathrm{i}\Gamma a & -\dfrac{\partial^2 h}{\partial \mathbf{u} \partial \mathbf{s}} \end{pmatrix},$$

where a is the same parameter as in equations (4.16). The set $\mathcal{B}^k$ is such subset of the set $\mathcal{A}^k$, where the matrix B_k is nilpotent, i. e. $B_k^{2n+k} = 0$.

Theorem 4.5. *Under Condition ω^k there exists such analytic canonical transformation* (4.2)*, which reduces the initial Hamiltonian to the normal form in the set $\mathcal{B}^k$ and that set is analytic.*

Theorem 4.6. *Generically $\mathcal{B}^k = \mathcal{A}^k_n$.*

Hence, the set $\operatorname{Re} \mathcal{B}^k$ consists of tori of dimension k, if all γ_j are zero, i.e. all α_j are linear combinations of frequencies $\mathbf{\Omega}$ with rational coefficients. It means that all α_j are commensurable with frequencies. Such set $\operatorname{Re} \mathcal{B}^k$ forms a family with $k + 1$ parameters.

5. Remarks

Neighborhood of the n-dimensional invariant torus in system with n degrees of freedom was studied in [Bruno, 1994, Ch. II, § 3]. There was shown that such irreducible torus lies at one-parameter family of irreducible invariant tori of dimension n.

If the variational system near a stationary point or near an invariant torus $\mathcal{T}^k$ or near a periodic solution has eigenvalues λ_j with $\operatorname{Re} \lambda_j \neq 0$, then all told above relates to their central manifolds and all Theorems are true. So, generically in real analytic Hamiltonian system with n degrees of freedom and without parameters:

(a) periodic solutions form one-parameter families,

(b) n-dimensional regular tori $\mathcal{T}^n$ form one-parameter families,

(c) k-dimensional irreducible regular tori $\mathcal{T}^k$ with $k < n$ form $(l+1)$-parameter families, if exactly $2l$ their eigenvalues have zero real parts, and imaginary parts of all of them are commensurable with frequencies.

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