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Averaging of the neutron transport equation over energy by the Lebesgue Moment Method

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Averaging of the neutron transport equation over energy by the Lebesgue Moment Method

The method of Lebesgue moments for simulating the reversal of resonances, resonance self-shielding, and block effect in the neutron spectra of heterogeneous extended objects, such as nuclear reactors, radiation shielding, and installations for studying the properties of matter, is developed. The method uses a more accurate averaging procedure over neutron energy than the group averaging. The main components of the method are the refinement of the resonance structure of neutron cross sections by dividing the energy scale into a series of sets called carriers of resonances, Lebesgue averaging of cross sections and neutron flux within carriers, and the expansion of the neutron flux in a series in basis functions that depend on the magnitude of the neutron cross sections. The expansion coefficients (the so-called Lebesgue moments) can be calculated by any available method for solving the neutron transport equation (the Sn method, the spherical harmonics method, and the Monte Carlo method).

Keywords: nuclear reactors, neutron diagnostics, neutron spectrum, resonance self-shielding, neutron transport equation

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Contents

Notation

A	Nuclide atomic number; p. 9
C	Index of the component; p. 15
c	Neutron capture reaction leading to α,β,γ-decay of a compound nucleus; p. 9
E	Neutron energy, [eV]; p. 7
$E_k^{\pm}(m)$, $k=1,2,...$	Boundary points of the Lebesgue set $\omega(g,m)$; p. 20
el	Neutron elastic scattering; pp. 9, 11
$F(E)$	Dimensionless weighting spectrum; p. 22
f	Nuclear fission reaction; p. 9
g	Index of the resonance carrier ω_g; p. 16
$\eta=\mathbf{\Omega}\mathbf{\Omega}'$	Cosine of the angle between the flight directions of neutrons; p. 7
$\theta(x)$	Heaviside function; p. 14
$\theta_r(E)$, $\theta_g(m)$	Symmetric resonance profile, [1/eV]; pp. 11, 31
i	Index of Cartesian coordinates; p. 7
in	Inelastic neutron scattering reactions; p. 9
K_g	Relative width of resonance carrier; p. 16
L_g	Dimention parameter of the order of object size; p. 32
λ^{τ}	Average constant of spontaneous decay of precursors of the group τ, [1/s]; pp. 7, 8
$m(S)$	Lebesgue measure of the set, [eV]; p. 20
$N_A(\mathbf{r},t)$	Density of nuclide A, [cm^{-3}]; p. 9
$N_C(\mathbf{r},t)$	Density of component C, [cm^{-3}]; p. 15
$n,m\times n$	Neutron scattering ($m=1$) and neutron multiplication ($m=2,3$) reactions; p. 9
$\nu^{\tau}(E,\mathbf{r},t)$, $\nu_A^{\tau}(E)$, $\nu_C^{\tau}(E)$; $\nu_g^{\tau}(m,\mathbf{r},t)$, $\nu_{C,g}^{\tau}(m)$	Multiplicity or total number of precursors of delayed neutrons in group τ at the output of fission reactions; pp. 8, 9, 16, 25, 28

$\nu^x(E,\mathbf{r},t)$, $\nu^x_A(E), \nu^x_C(E)$; $\nu^x_g(m,\mathbf{r},t)$, $\nu^x_{C,g}(m)$	Multiplicity or total number of prompt neutrons at the output of reactions *x*; pp. 7, 9, 16, 25, 28
$P_n(\xi)$, $n = 0,1,2,...$	Set of orthogonal polynomials on the interval $-1 < \xi < 1$; pp. 13, 31, 35
$P^\tau(\mathbf{r},t)$	Density of precursors of delayed neutrons, [cm^{-3}]; p. 8
$Q^x_g(m,\mathbf{\Omega},\mathbf{r},t)$	Arrival rate of prompt neutrons of reactions *x* into the Lebesgue set $E \in \omega(g,m)$, [$\mathrm{cm}^{-3}\mathrm{s}^{-1}$]; pp. 26, 34
$q^x_g(\varphi)$	Source of prompt neutrons of reactions *x* or the arrival rate of prompt neutrons of reactions *x* into the neutron beam $\varphi(E,\mathbf{\Omega},\mathbf{r},t)$, [$\mathrm{eV}^{-1}\mathrm{cm}^{-3}\mathrm{s}^{-1}$]; p. 7
$q^x_g(\psi)$	Lebesque source of prompt neutrons of reactions *x* or the arrival rate of prompt neutrons of reactions *x* into the neutron beam $\psi(m,\mathbf{\Omega},\mathbf{r},t)$, [$\mathrm{eV}^{-1}\mathrm{cm}^{-3}\mathrm{s}^{-1}$]; p. 26, 27, 34
$q^{ext}(\varphi)$, $q^{ext}_g(\psi)$	Source of external neutrons, [$\mathrm{eV}^{-1}\mathrm{cm}^{-3}\mathrm{s}^{-1}$]; pp. 7, 28
$q^r(E,\mathbf{\Omega},\mathbf{r},t)$, $q^r_g(m,\mathbf{\Omega},\mathbf{r},t)$	Independent (from φ) neutron source/sink, [$\mathrm{eV}^{-1}\mathrm{cm}^{-3}\mathrm{s}^{-1}$]; pp. 7, 28
r	Index of resonance; p. 11
$\rho(\xi)$	Weight function of the system of polynomials; pp. 31, 35
$S(E) \equiv S^t(E)$, $S^t_C(E)$; $S(m) \equiv S^t(g,m)$	Mean total neutron cross section, [1/cm]; pp. 17, 22
S_{0g} and S_g, $S_{0g} \le S(E), S(m) \le S_g$	Minimum and maximum values of the mean cross section on the resonance carrier $E \in \omega_g$; p. 20
s	Reactions resulting in the emergence of prompt neutrons; pp.7,9
$\Sigma^t(E,\mathbf{r},t)$; $\sigma^t_A(E)$ $\sigma^t_C(E)$; $\Sigma^t_g(m,\mathbf{r},t)$, $\sigma^t_{C,g}(m)$	Total neutron cross section, [cm^{-1}], [cm^2]; pp. 7, 9, 11, 16, 25, 28
$\Sigma^x(E,\mathbf{r},t)$; $\sigma^x_A(E), \sigma^x_C(E)$; $\Sigma^x_g(m,\mathbf{r},t)$, $\sigma^x_{C,g}(m)$	Cross section of a reaction or reaction group *x*, [cm^{-1}], [cm^2]; pp. 7, 9, 11, 16, 25, 28

$T(\mathbf{r},t)$	Temperature of the substance, [эВ]; pp. 14, 29
t	Symbol of the total neutron cross section; pp. 7, 9
t	Time; p. 7
τ	Group of precursors of delayed neutrons; pp. 7, 9
$\mathbf{u}$	Substance velocity, [cm/s]; pp. 8, 10
v	Neutron speed, [cm/s]; p. 7
$\varphi(E,\mathbf{\Omega},\mathbf{r},t)$	Neutron flux, [$\mathrm{eV}^{-1}\mathrm{cm}^{-2}\mathrm{s}^{-1}$]; p. 7
$\varphi^{ent}(E,\mathbf{\Omega},\mathbf{r}_\Gamma,t)$	External neutron flux through the outer boundary of the object, [$\mathrm{eV}^{-1}\mathrm{cm}^{-2}\mathrm{s}^{-1}$]; p. 8
$W_0^f(E'\to E,\mathbf{r},t)$, $w_{A,0}^f(E'\to E)$, $W_{h\to g,0}^f(m'\to m,\mathbf{r},t)$, $w_{C,h\to g,0}^f(m'\to m)$	Fission neutron spectrum, [1/eV]; pp. 13, 16, 27, 28
$W^x(E'\to E,\eta,\mathbf{r},t)$, $w_A^x, w_C^x(E'\to E,\eta)$, $W_{h\to g}^x(m'\to m,\eta,\mathbf{r},t)$, $w_{C,h\to g}^x(m'\to m,\eta)$	Indicatrix of prompt neutron emergence in reactions *x*, [1/eV]; pp. 7, 9, 16, 27, 28
x, y, z	Reaction or a group of reactions; p. 9
$\chi_r(E)$, $\chi_g(m)$	Antisymmetric resonance profile, [1/eV]; pp. 11, 31
$\chi^\tau(E,\mathbf{r},t)$, $\chi_A^\tau(E)$, $\chi_g^\tau(m,\mathbf{r},t)$, $\chi_{C,g}^\tau(m)$	Delayed neutron spectrum, [1/eV]; pp. 7, 9, 28
$\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t)$	Lebesgue moment of *n*th order of the neutron flux, [$\mathrm{eV}^{-1}\mathrm{cm}^{-2}\mathrm{s}^{-1}$]; p. 31
$\psi_g(m,\mathbf{\Omega},\mathbf{r},t)$	Lebesgue neutron flux, [$\mathrm{eV}^{-1}\mathrm{cm}^{-2}\mathrm{s}^{-1}$]; p. 22
$\mathbf{\Omega}$, $\lvert\mathbf{\Omega}\rvert=1$	Particle flight direction; p. 7
ω_g	Resonance carrier; p. 16
$\omega(g,m)\subseteq\omega_g$	Lebesgue set; p. 20

Introduction

The purpose of this paper is to describe the method of Lebesgue moments designed for simulating neutron fields in nonuniform heterogeneous objects, including such effects as resonance self-shielding of the neutron flux in the depth of zones containing homogeneous material and the block-effect (reducing the resonance self-shielding near the boundaries of zones containing different materials). The method has three important components:

(1) refinement of the resonance structure of neutron cross sections by dividing the energy scale $0 < E < \infty$ into resonance carriers that are analogs of groups that may consist a series of short intervals (rather than of a single interval);

(2) Lebesgue averaging of the neutron transport equation with respect to energy within the resonance carriers;

(3) expansion of the neutron flux in a series of basis functions that depend on the value of the neutron cross section.

We show that the division of the neutron energies into resonance carriers makes it possible to approximately separate the variables E and $(\mathbf{r},t)$ on which the neutron cross sections and the neutron flux depend. The Lebesgue averaging monotonizes the cross sections and the flux by replacing the neutron energy by a new variable that is the measure of Lebesgue sets. These sets are built within each resonance carrier. The gain obtained due to joining energy points into one computation point is approximately equal to the number of resonances on the carrier. This gain can be as high as two orders of magnitude. The expansion of the neutron flux in a series of basis functions of the spectrum defines the optimal finite grid of points on the measure of the Lebesgue sets; this grid describes the evolution of the neutron field in an object with the minimal (in a certain sense) error. In addition, the expansion makes it possible to approximately restore the neutron energy spectra lost as a result of the Lebesgue averaging. The name the *Lebesgue moments* denotes the dependence of the basis functions on the magnitude of the total cross section for the interaction of neutrons with matter.

The procedure of the Lebesgue averaging (the second component above) has discrete analogs—the exponential sum method for approximating the transmission function [1], the method of subgroups [2]–[4], the probability table method [5], and the multiband method [6], [7]. In distinction from these methods, the method proposed in the current paper is based first on the procedure of dividing the energy scale into resonance carriers and second on the transition from the Riemann integration to the Lebesgue integration of functions [8]–[10]. The resulting averaged particle transport equation depends on the *continuous* Lebesgue variable. Which is even more important is that this equation is «cross-cutting» for the object in which the neutron cross sections have spatial discontinuities. This equation does not require setting additional boundary conditions (in addition to conditions on the external boundaries) that describe neutron transitions between subgroups of different materials on the ma-

terial interfaces in order to maintain the continuity of the neutron flux through the boundaries. The presence of such internal conditions significantly reduces the efficiency of, e.g., the subgroup method. The Lebesgue averaging was described in the previous author's papers [8]–[10] simultaneously for the transport of photons and neutrons. In this paper, a version of the method that is adapted to the problems of neutron transport and adjusted for the use in the method of moments is described. A number of issues, e.g., the averaging of neutron differential cross sections, are considered in more detail. The effectiveness of the method in photon transfer problems is confirmed by calculations (see [11], [12] and references therein). An outline of the Lebesgue moment method for the photon and neutron transport can be found in [13].

In photon transfer problems, the discrete and semi-discrete analogs of the Lebesgue averaging method are the exponential sum method [14]–[16], the k-distribution method [17]–[19], and the correlated k-distribution method [20]–[24]. A more complete bibliography can be found in [11]. Like the method of subgroups, these methods require setting internal conditions on the interfaces of different materials in the object. For photons, there are other methods of monotonizing the cross sections [25]–[28]. The transport equation in the case of a periodic dependence of cross sections on the photon frequency was obtained in [25]. In [26] and independently in [27], a change of variables relating one of the cosines of the flight direction with the photon cross section was proposed. As a result, the coefficients of the averaged equation become monotonic functions of the new variable, and the dimension of the problem decreases by one. (Note that the replacement of Riemann integration by the Lebesgue integration of functions does not change the problem dimension). Unfortunately, this method is inapplicable in problems in which there is scattering of particles, and it has limitations related to the spatial symmetry of problems.

Basic equations. In this paper, we discuss how to solve the linear Boltzmann equation for the neutron flux

$$\left[\frac{1}{v}\frac{\partial}{\partial t}+\Omega_i\frac{\partial}{\partial r_i}+\Sigma^t(E,\mathbf{r},t)\right]\varphi(E,\mathbf{\Omega},\mathbf{r},t)=q^s(\varphi)+q^{ext}(\varphi)\,, \tag{1}$$

$$q^s(\varphi)=\int\limits_{4\pi}\frac{d\mathbf{\Omega}'}{2\pi}\int\limits_0^\infty\Sigma^s(E',\mathbf{r},t)\,\nu^s(E',\mathbf{r},t)W^s(E'\to E,\eta,\mathbf{r},t)\,\varphi(E',\mathbf{\Omega}',\mathbf{r},t)\,dE'\,,$$

$$q^{ext}(\varphi)=\sum_\tau\frac{\chi^\tau(E,\mathbf{r},t)}{4\pi}\lambda^\tau P^\tau(\mathbf{r},t)+q^r(E,\mathbf{\Omega},\mathbf{r},t)\,.$$

Here $\varphi(E,\mathbf{\Omega},\mathbf{r},t)$ is the neutron flux per unit volume-energy-angle at time t; $\mathbf{\Omega}$ ($|\mathbf{\Omega}|=1$) is the flight direction of particles, v is the velocity magnitude (speed);

$\Sigma^t(E,\mathbf{r},t)$ [1/cm] is the total macroscopic cross section for the interaction of a neutron with energy E [eV]; and $q^s(\varphi)$ is the prompt neutron source or the rate of prompt appearance of neutrons in the neutron beam $\varphi(E,\mathbf{\Omega},\mathbf{r},t)$. The prompt neutrons are born in reactions s of the interaction of the primary neutron $(E',\mathbf{\Omega}',\mathbf{r},t)$ with the nuclides of matter, $\Sigma^s(E',\mathbf{r},t)$ is the total macroscopic cross section for reactions s resulting in the emergence of prompt neutrons; $\nu^s(E',\mathbf{r},t)$ is the multiplicity or total number of prompt neutrons at the output of reactions regardless of their energies and flight directions; $W^s(E'\to E,\eta,\mathbf{r},t)$ [1/eV] is the indicatrix of reactions s (normalized differential cross section), $\eta=\mathbf{\Omega}\mathbf{\Omega}'$ $(-1\le\eta\le1)$ is the cosine the angle between the flight directions of the prompt neutrons and the primary neutron; and $q^{ext}(\varphi)$ is the source of external neutrons including the independent (from φ) neutron source/sink $q^r(E,\mathbf{\Omega},\mathbf{r},t)$ and the source of delayed neutrons $\sum\chi^\tau\lambda^\tau P^\tau/4\pi$. Delayed neutrons are emitted by fission fragments (the so-called precursors of delayed neutrons), τ is the index of the group of precursors, $\chi^\tau(E,\mathbf{r},t)$ [1/eV] is the spectrum of delayed neutrons, and λ^τ [1/s] is the constant of spontaneous decay of the precursors of group τ. The variations of precursor densities $P^\tau(\mathbf{r},t)$ [cm^{-3}] are described by the system of transport equations

$$\left[\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial r_i}+\lambda^\tau\right]P^\tau(\mathbf{r},t)=\int\limits_{4\pi}d\mathbf{\Omega}\int\limits_0^\infty\nu^\tau(E,\mathbf{r},t)\Sigma^f(E,\mathbf{r},t)\varphi(E,\mathbf{\Omega},\mathbf{r},t)\,dE\,, \qquad (2)$$

where $\nu^\tau(E,\mathbf{r},t)\ll1$ is the multiplicity (number) of precursors in group τ at the output of fission reactions, and $\mathbf{u}$ [cm/s] is the velocity of the substance.

Limitations. Equations (1)–(2) are solved subject to the initial and boundary conditions imposed on neutron beams that move from the outer boundary Γ into the depth of the object $\mathbf{\Omega n}<0$:

$$\left[\varphi(E,\mathbf{\Omega},\mathbf{r}_\Gamma,t)-\varphi^{ent}(E,\mathbf{\Omega},\mathbf{r}_\Gamma,t)\right]_{\mathbf{n\Omega}<0}=\int\limits_{\mathbf{n\Omega}'>0}G(\mathbf{\Omega}'\to\mathbf{\Omega})\varphi(E,\mathbf{\Omega}',\mathbf{r}_\Gamma,t)\,d\mathbf{\Omega}', \qquad (3)$$

where $\mathbf{n}$ is the external normal vector to the boundary, $\varphi^{ent}(E,\mathbf{\Omega},\mathbf{r}_\Gamma,t)$ is the flux of external neutrons passing through the boundary from the outside, and $G(E,\mathbf{\Omega}'\to\mathbf{\Omega},\mathbf{r}_\Gamma,t)$ is the cross section of the neutron reflection from the boundary.

Summation of cross sections. The macroscopic cross section of a reaction or group of reactions x will be denoted by the uppercase letter Σ^x, the microscopic cross

section per one nuclide with the atomic number A will be denoted by the lowercase letter σ_A^x [cm^2], and $N_A(\mathbf{r},t)$ [cm^{-3}] is the nuclide density. The total cross section $\Sigma^t(E,\mathbf{r},t)$ and the cross sections for the yield of reaction products are given by sums

$$\Sigma^t(E,\mathbf{r},t)=\Sigma^c(E,\mathbf{r},t)+\Sigma^s(E,\mathbf{r},t), \tag{4}$$

$$\Sigma^c(E,\mathbf{r},t)=\sum_A N_A(\mathbf{r},t)\sigma_A^c(E)=\sum_y \Sigma^y(E,\mathbf{r},t)=\sum_{y,A} N_A(\mathbf{r},t)\sigma_A^y(E),$$

$$\begin{Bmatrix}1\\ \nu^s(E,\mathbf{r},t)\end{Bmatrix}\times\Sigma^s(E,\mathbf{r},t)=\sum_x\begin{Bmatrix}1\\ \nu^x(E,\mathbf{r},t)\end{Bmatrix}\times\Sigma^x(E,\mathbf{r},t)=\sum_{x,A} N_A(\mathbf{r},t)\begin{Bmatrix}1\\ \nu_A^x(E)\end{Bmatrix}\sigma_A^x(E)=$$

$$=\begin{Bmatrix}1\\ \nu^f\end{Bmatrix}\Sigma^f+\sum_{m=1}^{3}\begin{Bmatrix}1\\ m\end{Bmatrix}\Sigma^{n,m\times n}=\sum_A N_A(\mathbf{r},t)\left[\begin{Bmatrix}1\\ \nu_A^f(E)\end{Bmatrix}\sigma_A^f(E)+\sum_{m=1}^{3}\begin{Bmatrix}1\\ m\end{Bmatrix}\sigma_A^{n,m\times n}(E)\right],$$

$$\nu^\tau(E,\mathbf{r},t)\Sigma^f(E,\mathbf{r},t)=\sum N_A(\mathbf{r},t)\nu_A^\tau(E)\sigma_A^f(E),$$

where Σ^c and σ_A^c are the cross sections of the neutron capture reactions leading to the death of the neutron due to α,β,γ-decay of the compound nucleus; Σ^s and σ_A^s are the cross sections of the reactions resulting in the emergence of prompt neutrons; Σ^f and σ_A^f are the cross sections of the nuclear fission reactions; $\Sigma^{n,1\times n}=\Sigma^{el}+\Sigma^{in}$ and $\sigma_A^{n,1\times n}=\sigma_A^{el}+\sigma_A^{in}$ $(m=1)$ are the cross sections of the elastic (*el*) and inelastic (*in*) scattering; $\Sigma^{n,m\times n}$ and $\sigma_A^{n,m\times n}$ $(m=2,3)$ are the cross sections of the neutron multiplication reactions; $\nu^x(E,\mathbf{r},t)$ and $\nu_A^x(E)$ are the multiplicities of prompt neutrons at the output of the reactions $x=s,f,m\times n,...$; $\nu^\tau(E,\mathbf{r},t)$ and $\nu_A^\tau(E)$ are the multiplicities of precursors of delayed neutrons in the group τ at the output of the fission reactions $x=f$. The differential cross sections are summed similarly:

$$\nu^s(E',\mathbf{r},t)\Sigma^s(E',\mathbf{r},t)W^s(E'\to E,\eta,\mathbf{r},t)= \tag{5}$$

$$=\sum_x \nu^x(E',\mathbf{r},t)\Sigma^x(E',\mathbf{r},t)W^x(E'\to E,\eta,\mathbf{r},t)=\sum_{x,A} N_A(\mathbf{r},t)\nu_A^x\sigma_A^x(E')\,w_A^x(E'\to E,\eta)=...,$$

$$\nu^{\tau}(E,\mathbf{r},t)\Sigma^{f}(E,\mathbf{r},t)\chi^{\tau}(E,\mathbf{r},t) \simeq \sum_{A} N_A(\mathbf{r},t)\,\nu_A^{\tau}(E)\,\sigma_A^{f}(E)\,\chi_A^{\tau}(E).$$

Here $W^{x}(E' \to E,\eta,\mathbf{r},t)$, $w_A^{x}(E' \to E,\eta)$ $(\eta = \mathbf{\Omega}\mathbf{\Omega}')$ are the indicatrices of the reaction group *x* and $\chi^{\tau}(E,\mathbf{r},t)$, $\chi_A^{\tau}(E)$ are the spectra of delayed neutrons. All indicatrices and spectra are normalized to unity

$$\int_0^{\infty} dE \int_{-1}^{1} W^{x}(E' \to E,\eta,\mathbf{r},t)\,d\eta = \int_0^{\infty} dE \int_{-1}^{1} w_A^{x}(E' \to E,\eta)\,d\eta = 1, \qquad (6)$$

$$\int \chi^{\tau}(E,\mathbf{r},t)\,dE = \int \chi_A^{\tau}(E)\,dE = 1.$$

Nuclide densities. If we want to consider the variations of the nuclide densities $N_A(\mathbf{r},t)$ in space and time, then Eqs. (1)–(3) are complemented with the coupled system of nuclear reaction equations

$$\left[\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial r_i} + \lambda_A + R_A(\mathbf{r},t)\right] N_A(\mathbf{r},t) = \sum_{z,B}\left[\xi_{B\to A}^{z}\lambda_{B\to A}^{z} + S_{B\to A}^{z}(\mathbf{r},t)\right] N_B(\mathbf{r},t), \qquad (7)$$

$$\lambda_A = \sum_{z,B} \lambda_{A\to B}^{z}, \qquad R_A(\mathbf{r},t) = \sum_{z,B} R_{A\to B}^{z}(\mathbf{r},t),$$

$$\begin{Bmatrix} R_{B\to A}^{z}(\mathbf{r},t) \\ S_{B\to A}^{z}(\mathbf{r},t) \end{Bmatrix} = \int_{4\pi} d\mathbf{\Omega} \int_0^{\infty} \begin{Bmatrix} 1 \\ \nu_{B\to A}^{z}(E) \end{Bmatrix} \times \sigma_{B\to A}^{z}(E)\,\varphi(E,\mathbf{\Omega},\mathbf{r},t)\,dE.$$

Here $\lambda_{B\to A}^{z}$ is the spontaneous decay constant of nuclide *B* with the emergence of nuclide *A* in the reaction *z*, $\sigma_{B\to A}^{z}(E)$ is the cross section of the corresponding induced reaction, $\xi_{B\to A}^{z}$ and $\nu_{B\to A}^{z}(E)$ are the multiplicities of nuclides *A* at the output of the reactions, and $\mathbf{u}$ is the velocity of the substance.

The dependence of the cross sections $\sigma_A^{x}(E)$, $x = t,s,f,m\times n,el,...$ **on the energy** includes a lot of resonances that arise when the neutron energy coincides with the difference in the energies of the quantum states of the compound nucleus. In the vicinity of the resonances, the cross sections sharply increase in magnitude, sometimes by several orders of magnitude. For heavy nuclei, the region of resonances extends from energies of $\sim 0.5\text{eV}$ to energies of $\sim 1\text{KeV} = 10^3\text{eV}$; and for light and medium nuclei, it extends to $\sim 1\text{MeV} = 10^6\text{eV}$. The number of resonances of each nu-

cleus is typically large, and the resonances are narrow. In heavy nuclei, the widths of narrow resonances are of the order $\sim 1\text{eV}$, and in medium and light nuclei, they are $\sim 1\text{KeV}$. For medium nuclei, in addition to the narrow resonances, medium resonances with widths of $\sim 10 \div 100$ KeV and giant resonances with widths up to 1MeV are sometimes observed. With increasing energy, the distance between the resonances decreases and becomes comparable with their width near the upper boundary of the region. This is called the unresolved resonance region. Fig. 1 shows the difference in the position of the resonance region for some nuclei.

The measurement of resonance parameters is the subject of many studies. The results are evaluated (e.g., see [29]–[31]) and published in nuclear data libraries [32]. The most complete libraries are ENDF (Evaluated Nuclear Data Files) [33], ROSFOND (Russian Library of Estimated Neutron Data Files) [34], JEF (Joined European Files) [35], and JENDL (Japanese Evaluated Nuclear Data Library) [36]. A description of the data, formulas, and computer codes performing the reconstruction of cross section can be found in the descriptions of libraries and in [31], [37]–[39].

The representation (formulas) of cross sections in terms of experimentally measured resonance parameters is given by the theory of nuclear reactions at low energies $E \ll mc^2$. This theory is thoroughly expounded in textbooks and monographs. To represent the cross sections, the Breit–Wigner, Rich–Moore, and Adler–Adler formulas are used. The total cross section σ_A^t, the cross section of elastic scattering σ_A^{el}, and the cross section of inelastic reactions σ_A^x $(x \neq t, el)$ in the single-level Breit–Wigner approximation are as follows:

$$\begin{Bmatrix} \sigma_A^t(E) \\ \sigma_A^x(E) \end{Bmatrix} = \begin{Bmatrix} \sigma^p(E) \\ 0 \end{Bmatrix} + 4\pi^2 \lambda\!\!\!^{-2} \sum_r \sum_{l,J} n_{lJ}\, g_{lJ}\, \Gamma_r^{el} \times \begin{Bmatrix} \theta_r(E)\cos 2\phi_l + \chi_r(E)\sin 2\phi_l \\ \theta_r(E)\, \Gamma_r^x / \Gamma_r, \qquad x \neq t, el \end{Bmatrix}, \quad (8)$$

$$\sigma_A^{el}(E) = \sigma_A^t(E) - \sum_{x \neq el} \sigma_A^x(E), \qquad \sigma^p(E) = 4\pi \lambda\!\!\!^{-2} \sum_{l,J} n_{lJ}\, g_{lJ} \sin^2 \phi_l ,$$

$$\begin{Bmatrix} \theta_r(E) \\ \chi_r(E) \end{Bmatrix} = \frac{1/\pi}{[E - E_r]^2 + \Gamma_r^2} \begin{Bmatrix} \Gamma_r \\ E - E_r \end{Bmatrix}, \quad \int \begin{Bmatrix} \theta_r(E) \\ \chi_r(E) \end{Bmatrix} dE = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad \Gamma_r = \Gamma_r^{el} + \sum_{x \neq el} \Gamma_r^x .$$

Here r is the resonance index; the quantum numbers l and J are the angular momentum of the incident neutron and the total spin of the compound nucleus, respectively; $\sigma^p(E)$ is the cross section of potential scattering; $\theta_r(E)$ and $\chi_r(E)$ are the symmetric and antisymmetric resonance profiles; $\lambda\!\!\!^{-} = \hbar / \sqrt{2\mu E}$ $(\mu = mA/[A+1])$ is the de Broglie wavelength; $n_{lJ}(E)$ is the probability of realization of the collision channel (l, J); and g_{lJ} is the spin statistical weight. The phase shift $\phi_l(E)$ and the

resonance half-widths $\Gamma_r(E)$, $\Gamma_r^{el}(E)$, and $\Gamma_r^{x}(E)$ depend on the neutron energy E, the collision channel, and the parameters of the nucleus. When approaching the resonance energy $E \approx E_r$, the symmetric resonance profile $\theta_r(E)$ changes faster than other functions.

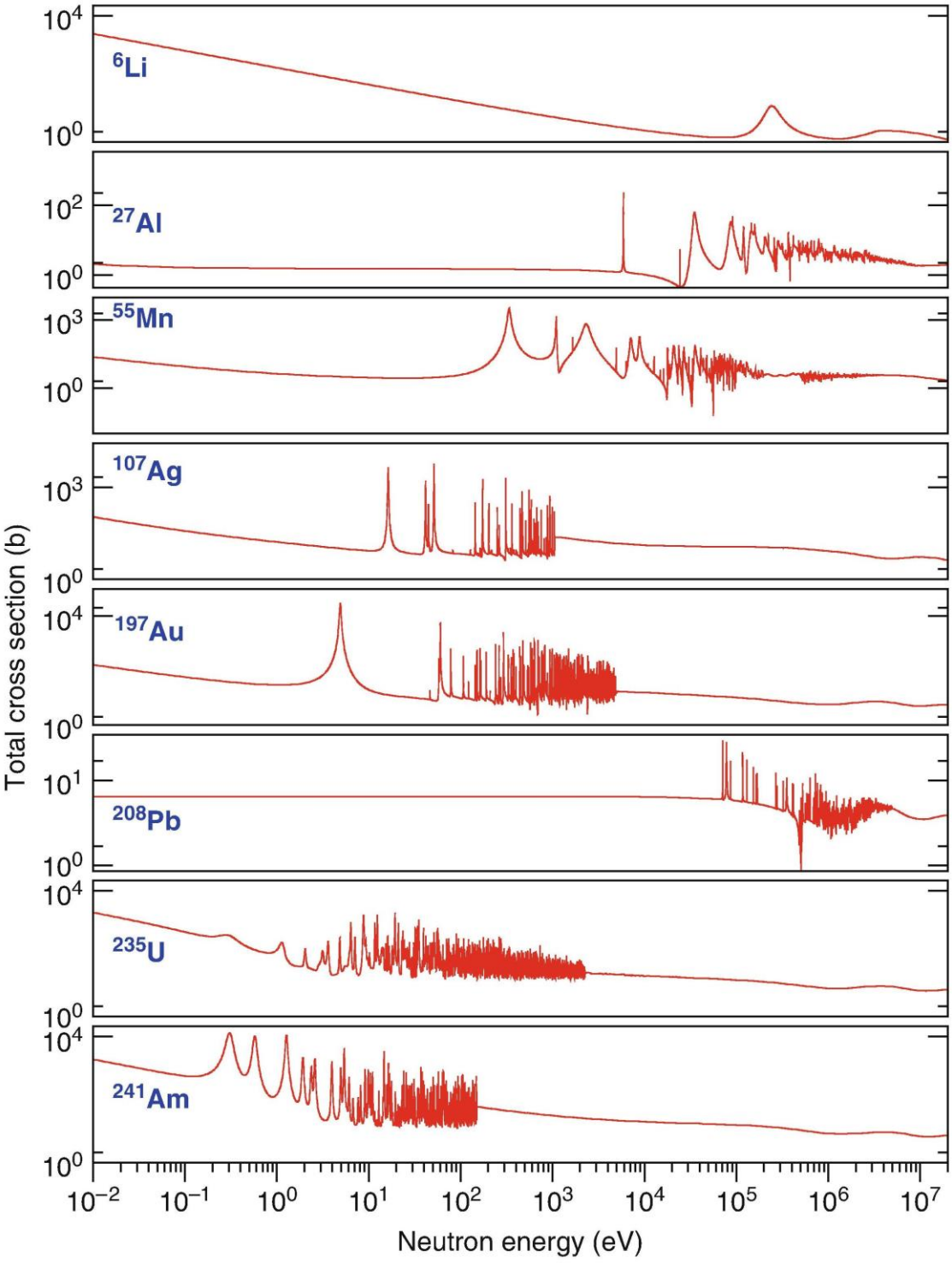

Fig. 1. The total cross sections of the interaction of the neutron with the nucleus (in barn = $10^{-24}\,\mathrm{cm}^2$) *for certain elements. The figure is borrowed from* [29].

Dependence of differential cross sections on their arguments. The cross section analysis and the verification of experimental data are carried out using the expansion of the reaction indicatrices in a Fourier series in Legendre polynomials $P_n(\eta)$ $(n = 0,1,...)$, which are orthogonal on the interval $-1 \le \eta \le 1$, where $\eta = \mathbf{\Omega\Omega'}$ is the reaction angle cosine

$$w_A^x(E' \to E, \eta) \simeq \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\eta) w_{A,n}^x(E' \to E), \qquad \eta = \mathbf{\Omega\Omega'}, \tag{9}$$

$$w_{A,n}^x(E' \to E) = \int_{-1}^{1} P_n(\eta) w_A^x(E' \to E, \eta) d\eta, \qquad \int_0^{\infty} w_{A,n}^x(E' \to E) dE = \delta_{n0},$$

$$P_0(\eta) = 1, \;\; P_1(\eta) = \eta, \;\; P_2(\eta) = \frac{3\eta^2 - 1}{2}, \ldots \qquad \int_{-1}^{1} P_n(\eta) P_m(\eta) d\eta = \frac{2}{2n+1} \delta_{mn},$$

and $w_{A,n}^x$ are the expansion coefficients.

The indicatrix of fission reactions of nuclides of thorium, uranium and transuranium elements $w_A^f(E' \to E, \eta)$ $(x = f)$ weakly depends on η and the energy of the primary neutron E', and it strongly depends on the energy of prompt neutrons E. The zero-order expansion coefficient $w_{A,0}^f(E' \to E)$ is called the spectrum of fission neutrons. The neutrons are born with energies $0.5 \le E \le 10\,\text{MeV}$, which are much higher than the resonance region. The average energy of 235U fission neutrons is $E \approx 2\,\text{MeV}$. All the spectra are normalized to unity:

$$W_0^f(E' \to E, \mathbf{r}, t) = \frac{\sum N_A(\mathbf{r}, t) \sigma_A^f(E') w_{A,0}^f(E' \to E)}{\sum N_A(\mathbf{r}, t) \sigma_A^f(E')}, \qquad \int_0^{\infty} W_0^f(E' \to E) dE = 1.$$

The elastic neutron scattering indicatrix $w_A^{el}(E' \to E, \eta)$ $(x = el)$ is a function with bounded support. It strongly depends on on the energy difference $E' - E$. The dependence on $\eta = \mathbf{\Omega\Omega'}$ is significant only for scattering by light nuclei. An approximate formula can be obtained from the laws of energy and momentum conservation under the assumption of scattering isotropy in the system of the center of mass of the neutron and nucleus (see, [40, p. 280]). If we change to the observer's coordinate system, a strong relation arises between the neutron energy loss and the scattering angle η_A^{el} in the observer's system expressed by the δ- function:

$$w_A^{el}(E' \to E, \eta) \approx \frac{\theta(1-|\eta|)}{\alpha_A E'} \delta(\eta - \eta_A^{el}), \qquad E' \text{ t } 10T_{[эв]}, \tag{10}$$

$$\theta(x) = \begin{Bmatrix} 1, x \geq 0 \\ 0, x < 0 \end{Bmatrix}, \quad \alpha_A = \frac{4A}{[A+1]^2}, \quad \eta_A^{el}(E', E) = \frac{A+1}{2}\left[\frac{E}{E'}\right]^{1/2} - \frac{A-1}{2}\left[\frac{E'}{E}\right]^{1/2},$$

$$w_{A,n}^{el}(E' \to E) \approx \frac{P_n(\eta_A^{el})}{\alpha_A E'} \theta(E' - E)\theta(E - E' + \alpha_A E'), \qquad \Delta_A(E') = \alpha_A E'.$$

Here $\Delta_A(E')$ is the width of the indicatrix support (deceleration step).

Dependence of cross sections on coordinates. The microscopic cross sections $\sigma_A^x(E)$ $(x = t, s, c, f, m \times n, el, ...)$ depend on the coordinates $(\mathbf{r}, t)$ through the temperature of the substance $T(\mathbf{r}, t)$. This dependence is not indicated in the list of arguments. The thermal motion of atoms and molecules broadens narrow resonances. Accordingly, in formulas (8) (and in similar formulas), the resonance profiles $\theta_r(E)$ and $\chi_r(E)$ are convolved with the rapidly decreasing Gaussian distribution. If the temperature does not exceed $10\,\text{eV}$, then distortions affect the cores of narrow resonances within the Doppler width leaving the wings unchanged. The temperature dependence is weak "almost everywhere".

Unlike the microscopic cross sections, the macroscopic cross sections $\Sigma^x(E, \mathbf{r}, t)$ $(x = t, s, c, f, m \times n, el, ...)$ can strongly depend on the coordinates $(\mathbf{r}, t)$. An object in which the neutrons move usually consists of spatial zones filled with materials with different nuclide composition. The nuclide densities $N_A(\mathbf{r}, t)$ can experience a discontinuity on the zone interfaces; e.g., they can vanish. Furthermore, the densities of nuclides can vary in space and time within the same material. For example, in the fuel elements of a nuclear reactor, a non-uniform burn-up of 235U occurs due to the self-shielding effect.

Inversion of resonances. Consider a heterogeneous object. Let the neutron flux $\varphi(E, \mathbf{\Omega}, \mathbf{r}, t)$ on the interfaces between the zones change relatively slowly with E. In the depth of the zone filled with a homogeneous material, the solution to Eq. (1) asymptotically tends to the stationary point of the equation, i.e. to the regime in which the balance between the rate of departure and the rate of arrival of neutrons into the beam is maintained:

$$\varphi(E, \mathbf{\Omega}, \mathbf{r}, t) \to \varphi_0(E, \mathbf{r}, t) = q(\varphi_0) \big/ \Sigma^t(E, \mathbf{r}, t) \sim F(E) \big/ \Sigma^t(E, \mathbf{r}, t). \tag{11}$$

The neutron arrival rate $q(\varphi_0)$ varies with E relatively slowly compared with

the function of large and sharp variation $\Sigma^t(E)$. Therefore, in the depth of the zone, the neutron flux $\varphi(E,\mathbf{\Omega},\mathbf{r},t)$ is rugged by deep dips in the form of «inverted resonances» in the vicinity of the resonance energies $E \approx E_r$. These dips decrease the rates of reactions $\int \Sigma^x(E)\varphi(E)dE$, including fission reactions. This effect is called the resonance self-shielding of the neutron field. A similar effect—«resonance damping» of the flux—is observed when the radiation of a boundary source propagates into the depth of the object.

The specific dependence of the fission indicatrix and elastic scattering indicatrix (10) on the energy of prompt neutrons E allows one to estimate the E- dependence of the functional $q(\varphi_0) \sim F(E)$ on E (see [40, p. 286]). In the first approximation («narrow» resonances, the Wigner approximation), the synthetic spectrum $F(E)$ is a certain crosslinking of the fission neutron spectrum, the Fermi spectrum $\sim 1/E$, and the spectrum of the neutron thermalization region. In problems of radiation shielding and neutron diagnostics, if the object does not contain fissile materials, then the fission neutron spectrum is replaced by the spectrum of the boundary source φ^{ent}. Wide resonances with a width exceeding the width of the scattering indicatrix support (see (10)) can introduce small-scale disturbances into the synthetic spectrum $F(E)$. These disturbances can be found analytically or numerically. In the data processing system NJOY [37], [38], the calculation is performed by the Flux calculator.

The asymptotic spectrum φ_0 is used to prepare the group constants [7], [37]–[41] by Bondarenko's method [42]. These constants are used in calculations of neutron transport in the multigroup approximation [43]–[46].

1. Separation of variables and resonance carriers

Outside the resonance region, where the neutron cross sections are slowly varying functions of energy, we will use group averaging of cross sections and the multigroup approximation of neutron transport. In the region of resonances, we will perform more accurate averaging. Let us reduce the dimension of the problem by combining the nuclides into several components.

The component C is the set of nuclides the densities of which vary similarly to each other within the object:

$$\frac{N_A(\mathbf{r},t)}{N_C(\mathbf{r},t)} \approx \text{const}, \quad N_C(\mathbf{r},t) = \sum_{A \subset C} N_A(\mathbf{r},t), \qquad \mathbf{r} \in V(t), \ \ 0 \le t \le T. \tag{12}$$

Here N_C is the component density, $V(t)$ is the object volume, and $0 \le t \le T$ is the time interval characteristic of the problem. If the nuclide densities satisfy conditions (12) in certain zones of the object and in other zones the densities are close to zero, then the nuclides make up a component. The nuclides that are not involved in nuclear

transformations can be combined into components. These are light nuclides of neutron moderators and nuclides of structural materials (concrete, steel, …). The fission fragments and transuranic nuclides that occur in chains of nuclear reactions in similar proportions from the zero or small initial density are also combined into components. The main fissile nuclides 233U, 235U, 238U, 239Pu, and 232Th are considered to be separate components if they cannot be assigned to one of groups (12). In terms of components, the neutron cross sections have form (4), (5) with the replacement of the nuclide index *A* by the index of the component *C* and the replacement of the microscopic cross sections

$$\begin{Bmatrix} 1, & \nu_C^x(E') \\ w_C^x(E' \to E, \eta)\nu_C^x(E') \end{Bmatrix} \sigma_C^x(E') = \sum_{A \subset C} \frac{N_A(\mathbf{r},t)}{N_C(\mathbf{r},t)} \begin{Bmatrix} 1, & \nu_A^x(E') \\ w_A^x(E' \to E, \eta)\nu_A^x(E') \end{Bmatrix} \sigma_A^x(E'), \quad (13)$$

$$\chi_C^\tau(E)\nu_C^\tau(E)\sigma_C^f(E) = \sum_{A \subset C} \frac{N_A(\mathbf{r},t)}{N_C(\mathbf{r},t)} \chi_A^\tau(E)\nu_A^\tau(E)\sigma_A^f(E).$$

The total macroscopic cross section $x = t$ can be written as the sums

$$\Sigma^t(E,\mathbf{r},t) = \Sigma_C^t(E,\mathbf{r},t) + M_C^t(E,\mathbf{r},t) = N_C(\mathbf{r},t)\sigma_C^t(E) + M_C^t(E,\mathbf{r},t), \quad (14)$$

$$M_C^t(E,\mathbf{r},t) = \sum_{D \neq C} \Sigma_D^t(E,\mathbf{r},t) = \sum_{A \not\subset C} N_A(\mathbf{r},t)\sigma_A^t(E), \qquad C = C_1, C_2, ...$$

where $\Sigma_C^t(E,\mathbf{r},t)$ is the macroscopic cross section of the component *C* and $M_C^t(E,\mathbf{r},t)$ is the cross section of the component dilution.

Resonance carriers. *Proposition 1*. The region of neutron resonances can be partitioned into a number of sets $\omega_g = \{E_p^- < E < E_p^+,\ p = 1,2,...,P\},\ 1 \le g \le gg$ called the carriers of resonances. A carrier generally consists of several intervals. The relative width of the resonance carrier does not exceed a given number K_g:

$$2[E_P^+ - E_1^-]/[E_P^+ + E_1^-] < K_g, \qquad 1 \le g \le gg. \quad (15)$$

(If condition (15) is not satisfied, then the carrier can always be partitioned into more compact carriers.) Within the carrier, the total macroscopic cross section allows an approximate separation of the variables *E* and $(\mathbf{r},t)$ that is valid at all points of the object:

$$\Sigma^t(E,\mathbf{r},t) = a_g(\mathbf{r},t)s_g(E) + b_g(E,\mathbf{r},t), \qquad E \in \omega_g,\ \mathbf{r} \in V(t),\ 0 \le t \le T, \qquad (16)$$

where $a_g, b_g, s_g \ge 0$ are nonnegative functions and $b_g(E,\mathbf{r},t)$ is a bounded function of small variation in neutron energy E

$$\int_{\omega_g} [b_g(E,\mathbf{r},t) - \bar{b}_g(\mathbf{r},t)]^2 dE \ll \left[a_g(\mathbf{r},t)\bar{s}_g + \bar{b}_g(\mathbf{r},t)\right]^2 \int_{\omega_g} dE, \qquad (17)$$

$$\bar{b}_g(\mathbf{r},t) = \int_{\omega_g} b_g(E,\mathbf{r},t)dE \Big/ \int_{\omega_g} dE, \qquad \bar{s}_g = \int_{\omega_g} s_g(E)dE \Big/ \int_{\omega_g} dE.$$

If the function $b_g(E,\mathbf{r},t) = \bar{b}_g(\mathbf{r},t)$ is independent of energy, then the variables are exactly separated on the resonance carrier.

The existence of partitions satisfying properties (16) and (17) follows from (14) and the presence of resonances. The resonance carrier $\omega_g(C)$ of the component C includes the intervals of energy ΔE that are close to the strong resonances of C and do not include the resonances of other components. Then $\omega_g(C)$ contains the resonances of the component C and the background cross section (wings of the resonances) of all other components. Formula (16) coincides with (14) for $a_g = N_C(\mathbf{r},t)$, $s_g = \sigma_C^t(E)$, and $b_g = M_C^t(E,\mathbf{r},t)$, where b_g has small variations on $\omega_g(C)$ at any point of the object $(\mathbf{r},t)$ regardless of whether the material contains the component C ($N_C(\mathbf{r},t) \ne 0$) or does not contain it ($N_C(\mathbf{r},t) = 0$).

Remark. In the theory of photon transfer, the assumption that the variables E and $(\mathbf{r},t)$ are separable in the entire spectrum of the particle energies $\Sigma^t(E,\mathbf{r},t) \approx a_g(\mathbf{r},t)s_g(E)$ $(0 < E < \infty)$ is called the Milne–Eddington model.

Mean cross sections. In what follows, the term *mean* cross sections denotes the total macroscopic cross section $x = t$ averaged over the volume of the object and the time interval, unless otherwise specified. The mean total cross section $S^t(E)$ [1/cm] is the sum of the mean total cross sections of the components $S_C^t(E)$. The means depend only on the neutron energy E:

$$S^t(E) = \langle \Sigma^t \rangle = \frac{1}{T}\int_0^T \frac{dt}{V(t)} \int_{V(t)} K(\mathbf{r},t)\,\Sigma^t(E,\mathbf{r},t)\,d\mathbf{r} = \sum_C S_C^t(E), \qquad (18)$$

$$S_C^t(E)=\frac{1}{T}\int_0^T\frac{dt}{V(t)}\int_{V(t)}K(\mathbf{r},t)\,N_C(\mathbf{r},t)\sigma_C^t(E)\,d\mathbf{r}\,,\qquad \frac{1}{T}\int_0^T\frac{dt}{V(t)}\int_{V(t)}K(\mathbf{r},t)\,d\mathbf{r}=1.$$

$K(\mathbf{r},t)$ is the weight function that selects cross sections in regions of the object if this is required by specific properties of the problem. In the nonspecific case, $K(\mathbf{r},t)=1$.

The algorithm of partitioning the resonance region into resonance carriers uses the mean cross sections (18).

(A) The interval of energies ΔE is included in the resonance carrier $\omega_g(C)$ that accumulates the resonances of the component C if the mean macroscopic cross section of this component on this interval is greater than the macroscopic cross section of any other resonance component: $S_C^t(E)>S_D^t(E)$, $D\neq C$ (see Fig. 2).

(B) If the mean macroscopic cross section of a resonance-free component (e.g., neutron moderator) dominates on the interval ΔE, then this interval is included in the nearest carrier. The low density components may remain without carriers of their resonances.

(C) If a resonance carrier contains wide resonances with the width exceeding the width of the scattering indicatrix $\Delta(E')$ (see (10)), then it is divided into two carriers: the carrier of the left resonance wings consisting of the intervals on which the cross section $S^t(E)$ grows and the carrier of the right resonance wings consisting of the intervals on which this cross section decreases. This makes it possible to take into account the asymmetry of the the spectrum of moderating neutrons arising in the wings of wide resonances.

Example. If an object consists of zones filled with a resonance-free moderator component and a resonance component that has narrow resonances, then, according to the algorithm, the resonance region will be partitioned into ordinary groups $\omega_g=\{E_{g-1}<E<E_g\}$. The number of groups $gg=I$ depends on the choice of the number K_g in (15). If an object consists of zones filled with moderator and resonant components C and D, then the resonance region is partitioned into the resonance carriers of the components C and D (Fig. 2). The total number of carriers gg does not exceed $2I$.

Proposition 2. On the resonance carrier, the local macroscopic cross section of any group of reactions *x* at point $(\mathbf{r},t)$ is related to the mean total cross section $S^t(E)$ by a dependence that is close to the linear dependence

$$\Sigma^x(E,\mathbf{r},t)=A_g^x(\mathbf{r},t)\,S^t(E)+B_g^x(E,\mathbf{r},t),\qquad E\in\omega_g,\ \mathbf{r}\in V(t),\ 0\le t\le T,\qquad (19)$$

where $B_g^x(E,\mathbf{r},t)$ is a function of small variation in the argument E at any point of the object. If the function $A_g^x(\mathbf{r},t)$ is zero in certain zones of the object, then the mac-

roscopic cross section is determined only by the function B_g^x.

To prove this proposition in the case $x = t$, we average (16) over the object, express $s_g(E)$ in terms of $S^t(E)$, and substitute it into (16). This gives (19) with the coefficients $A_g^t(\mathbf{r},t) = a_g(\mathbf{r},t)/\langle a_g \rangle$ and $B_g^t(E,\mathbf{r},t) = b_g(E,\mathbf{r},t) - A_g^t(\mathbf{r},t)\langle b_g \rangle$. The Breit–Wigner formulas (8) imply that the microscopic cross section of the reaction group *x* and the total microscopic cross section are related by a dependence that is close to the linear dependence $\sigma_A^x(E) = d_A^x \sigma_A^t(E) + e_A^x$. Here $d_A^x \sim \Gamma^x/\Gamma$ and e_A^x are slowly varying functions of the neutron energy compared with $\sigma_A^t(E)$. Therefore, (19) can be extended to any group of reactions *x*. Below, the superscript *t* for the mean total macroscopic cross section will be sometimes omitted: $S(E) \equiv S^t(E)$.

Proposition 3. All materials and geometric zones of the object are involved in the formation of the neutron field. The neutron flux at any point $(\mathbf{r},t)$ of the object can be represented by the function

$$\varphi(E,\mathbf{\Omega},\mathbf{r},t) = F(E)\cdot\left[Z_g(S(E),\mathbf{\Omega},\mathbf{r},t) + \varepsilon_g(E,\mathbf{\Omega},\mathbf{r},t)\right], \quad E \in \omega_g, \tag{20}$$

where $\varepsilon_g(E)$ and $F(E)$ are functions of small variation in *E*, $F(E)$ is the the representative synthetic spectrum of the problem (see (11)), and $Z_g > 0$ is a complicated function depending on the neutron energy through the mean (over the object) cross section $S(E)$.

We restrict ourselves to considering the stationary problem without reflection of particles from external boundaries. In the more general case, the proof is similar. Let us write the solution to the transport equation (1) along the characteristic of the transport operator within the zone of the object containing a homogeneous material:

$$\varphi(E,l) = \int_{l_{in}}^{l} q(\varphi)\exp\left[-\int_{l'}^{l}\Sigma^t(E,l'')dl''\right]dl' + \varphi(E,l_{in})\exp\left[-\int_{l_{in}}^{l}\Sigma^t(E,l')dl'\right] \approx \tag{21}$$

$$\approx \frac{q(\varphi)}{\Sigma^t(E,l)}\left[1-\exp\left[-\int_{l_{in}}^{l}\Sigma^t(E,l')dl'\right]\right] + \varphi(E,l_{in})\exp\left[-\int_{l_{in}}^{l}\Sigma^t(E,l')dl'\right].$$

Here l and l_{in} are the characteristic coordinates of the observation point $\mathbf{r}$ and the entry point of the characteristic into the zone $\mathbf{r}_{in} = \mathbf{r} - \mathbf{\Omega}[l - l_{in}]$ and $\varphi(E,l_{in})$ is the neutron flux on the boundary of the zone. The approximate estimate of the neutron

flux is valid because the cross section $\Sigma^t(E,l)$ varies continuously within each zone. Formula (21) gives the rate at which the solution reaches the asymptotics of the stationary regime (11), which settles deep inside the zone of infinite length. The rate depends on the cross section $\Sigma^t(E,l)$ at different neutron energies. Let us substitute the expression of the cross section on the resonance carrier in form (19) into (21). If the neutron flux on the boundary of the zone $\varphi(E,l_{in})$ can be represented in form (20), then (21) can also be reduced to (20). Let us move along the characteristic from the entry point into the object while this condition holds true. By sequentially calculating the functions $\varphi(E,l_{in})$ at the points of intersection of the interfaces between the zones, we can always reduce solution (21) to form (20).

2. Lebesgue averaging

This section describes the averaging of resonance cross sections and the neutron flux over a system of Lebesgue sets. The sets are constructed independently within each resonance carrier. The reader is not required to know measure theory and the Lebesgue integral. We will use only the construction of the integral that is suitable for averaging resonance cross sections. All functions under consideration are assumed to be bounded, Riemann and Lebesgue integrable, and both integrals are equal.

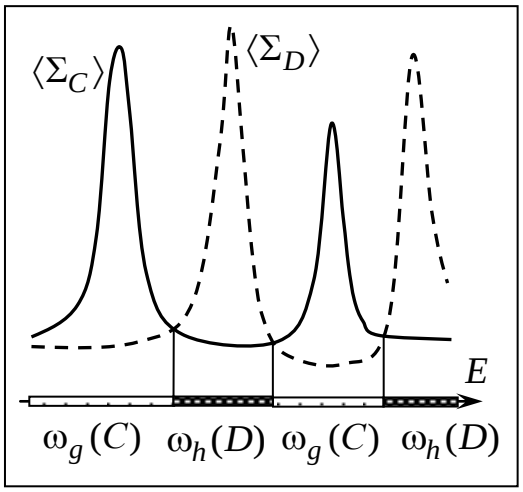

Fig. 2. Partitioning the spectrum into the resonance carriers of the components C and D.

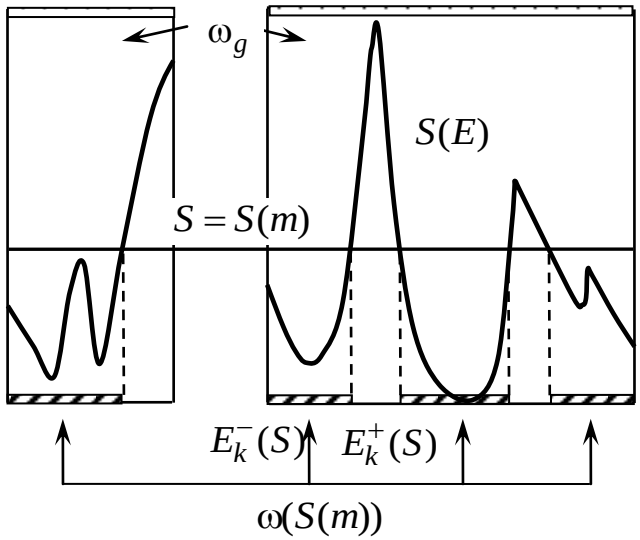

Fig. 3. A fragment of the resonance carrier g consists of two intervals. For the given level $S(m)$, the Lebesgue set $\omega(S(m))$ consists of four intervals.

System of embedded sets. We construct within the carrier *g* a system of sets (see Fig. 3) by including in the set $\omega(g,S)$ the energies E at which the mean cross section (averaged over the object) does not exceed the S, $S(E)<S$. This set consists of the set of intervals $E_k^-(S)<E<E_k^+(S)$, $k=1,2,...$ The left and right boundaries of

the intervals $E_k^{\pm}(S)$ are the points with the same mean cross section $S = S(E_k^{\pm})$ or the boundaries of resonance carriers. As the level S increases, the left boundaries can only move to the left and the right boundaries can move only to the right. Therefore, we have the embedding of sets $\omega(S_{0g}) \subseteq \omega(S) \subseteq \omega(S_g)$, $S_{0g} \le S \le S_g$, where S_{0g} is the level at which the set is empty $\omega(S_{0g}) = \varnothing$ and S_g is the level at which the set corresponds with the resonance carrier ω_g.

Define the measure $m(S)$ ([eV]) of the set $\omega(g,S)$ using the integral

$$m(S) = \int\limits_{\omega(g,S)} F(E)dE = \sum_k \int\limits_{E_k^-(S)}^{E_k^+(S)} F(E)dE, \qquad \begin{array}{l} 0 \le m(S) \le m_g, \\ S_{0g} \le S \le S_g, \end{array} \tag{22}$$

$$m(S_{0g}) = 0, \qquad m_g = m(S_g) = \int\limits_{\omega_g} F(E)dE,$$

where $F(E)$ is the dimensionless weighting spectrum that is proportional, e.g., to the Wigner synthetic spectrum (see (11). If the weighting spectrum is difficult to determine, then set $F = 1$. The spectrum can be adjusted when typical problems are solved. The measure varies from zero to the integral of the weighting spectrum over the resonance carrier. If $F = 1$, then the measure equals the sum of lengths of the intervals on which $S(E) < S$.

Remark. The Lebesgue averaging can be performed using the reference functions $S(E,\mathbf{r},t)$ and $F(E,\mathbf{r},t)$ that depend on the energy of particles and on the coordinates [8]–[11]. In the neutron transport problems, it suffices to consider the case $S = S(E)$, $F = F(E)$.

Since the measure is a nondecreasing function of level *S*, then there exists a nondecreasing inverse function specifying the dependence of the level on the measure $S = S(m)$ ($S_{0g} \le S(m) \le S_g$). We parameterize the system of sets $\omega(g,S)$ in terms of their measure *m* making the change of variables by means of the inverse function $\omega(g,S(m)) = \omega(g,m)$. If $m = 0$, then the Lebesgue set is empty—$\omega(0) = \varnothing$. If $m = m_g$, then the set $\omega(m_g)$ densely fills the resonance carrier ω_g. A part of the boundary points of the set $E_k^{\pm}(S(m)) = E_k^{\pm}(m)$ ($k = 1,2,...$) are the points with the same mean cross section $S(E_k^{\pm}) = S(m)$. At these points, $\partial E_k^+ / \partial m \ge 0$ and $\partial E_k^- / \partial m \le 0$. The remaining points are the boundaries of the resonance carrier. At them, we have $\partial E_k^{\pm} / \partial m = 0$.

By differentiating (22) with respect to the measure *m* under the condition that it is an independent variable, we obtain an identity expressing the relationship between

the derivatives of the direct and inverse functions

$$\frac{\partial m}{\partial m} = \frac{\partial}{\partial m}\sum_k \int_{E_k^-(m)}^{E_k^+(m)} F(E)\,dE = \sum_k \left[F(E_k)\frac{\partial E_k}{\partial m} \right]_-^+ = 1, \qquad (23)$$

where $\left[F(E_k)\dfrac{\partial E_k}{\partial m} \right]_-^+ = F(E_k^+)\dfrac{\partial E_k^+}{\partial m} - F(E_k^-)\dfrac{\partial E_k^-}{\partial m}$, $\qquad \dfrac{\partial E_k^+}{\partial m} \ge 0$, $\dfrac{\partial E_k^-}{\partial m} \le 0$.

The summation is over the boundary points $E_k^\pm$ with the same mean cross section $S(E_k^\pm) = S(m)$. The boundaries of the resonance carrier are not included in the sum because $\partial E_k^\pm / \partial m = 0$ at them. All terms in the sum are nonnegative.

Lebesgue neutron flux. Let $T_g(m, \mathbf{\Omega}, \mathbf{r}, t)$ [$\mathrm{cm}^{-2}\mathrm{sec}^{-1}$] be the cumulative distribution of neutrons on the system of embedded sets built inside the carrier *g*:

$$T_g(m, \mathbf{\Omega}, \mathbf{r}, t) = \int_{\omega(g,m)} \varphi(E, \mathbf{\Omega}, \mathbf{r}, t)\,dE = \sum_k \int_{E_k^-(m)}^{E_k^+(m)} \varphi(E, \mathbf{\Omega}, \mathbf{r}, t)\,dE\,. \qquad (24)$$

If $m = 0$, then the distribution is zero; if $m = m_g$, then it equals the integral of the neutron flux $\varphi(E)$ over the resonance carrier *g*. We will call the distribution density

$$\psi_g(m, \mathbf{\Omega}, \mathbf{r}, t) = \frac{\partial T_g}{\partial m} = \frac{\partial}{\partial m}\sum_k \int_{E_k^-(m)}^{E_k^+(m)} \varphi(E, \mathbf{\Omega}, \mathbf{r}, t)\,dE = \sum_k \left[\varphi(E_k)\frac{\partial E_k}{\partial m} \right]_-^+ = \qquad (25)$$

$$= \sum_k \left[\frac{\varphi(E_k)}{F(E_k)} F(E_k)\frac{\partial E_k}{\partial m} \right]_-^+ = \left\langle \frac{\varphi(E_k^\pm)}{F(E_k^\pm)} \right\rangle \cdot \sum_k \left[F(E_k)\frac{\partial E_k}{\partial m} \right]_-^+ = \left\langle \frac{\varphi(E_k^\pm)}{F(E_k^\pm)} \right\rangle .$$

the Lebesgue neutron flux $\psi_g(m, \mathbf{\Omega}, \mathbf{r}, t)$ [$\mathrm{eV}^{-1}\mathrm{cm}^{-2}\mathrm{s}^{-1}$]. The last equality is obtained taking into account identity (23). It is clear that the Lebesgue flux $\psi_g(m, \mathbf{\Omega}, \mathbf{r}, t)$ is the ratio of the neutron flux $\varphi(E, \mathbf{\Omega}, \mathbf{r}, t)$ to the weighting spectrum $F(E)$ averaged over the set of boundary points of the Lebesgue set $E_k^\pm = E_k^\pm(m)$ ($k = 1, 2, ..$) at which the mean macroscopic cross section takes the same value $S(E_k^\pm) = S(m)$. This definition is correct because all the terms in the sum are nonnegative. The depend-

ence of the Lebesgue flux $\psi_g(m)$ on the measure m corresponds to the dependence of the neutron flux $\varphi(E)$ on the energy E. The summation over the points $E_k^{\pm}(m)$ with the weight $|\partial E_k^{\pm}/\partial m|$ does not reduce the number of independent arguments of functions.

Example. Let the Lebesgue flux be independent of m:

$$T_g(m,\mathbf{\Omega},\mathbf{r},t)=\int_0^m \frac{\partial T_g}{\partial m}\,dm=\int_0^m \psi_g(m,\mathbf{\Omega},\mathbf{r},t)dm=\psi_g(\mathbf{\Omega},\mathbf{r},t)\cdot m\,.$$

Assuming $m=m_g$, we find that $\psi_g(\mathbf{\Omega},\mathbf{r},t)=T_g(m_g,\mathbf{\Omega},\mathbf{r},t)/m_g$, i.e., the Lebesgue flux equals the ratio of the integral of the neutron flux on the resonance carrier ω_g to the corresponding integral of the weighting spectrum.

Evaluation of integrals. An important useful property of the Lebesgue integral is the ability to efficiently evaluate integrals of nonmonotone resonance functions of many variables

$$I(m,\mathbf{\Omega},\mathbf{r},t)=\int\limits_{\omega(g,m)} Y(E,\mathbf{r},t)\,\varphi(E,\mathbf{\Omega},\mathbf{r},t)\,dE\,,\qquad 0<m\le m_g\,, \tag{26}$$

where $Y(E,\mathbf{r},t)=X(S(E),\mathbf{r},t)$ is a given complicated function depending on the energy E through a relatively «good» function $X(S)$ and a strongly nonmonotone function $S(E)$; and φ is the neutron flux. For $m=m_g$, the integration is carried out over the resonance carrier g. The numerical evaluation of integrals (26) according to the Riemann scheme at points in space $E\times\mathbf{\Omega}\times\mathbf{r}\times t$ requires a large amount of computations on a fine nonuniform grid. The accuracy of the computations depends on the art of grid selection. On the other hand, if we construct a system of Lebesgue sets on the basis of the function $S(E)$, then the integrals can be represented in the form

$$I=\int_0^m \frac{\partial}{\partial m}\left[\int\limits_{\omega(g,m)} Y(E)\,\varphi(E)\,dE\right]dm=\int_0^m\sum_k\left[X(S(E_k))\,\varphi(E_k)\,\frac{\partial E_k}{\partial m}\right]_-^+dm= \tag{27}$$

$$=\int_0^m X(S(m))\sum_k\left[\varphi(E_k)\,\frac{\partial E_k}{\partial m}\right]_-^+dm=\int_0^m X(S(m),\mathbf{r},t)\,\psi_g(m,\mathbf{\Omega},\mathbf{r},t)\,dm\,.$$

The function $X(S(E_k^{\pm}))$ is taken out of the sum because it takes the same value $X(S(m))$ at the points $E_k^{\pm}$. The level $S(m)$ is a nondecreasing function of m inde-

pendently of how many resonances $S(E)$ has on the carrier ω_g. The functions $X(S(m))$ and $\psi_g(m)$ have good monotonicity properties with respect to the argument *m*. Therefore, the integrals may be evaluated on a coarse grid at a low computational cost.

The gain from the transition to Lebesgue distributions is achieved by combining points with the same macroscopic cross section $S(E_k^{\pm}) = S(m)$ into one computation point and due to momotonization of sections. The gain magnitude is approximately equal to the number nonmonotonicities (resonances) on the carrier. The functions $X(S)$ and $S(E)$ must be known in advance. The relationship between the evaluation of integrals (26) of nonmonotone functions according to Lebesgue and Riemann is similar to the relationship between positional and nonpositional number systems—eggs may be counted one-by-one or by dozens.

Let us calculate **the departure rate** of neutrons from beams whose energy lies within the set $E \in \omega(g,m) \subseteq \omega_g$, and **the emergence rates** of prompt neutrons, fission fragments (see (2)), and nuclides (see (7)) in reactions *x*:

$$\begin{Bmatrix} I_g^x(m,\mathbf{\Omega},\mathbf{r},t) \\ J_g^x(m,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix} = \int\limits_{\omega(g,m)} \begin{Bmatrix} 1 \\ \nu^x(E,\mathbf{r},t) \end{Bmatrix} \times \Sigma^x(E,\mathbf{r},t)\,\varphi(E,\mathbf{\Omega},\mathbf{r},t)\,dE\,, \qquad 0 < m \le m_g\,.$$

Let us write the integrals of interest as the Lebesgue integrals (27)

$$\begin{Bmatrix} I_g^x \\ J_g^x \end{Bmatrix} = \int\limits_0^m \frac{\partial}{\partial m} \left[\int\limits_{\omega(g,m)} \begin{Bmatrix} 1 \\ \nu^x(E,\mathbf{r},t) \end{Bmatrix} \times \Sigma^x(E,\mathbf{r},t)\,\varphi(E,\mathbf{\Omega},\mathbf{r},t)dE \right] dm = \qquad (28)$$

$$= \int\limits_0^m \sum_k \left[\begin{Bmatrix} 1 \\ \nu^x \end{Bmatrix} \Sigma^x F \frac{\varphi}{F} \frac{\partial E_k}{\partial m} \right]_-^+ dm \approx \int\limits_0^m \psi_g(m) \sum_k \left[\begin{Bmatrix} 1 \\ \nu^x(E_k) \end{Bmatrix} \Sigma^x(E_k) F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ dm \approx$$

$$\approx \int\limits_0^m \begin{Bmatrix} 1 \\ \nu^x(m,\mathbf{r},t) \end{Bmatrix} \times \Sigma_g^x(m,\mathbf{r},t)\,\psi_g(m,\mathbf{\Omega},\mathbf{r},t)\,dm\,.$$

Here the ratios $\varphi(E_k^{\pm})/F(E_k^{\pm})$ at the boundary points $E_k^{\pm}$ are replaced by the average $\langle \varphi(E_k^{\pm})/F(E_k^{\pm}) \rangle = \psi_g(m)$ calculated at the same points. The functions $\Sigma_g^x(m,\mathbf{r},t)$ and $\nu^x(m,\mathbf{r},t)$ are called the *Lebesgue cross section* and the *Lebesgue multiplicity* of the reactions of group *x*, respectively*:*

$$\left\{1, \quad \nu_g^x(m,\mathbf{r},t)\right\} \times \Sigma_g^x(m,\mathbf{r},t) = \frac{\partial}{\partial m} \int\limits_{\omega(m)} \left\{1, \quad \nu^x(E,\mathbf{r},t)\right\} \times \Sigma^x(E,\mathbf{r},t) F(E) dE = \quad (29)$$

$$= \sum_k \left[\left\{1, \quad \nu^x(E_k,\mathbf{r},t)\right\} \times \Sigma^x(E_k,\mathbf{r},t) F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ .$$

The Lebesgue cross section is the average cross section calculated on the set of boundary points $E_k^{\pm}$, $S(E_k^{\pm}) = S(m)$ with the weight $F(E_k^{\pm}) \times |\partial E_k^{\pm} / \partial m|$.

In distinction from (27), equality (28) is approximate. Let us calculate the error

$$\Delta(m,\mathbf{\Omega},\mathbf{r},t) = \int_0^m \delta(m,\mathbf{\Omega},\mathbf{r},t)\, dm, \quad (30)$$

$$\delta(m,\mathbf{\Omega},\mathbf{r},t) = \sum_k \left[\left[\Sigma^x(E_k) - \Sigma_g^x(m) \right] \left[\frac{\varphi(E_k)}{F(E_k)} - \psi_g(m) \right] F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ =$$

$$= \sum_k \left[\left[B_g^x(E_k,\mathbf{r},t) - B_g^x(m,\mathbf{r},t) \right] \left[\varepsilon_g(E_k,\mathbf{\Omega},\mathbf{r},t) - \varepsilon_g(m,\mathbf{\Omega},\mathbf{r},t) \right] F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ ,$$

$$\begin{Bmatrix} B_g^x(m,\mathbf{r},t) \\ \varepsilon_g(m,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix} = \sum_k \left[\begin{Bmatrix} B_g^x(E_k) \\ \varepsilon_g(E_k) \end{Bmatrix} F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ = \frac{\partial}{\partial m} \int\limits_{\omega(m)} \begin{Bmatrix} B_g^x(E,\mathbf{r},t) \\ \varepsilon_g(E,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix} F(E) dE .$$

Here we used representations of the cross section (19) and neutron flux (20) on the resonance carrier g in the form of separated variables. The terms $A_g^x(\mathbf{r},t) S(E_k^{\pm})$ and $Z_g(S(E_k^{\pm}),\mathbf{\Omega},\mathbf{r},t)$ are not included in the sum because they take equal values $A_g^x(\mathbf{r},t) S(m)$ and $Z_g(S(m),\mathbf{\Omega},\mathbf{r},t)$ at the summation points $E_k^{\pm}$. The error depends only on the functions of small variation $B_g^x(E_k^{\pm})$ and $\varepsilon_g(E_k^{\pm})$. The quantities $B_g^x(m)$ and $\varepsilon_g(m)$ are the mean values of the functions calculated over the summation points.

For the error δ to be small, it is sufficient that at least one of the following inequalities holds:

$$\begin{Bmatrix} D(m,B) \\ D(m,\varepsilon) \end{Bmatrix} = \sum_k \left[\begin{Bmatrix} [B_g^x(E_k) - B_g^x(m)]^2 \\ [\varepsilon_g(E_k) - \varepsilon_g(m)]^2 \end{Bmatrix} F(E_k) \frac{\partial E_k}{\partial m} \right]_-^+ \ll \begin{Bmatrix} [\Sigma_g^x(m,\mathbf{r},t)]^2 \\ \psi_g^2(m,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix}, \qquad (31)$$

$$\begin{Bmatrix} \Sigma_g^x(m,\mathbf{r},t) \\ \psi_g(m,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix} = \begin{Bmatrix} A_g^x(\mathbf{r},t)S(m) + B_g^x(m,\mathbf{r},t) \\ Z_g(S(m),\mathbf{\Omega},\mathbf{r},t) + \varepsilon_g(m,\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix}.$$

Here $D(m,B)$ and $D(m,\varepsilon)$ are the variances of the distributions $B_g^x(E_k^{\pm})$, $\varepsilon_g(E_k^{\pm})$ on the set of boundary points $E_k^{\pm}$, $k = 1,2,...$ Since $B_g^x(E)$ and $\varepsilon_g(E)$ are functions of small variation (this is a consequence of the separation of variables on the resonance carrier), both inequalities in (31) are simultaneously satisfied. The error in the calculation of rates (28) is negligibly small.

Moreover, the error $\delta(m,\mathbf{\Omega},\mathbf{r},t)$ vanishes if the sums over *k* consist of only one term. This happens in the corridors of monotonicity $S_{gl} < S^t(E) < S_{gl+1}$, $l = 1,2,...$ If we will decrease the width of the resonance carriers (the number K_g in (15)), then the mean cross section $S^t(E)$ will at some point become a monotone function on the entire resonance carrier, the corridors will merge into a single corridor, and the error $\Delta(m,\mathbf{\Omega},\mathbf{r},t)$ will vanish on the entire resonance carrier.

The arrival rates of prompt neutrons $Q_g^x(m,\mathbf{\Omega},\mathbf{r},t)$ and $q_g^x(\psi) = \partial Q_g^x(m) \big/ \partial m$ into the set $E \in \omega(g,m)$ and the unit interval of the measure *m* are calculated in accordance with the general rule of *averaging over the initial states of transitions and summing over final states.* First, we perform the Lebesgue summation over the points of the resonance carrier $E' \in \omega(h,m') \subseteq \omega_h$ from which the primary neutrons depart; m' $(0 \le m' \le m_h)$ is the measure of sets embedded into it. The dependence of the quantities on $(\mathbf{r},t)$ is temporarily omitted:

$$Q_g^x(m,\mathbf{\Omega}) = \int\limits_{\omega(g,m)} dE \int\limits_{4\pi} \frac{d\mathbf{\Omega}'}{2\pi} \int\limits_0^{\infty} \Sigma^x(E')\nu^x(E')W^x(E' \to E,\eta)\varphi(E',\mathbf{\Omega}')dE' = \qquad (32)$$

$$= \int\limits_{\omega(g,m)} dE \int\limits_{4\pi} \frac{d\mathbf{\Omega}'}{2\pi} \sum_h \int\limits_0^{m_h} \sum_k \left[\Sigma^x(E_k')\nu^x(E_k')W^x(E_k' \to E,\eta)\varphi(E_k',\mathbf{\Omega}') \frac{\partial E_k'}{\partial m'} \right]_-^+ dm' \approx$$

$$\approx \int\limits_{\omega(g,m)} dE \int\limits_{4\pi} \frac{d\mathbf{\Omega}'}{2\pi} \sum_h \int\limits_0^{m_h} \Sigma_h^x(m')\nu_h^x(m')W_h^x(m' \to E,\eta)\psi_h(m',\mathbf{\Omega}')\,dm' = \int\limits_0^m q_g^x(\psi)dm,$$

$$\Sigma_h^x(m')\nu_h^x(m')W_h^x(m'\to E,\eta)=\frac{\partial}{\partial m'}\int\limits_{\omega(h,m')}\Sigma^x(E')\nu^x(E')W^x(E'\to E,\eta)F(E')dE'.$$

In the second stage, we perform the summation over the boundary points of the set $E\in\omega(g,m)$ into which the prompt neutrons arrive:

$$q_g^x(\psi)=\frac{\partial}{\partial m}\int\limits_{\omega(g,m)}dE\int\limits_{4\pi}\frac{d\mathbf{\Omega}'}{2\pi}\sum_h\int\limits_0^{m_h}\Sigma_h^x(m')\nu_h^x(m')W_h^x(m'\to E,\eta)\psi_h(m',\mathbf{\Omega}')\,dm'= \quad (33)$$

$$=\int\limits_{4\pi}\frac{d\mathbf{\Omega}'}{2\pi}\sum_h\int\limits_0^{m_h}\Sigma_h^x(m',\mathbf{r},t)\nu_h^x(m',\mathbf{r},t)W_{h\to g}^x(m'\to m,\eta,\mathbf{r},t)\psi_h(m',\mathbf{\Omega}',\mathbf{r},t)dm'.$$

The indicatrix of reactions $W_{h\to g}^x(m'\to m,\eta,\mathbf{r},t)$ is determined by the formula

$$\Sigma_h^x(m',\mathbf{r},t)\,\nu_h^x(m',\mathbf{r},t)\,W_{h\to g}^x(m'\to m,\eta,\mathbf{r},t)= \quad (34)$$

$$=\frac{\partial^2}{\partial m\partial m'}\int\limits_{\omega(g,m)}dE\int\limits_{\omega(h,m')}\Sigma^x(E',\mathbf{r},t)\,\nu^x(E',\mathbf{r},t)W^x(E'\to E,\eta,\mathbf{r},t)F(E')dE'.$$

It describes the transition of neutrons up and down the spectrum variable m within the resonance carrier $h=g$ and the transition of neutrons between different carriers $h\neq g$; the indicatrix satisfies the normalization relation (cf. (6))

$$\sum_g\int\limits_0^{m_g}dm\int\limits_{-1}^{1}W_{h\to g}^x(m'\to m,\eta,\mathbf{r},t)\,d\eta=1,\qquad \eta=\mathbf{\Omega}\mathbf{\Omega}'. \quad (35)$$

The error in calculating the arrival rates of neutrons (33) is the sum of negligibly small errors in calculating the departure rates of neutrons (28).

The transport equation for the Lebesgue neutron flux $\psi_g(m,\mathbf{\Omega},\mathbf{r},t)$ is derived by the direct integration of Eq. (1) over the system of embedded sets $E\in\omega(g,m)$. The integration gives the equation for the cumulative distribution (24). Taking the derivative with respect to the measure *m*, we obtain the desired equation

$$\left[\frac{1}{v_g}\frac{\partial}{\partial t}+\Omega_i\frac{\partial}{\partial r_i}+\Sigma_g^t(m,\mathbf{r},t)\right]\psi_g(m,\mathbf{\Omega},\mathbf{r},t)=q_g^s(\psi)+q_g^{ext}(\psi). \qquad (36)$$

The Lebesgue cross sections Σ_g^x, ν_g^x, and the neutron arrival rate $q_g^s(\psi)$ are given by formulas (29), (33), (34), and $q_g^{ext}(\psi)$, and v_g are given by the formulas

$$q_g^{ext}(\psi)=\sum_\tau\frac{\chi_g^\tau(m,\mathbf{r},t)}{4\pi}\lambda^\tau P^\tau(\mathbf{r},t)+q_g^r(m,\mathbf{\Omega},\mathbf{r},t),\quad \frac{1}{v_g}=\frac{1}{m_g}\int\limits_{\omega_g}\frac{F(E)}{v}dE, \qquad (37)$$

$$\begin{Bmatrix}\chi_g^\tau(m,\mathbf{r},t)\\ q_g^r(m,\mathbf{\Omega},\mathbf{r},t)\end{Bmatrix}=\frac{\partial}{\partial m}\int\limits_{\omega(g,m)}\begin{Bmatrix}\chi^\tau(E,\mathbf{r},t)\\ q^r(E,\mathbf{\Omega},\mathbf{r},t)\end{Bmatrix}dE,\qquad \sum_g\int\limits_0^{m_g}\chi_g^\tau(m,\mathbf{r},t)dm=1.$$

The precursor densities $P^\tau(\mathbf{r},t)$ satisfy the transport equations

$$\left[\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial r_i}+\lambda^\tau\right]P^\tau(\mathbf{r},t)=\int\limits_{4\pi}d\mathbf{\Omega}\sum_g\int\limits_0^{m_g}\nu_g^\tau(m,\mathbf{r},t)\Sigma_g^f(m,\mathbf{r},t)\psi_g(m,\mathbf{\Omega},\mathbf{r},t)dm. \qquad (38)$$

The boundary conditions are derived from conditions (3) by the similar summation

$$\left[\psi_g(m,\mathbf{\Omega},\mathbf{r}_\Gamma)-\psi_g^{ent}(m,\mathbf{\Omega},\mathbf{r}_\Gamma)\right]_{\mathbf{n\Omega}<0}=\int\limits_{\mathbf{n\Omega}'>0}G_g(\mathbf{\Omega}'\to\mathbf{\Omega},m)\psi_g(m,\mathbf{\Omega}',\mathbf{r}_\Gamma)\,d\mathbf{\Omega}'. \qquad (39)$$

The reaction equations (7) are averaged using (28) and (29).

We discuss the averaged equations (36)-(39) later together with the equations of the Lebesgue moment method.

Summation of Lebesgue cross sections. The Lebesgue macroscopic cross sections of reactions are the sums of the Lebesgue microscopic cross sections with the weight of component (nuclide) densities similar to sums (4), (5)

$$\begin{Bmatrix}1, & \nu_h^y(m',\mathbf{r},t)\\ \nu_h^y(m',\mathbf{r},t)W_{h\to g}^y(m'\to m,\eta,\mathbf{r},t)\end{Bmatrix}\times\Sigma_h^y(m',\mathbf{r},t)= \qquad (40)$$

$$=\sum_{x,C}N_C(\mathbf{r},t)\times\begin{Bmatrix}1, & \nu_{C,h}^x(m')\\ \nu_{C,h}^x(m')\,w_{C,h\to g}^x(m'\to m,\eta)\end{Bmatrix}\times\sigma_{C,h}^x(m'),$$

$$\chi_g^{\tau}(m,\mathbf{r},t)\,\nu_g^{\tau}(m,\mathbf{r},t)\,\Sigma_g^{f}(m,\mathbf{r},t) \simeq \sum_C N_C(\mathbf{r},t)\,\chi_{C,g}^{\tau}(m)\,\nu_{C,g}^{\tau}(m)\,\sigma_{C,g}^{f}(m).$$

The cross sections in (40) are calculated by averaging the neutron cross sections (8), (13) over the Lebesgue sets $\omega(g,m)$

$$\left\{1, \quad \nu_{C,g}^{x}(m)\right\} \times \sigma_{C,g}^{x}(m) = \frac{\partial}{\partial m} \int\limits_{\omega(g,m)} \left\{1, \quad \nu_C^{x}(E)\right\} \times \sigma_C^{x}(E) F(E) dE = \qquad (41)$$

$$= \sum_k \left[\left\{1, \quad \nu_C^{x}(E_k)\right\} \times \sigma_C^{x}(E_k) F(E_k) \frac{\partial E_k}{\partial m} \right]_{-}^{+},$$

$$w_{C,h\to g}^{x}(m' \to m, \eta)\,\nu_{C,h}^{x}(m')\,\sigma_{C,h}^{x}(m') =$$

$$= \frac{\partial^2}{\partial m \partial m'} \int\limits_{\omega(g,m)} dE \int\limits_{\omega(h,m')} w_C^{x}(E' \to E, \eta)\,\nu_C^{x}(E')\,\sigma_C^{x}(E')\,F(E') dE',$$

$$\chi_{C,g}^{\tau}(m)\,\nu_{C,g}^{\tau}(m)\,\sigma_{C,g}^{f}(m) = \frac{\partial}{\partial m} \int\limits_{\omega(g,m)} \chi_C^{\tau}(E)\nu_C^{\tau}(E)\sigma_C^{\tau}(E) F(E) dE.$$

Recall that the sets $\omega(g,m)$ are constructed on the basis of the mean cross section $S^t(E)$ (18) calculated for a certain class of applied problems.

Analytical formulas. Let us discuss the storage of the Lebesgue microscopic cross sections (41). The cross sections depend on the spectrum variable (g,m), and they relatively weakly depend on the temperature of the substance $T(\mathbf{r},t)$. Therefore, they can be stored in the files of the cross section library in the form of analytical formulas as in the method of storing the initial neutron cross sections.

Due to the partition of the energy scale into resonance carriers described in Section 1, each carrier contains the resonances of a certain component *C* and the background cross section of the other components. Therefore, the Lebesgue cross sections of the other components $D \neq C$ can be expanded on the carrier $\omega_g(C)$ into rapidly converging Fourier series in trigonometric or power polynomials of the variable *m*. Only the first few expansion coefficients need to be stored. These coefficients may slightly depend on the temperature.

Now we derive analytical formulas for the Lebesgue cross sections of the component *C* on the carrier of its own resonances $\omega_g(C)$. Assume that within the resonance carrier and its vicinity, the resonances are equally spaced at distance *d* from

each other and have the same strength and half-widths Γ, Γ^x. In the theory of photon transfer, the model of identical resonances is known as the Elsasser approximation [49]

$$E_r = E_g + \Delta + rd\,, \qquad e = [E - E_g - \Delta]/d \qquad \gamma_g = \Gamma/d\,.$$

Here E_r is the position of the resonance with the index *r*, E_g is the center of the carrier, Δ is the shift, d is the period, e is the dimensionless energy, and γ_g is the dimensionless half-width of the resonance in the vicinity of the carrier ω_g.

Let us perform the summation over the resonances in the neutron cross sections (8) using the formulas (e.g., see [48, p. 652])

$$\frac{1}{\pi}\sum_{r=-\infty}^{\infty}\frac{\gamma_g}{[e-r]^2+\gamma_g^2} = \frac{i}{2\pi}\sum_{r=-\infty}^{\infty}\left[\frac{1}{e-r+i\gamma_g} - \frac{1}{e-r-i\gamma_g}\right] = \frac{\sinh 2\pi\gamma_g}{\cosh 2\pi\gamma_g - \cos 2\pi e},$$

$$\frac{1}{\pi}\sum_{r=-\infty}^{\infty}\frac{e-r}{[e-r]^2+\gamma_g^2} = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\left[\frac{1}{e-r+i\gamma_g} + \frac{1}{e-r-i\gamma_g}\right] = \frac{\sin 2\pi e}{\cosh 2\pi\gamma_g - \cos 2\pi e}.$$

Thus, within the resonance carrier $E \in \omega_g$, we have

$$\begin{Bmatrix}\sigma_C^t(E)\\ \sigma_C^x(E)\end{Bmatrix} = \begin{Bmatrix}\sigma_g^p\\ 0\end{Bmatrix} + H_g \times \begin{Bmatrix}p_g\theta_g(E) + q_g\chi_g(E)\\ u_g^x\,\theta_g(E), \quad x \neq t, el\end{Bmatrix}, \qquad \sigma_C^{el}(E) = \sigma_C^t(E) - \sum_{x\neq el}\sigma_C^x(E)\,,$$

$$\begin{Bmatrix}\theta_g(E)\\ \chi_g(E)\end{Bmatrix} = \frac{1}{\cosh 2\pi\gamma_g - \cos 2\pi e} \times \begin{Bmatrix}\sinh 2\pi\gamma_g\\ \sin 2\pi e\end{Bmatrix}, \qquad \frac{1}{d}\int_E^{E+d}\begin{Bmatrix}\theta_g(E')\\ \chi_g(E')\end{Bmatrix}dE' = \begin{Bmatrix}1\\ 0\end{Bmatrix},$$

$$H_g \times \left\{1,\ u_g^x,\ p_g,\ q_g\right\} = 4\pi^2\lambda\!\!\!^{-2}\sum_{l,J} n_{lJ}\,g_{lJ}\,\frac{\Gamma^{el}}{d} \times \left\{1,\ \Gamma^x/\Gamma,\ \cos 2\phi_l,\ \sin 2\phi_l\right\}.$$

Here H_g is the resonance strength per unit energy interval, $\theta_g(E)$ and $\chi_g(E)$ are the symmetric and antisymmetric resonance profiles determined on the cross section period, p_g and q_g are the phase shift coefficients, and u_g^x is the partial coefficient of the reaction *x*. The other quantities were described in the comments to (8). On the pe-

riod d, the equation $\sigma_C^t(E)=\sigma$ has two roots E_1 and E_2. The measure μ of the set $\omega(g,\mu)=\{E:\sigma_C^t(E)<\sigma\}$ is $\mu=d-[E_2-E_1]$. Put $\mu/d\approx m/m_g$. In this model, we can derive for the Lebesgue sections the analytical formulas

$$\begin{Bmatrix}\sigma_{C,g}^t(m)\\ \sigma_{C,g}^x(m)\end{Bmatrix}\approx\begin{Bmatrix}\sigma_g^p\\ 0\end{Bmatrix}+H_g\times\begin{Bmatrix}p_g\theta_g(m)+q_g\chi_g(m)\\ u_g^x\theta_g(m),\quad x\neq t,el\end{Bmatrix},\qquad 0\le m\le m_g,\qquad(42)$$

$$\theta_g(m)=\frac{\sinh 2\pi\gamma_g}{\cosh 2\pi\gamma_g+\cos(\pi m/m_g)},\qquad \chi_g(m)\approx-\frac{q_g\cos(\pi m/m_g)}{[1+p_g]\sinh 2\pi\gamma_g}.$$

The same formulas are obtained if the resonance carrier contains only the left or only the right wings of the resonances (intervals of growth or decrease of the cross section $\sigma_C^t(E)$). Formulas (42) can be extended to the general case of arbitrary resonances if the parameters $\sigma_g^p(m)$, $p_g(m)$, $q_g(m)$, and $u_g^x(m)$ are considered as functions of the variable *m*, and the parameters $\gamma_g(m,T)$ and $h_g(m,T)$ as functions of *m* and temperature. The cross sections of a nuclide *A* that appear in the component *C* can also be represented by formulas (42). Note that these formulas are valid in the entire resonance region, including the region of unresolved resonances.

To store the indicatrices $w_{C,h\to g}^x(m'\to m,\eta)$ and the multiplicities of reaction products $\nu_{C,h}^x(m')$, a tabulated data representation can be used.

3. The Lebesgue moment method

The numerical solution to Eq. (36) is sought on a finite grid in the new spectrum variable (g,m_i), $i=0,...,Ng$, $1\le g\le gg$. Let us raise the question of selecting the optimal grid m_i with a fixed total number of nodes Ng that would describe the evolution of the neutron field in the object with minimal (in a certain sense) error. The grid optimization problem and the problem of the neutron flux reconstruction in the *E*-space after finding the Lebesgue neutron flux is solved using the Lebesgue moment method.

Expansion of the neutron flux. We will seek the solution to the transport equation (36) on sets $\omega(g,m)\subset\omega_g$, $1\le g\le gg$ as a sum of the main function ψ_{Ng} and the small correction ε_{Ng}

$$\psi_g(m,\mathbf{\Omega},\mathbf{r},t)=\psi_{Ng}(m,\mathbf{\Omega},\mathbf{r},t)+\varepsilon_{Ng}(m,\mathbf{\Omega},\mathbf{r},t),\qquad(43)$$

$$\psi_{Ng}(m,\mathbf{\Omega},\mathbf{r},t)=\sum_{n=0}^{Ng}\frac{P_n(\xi(m))}{d_n^2}\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t),$$

$$\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t)=\int_{-1}^{1}\rho(\xi)P_n(\xi)\psi_g(m,\mathbf{\Omega},\mathbf{r},t)\,d\xi,\qquad \int_{-1}^{1}\rho(\xi)\,P_n(\xi)P_k(\xi)\,d\xi=d_n^2\,\delta_{nk}.$$

$$\xi(m)=\frac{e^{-S_{0g}L_g}+e^{-S_gL_g}-2e^{-S(m)L_g}}{e^{-S_{0g}L_g}-e^{-S_gL_g}},\qquad \begin{matrix}0\le m\le m_g,\\ S_{0g}\le S(m)\le S_g,\end{matrix}\quad -1\le\xi(m)\le 1. \qquad (44)$$

The main function is a segment of the expansion of the Lebesgue neutron flux $\psi_g(m)$ in a series in a system of orthogonal polynomials $P_n(\xi)$ $(n=0,1,\ldots,)$ on the interval $-1\le\xi(m)\le 1$, Ng is the expansion order, $\rho(\xi)$ is the weight function of the polynomial system, and d_n are the normalization factors of the polynomials. The expansion coefficients $\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t)$ will be called *Lebesgue moments*. The argument of the polynomials $\xi(m)=\xi(S(m))$ is a complicated function that monotonically increases in the argument $S=S(m)$, does not decrease in *m*, and takes the values $\xi(0)=-1$ and $\xi(m_g)=1$ on the interval endpoints. In the original *E*-space, the expansion (43) corresponds to the following expansion of the neutron flux within the resonance carrier:

$$\varphi(E,\mathbf{\Omega},\mathbf{r},t)=F(E)\sum_{n=0}^{Ng}\frac{P_n(\xi(E))}{d_n^2}\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t)+\varepsilon_{Ng}(E,\mathbf{\Omega},\mathbf{r},t), \qquad (45)$$

$$\xi(E)=\xi(S(E))=\frac{e^{-S_{0g}L_g}+e^{-S_gL_g}-2e^{-S(E)L_g}}{e^{-S_{0g}L_g}-e^{-S_gL_g}},\qquad \begin{matrix}E\in\omega_g,\\ G_1\le g\le G_2.\end{matrix}$$

This can be easily verified by substituting (45) into (25).

If we substitute (44) into (43) and collect similar terms, then the main part of the flux ψ_{Ng} (or φ_{Ng}) can be written as a linear combination of the exponential functions $\exp(-nS(u)L_g)$, $0\le n\le Ng$, $u=E,m$. The factor n in the exponent specifies the partial rate of neutron departure from the beam associated with this exponential function, $S(u)$ $(u=E,m)$ is the mean cross section (18), and $L=L_g$ is the dimensional parameter adjusted for each specific problem (class of problems), which has the order of the object size. Grouping the exponential functions into polynomials is a way of orthogonalizing the basis functions. The inclusion of a new exponential func-

tion in the basis is equivalent to adding one more higher order harmonic to ψ_{Ng}. The choice of exponential functions as basis functions is motivated by the form of the solution to the particle transport equation (21), which is the sum of exponential functions.

Another set of basis functions that is suitable for describing the effects of inversion of resonances in neutron spectra is the system of rational functions $1/[1+S(u)L_g]^n$, $n=0,1,2,\ldots$, $u=E,m$. To use this system, it is sufficient to carry out a rational approximation of the exponential function $\exp(-S(u)L_g)\approx 1/[1+S(u)L_g]$. Then the argument of the polynomials in expansions (43), (45) is replaced by the function

$$\xi(u)=\xi(S(u))=1-2\frac{S_g-S(u)}{S_g-S_{0g}}\frac{1+S_{0g}L_g}{1+S(u)L_g}, \qquad \begin{matrix}-1\le\xi(u)\le 1,\\ u=E,m.\end{matrix} \tag{46}$$

Probably, there are other convenient systems of basis functions for describing the inversion of resonances. The name *Lebesgue moments* reflects the dependence of the basis functions of expansions (43), (44) and (43), (46) on the magnitude of the cross section for the interaction of neutrons with matter.

Assume that the principal part of expansions (43), (45) converges to the neutron flux at the point $(\mathbf{\Omega},\mathbf{r},t)$ on the resonance carriers $E\in\omega_g$ as Ng increases and the carrier width K_g decreases (see (15)), e.g. in the norm L_2: $\|\varepsilon_{Ng}(u)\|_2\to 0$, $u=E,m\in\omega_g$. The convergence rate depends on the fortunate choice of the basis functions (44), (46) for describing the resonance inversion, on the choice of the scaling parameter L_g in these functions, and on the problem to be solved—material and geometry of the object.

Then, the small error made in deriving the transport equation (36) for the Lebesgue neutron flux rapidly tends to zero. Indeed, the main expansion function $\varphi_{Ng}(E)$ depends on energy E only through the mean macroscopic cross section $S(E)$. Therefore, the variance $D(m,\varepsilon_{Ng})$ in the second inequality in (31) tends to zero. This accelerates the ultimate convergence of equations (36) and (1) in the norm L_2 due to the simultaneous fulfillment of inequalities (31). The convergence rate is much higher than the convergence rate of the equation of the multigroup approximation to (1).

Reconstruction of the neutron spectrum. If the Lebesgue moments $\Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t)$ $(0\le n\le Ng,\ 1\le g\le gg)$ are known, then the neutron flux in the object in the approximation $\varepsilon_{Ng}\approx 0$ is restored using formulas (43), (45) in both (g,m)-space and *E*-space (if required).

Evaluation of integrals. The Lebesgue moments give additional Lebesgue cross sections that simplify the calculations of the neutron field functionals, such as the

neutron flux on the resonance carrier *g*, the neutron departure rates from the carrier, the arrival rates of prompt neutrons, precursors (see (2)), and nuclides (see (7)) in the reactions of group *x*

$$\begin{Bmatrix} \varphi_g(\mathbf{\Omega},\mathbf{r},t), & I_g^x(\mathbf{\Omega},\mathbf{r},t) \\ & J_g^x(\mathbf{\Omega},\mathbf{r},t) \end{Bmatrix} = \int\limits_{\omega_g} \begin{Bmatrix} 1, & \Sigma^x(E,\mathbf{r},t) \\ \nu^x(E,\mathbf{r},t)\Sigma^x(E,\mathbf{r},t) \end{Bmatrix} \times \varphi(E,\mathbf{\Omega},\mathbf{r},t)\,dE \approx \qquad (47)$$

$$\approx \int\limits_0^{m_g} \begin{Bmatrix} 1, & \Sigma_g^x(m,\mathbf{r},t) \\ \nu_g^x\Sigma_g^x(m,\mathbf{r},t) \end{Bmatrix} \times \psi_{Ng}(m,\mathbf{\Omega},\mathbf{r},t)dm = \sum_{n=0}^{Ng} \frac{F_g^{(n)}}{d_n^2} \begin{Bmatrix} 1, & \Sigma_g^{x(n)}(\mathbf{r},t) \\ \nu_g^{x(n)}\Sigma_g^{x(n)}(\mathbf{r},t) \end{Bmatrix} \times \Psi_g^{(n)}(\mathbf{\Omega},\mathbf{r},t).$$

Here $F_g^{(n)} = F_g^{(n)}(L_g)$ is the weight of the *n*th order Lebesgue moment on the carrier. The *n*th order macroscopic cross section $\Sigma_g^{x(n)}(\mathbf{r},t)$ and the multiplicity $\nu_g^{x(n)}(\mathbf{r},t)$ in reactions *x* are given by

$$\begin{Bmatrix} 1, & \Sigma_g^{x(n)}(\mathbf{r},t) \\ \nu_g^{x(n)}(\mathbf{r},t)\Sigma_g^{x(n)}(\mathbf{r},t) \end{Bmatrix} \times F_g^{(n)} = \int\limits_{\omega_g} P_n(\xi(E)) \begin{Bmatrix} 1, & \Sigma^x(E,\mathbf{r},t) \\ \nu^x(E,\mathbf{r},t)\Sigma^x(E,\mathbf{r},t) \end{Bmatrix} F(E)\,dE = \qquad (48)$$

$$= \int\limits_0^{m_g} P_n(\xi(m)) \times \begin{Bmatrix} 1, & \Sigma_g^x(m,\mathbf{r},t) \\ \nu_g^x(m,\mathbf{r},t)\Sigma_g^x(m,\mathbf{r},t) \end{Bmatrix} dm\,.$$

The weight of the zero-order moment is equal to the integral of the weighting spectrum $F_g^{(0)} = F_g$. The macroscopic cross section of order $n = 0$ corresponds to the macroscopic cross section of the multigroup approximation.

The differential arrival rates of prompt neutrons $Q_g^x(m,\mathbf{\Omega},\mathbf{r},t)$ and $q_{Ng}^x(\psi)$ (32) into the set $E \in \omega(g,m)$ and into the unit interval of the variable *m* are

$$Q_g^x(m,\mathbf{\Omega},\mathbf{r},t) \approx \int\limits_0^m q_{Ng}^x(\psi)\,dm, \qquad 0 \le m \le m_g, \qquad (49)$$

$$q_{Ng}^x(\psi) = \int\limits_{4\pi} \frac{d\mathbf{\Omega}'}{2\pi} \sum_h \sum_{n=0}^{N} \frac{F_h^{(n)}}{d_n^2} \Sigma_h^{x(n)}(\mathbf{r},t)\nu_h^{x(n)}(\mathbf{r},t)W_{h\to g}^{x(n)}(m,\eta,\mathbf{r},t)\Psi_h^{(n)}(\mathbf{\Omega}',\mathbf{r},t),$$

$$\nu_h^{x(n)}(\mathbf{r},t)W_{h\to g}^{x(n)}(m,\eta,\mathbf{r},t)\Sigma_h^{x(n)}(\mathbf{r},t)F_h^{(n)} = \tag{50}$$

$$= \frac{\partial}{\partial m}\int\limits_{\omega(g,m)} dE \int\limits_{\omega_h} P_n(\xi'(E'))\Sigma^x(E',\mathbf{r},t)\nu^x(E',\mathbf{r},t)W^x(E'\to E,\eta,\mathbf{r},t)F(E')dE',$$

$$\sum_g \int\limits_0^{m_g} dm \int\limits_{-1}^{1} W_{h\to g}^{x(0)}(m,\eta,\mathbf{r},t)\,d\eta = 1, \qquad (n=0).$$

Here $W_{h\to g}^{x(n)}(m,\eta,\mathbf{r},t)$ ($\eta = \mathbf{\Omega\Omega'}$) is the *n*th order Lebesgue moment of the indicatrix of reactions *x* (not to be confused with the angular moments of the indicatrix, see (9)).

The moments of cross sections (48), (50) are calculated from the Lebesgue microscopic cross sections of the components by formulas (40), (41).

Calculation of the Lebesgue moments. The distribution moments can be calculated following their formal definition (43) using non-optimized quadrature formulas, such as the rectangle, trapezoid, etc. rules. In addition to the primitive method, two direct methods can be used for the calculation of moments. The first method is to write a system of equations for the moments and solve it. The second method uses optimized quadrature formulas such as Gaussian formulas. The solution to the particle transport equation is sought on a finite grid consisting of *N+1* nodes of the quadrature formula. This grid is optimal in the sense that it provides the ability to accurately calculate M ($M \le N$) power moments if the distribution is continuous in the given variable and M is the quality parameter of the formula. In the case $M = N$, both methods are equivalent and the grid consisting if *N+1* nodes is a necessary but not sufficient condition for the exact calculation of the moments of particle distribution in the object. To calculate the Lebesgue moments, we will further follow a version of quadrature formulas.

Remark. In the theory of particle transport, direct methods of calculation of the angular moments of distributions over the flight directions of particles are known as the method of spherical harmonics and as the Vick–Chandrasekhar method of discrete ordinates, and also the optimized Carlson's Sn-method (in multidimensional problems).

The Gauss–Christoffel quadrature formula is the formula for the approximate evaluation of integrals [47]

$$\int\limits_{-1}^{1} \rho(\xi)Y(\xi)\,d\xi \approx \sum_{i=0}^{N} \lambda_i\, Y(\xi_i), \qquad \sum_{i=0}^{N} \lambda_i = \int\limits_{-1}^{1} \rho(\xi)\,d\xi. \tag{51}$$

Here $Y(\xi)$ is an integrable function from a certain class of functions, $\rho(\xi)$ is a given

weight function that emphasizes the features of the behavior of functions of this class on parts of the interval $-1 \le \xi \le 1$, ξ_i and λ_i are the nodes and weights of the formula, and $Y(\xi_i)$ are the values of the integrand at the nodes. The nodes and weights are chosen so that the approximate equality (51) becomes exact if $Y(\xi)$ is an arbitrary polynomial of degree not higher than $2N+1$. The weights ξ_i $(i = 0,1,..., N)$ are the roots of the equation $P_{N+1}(\xi_i) = 0$, where P_{N+1} belongs to the system of orthogonal polynomials on the interval $-1 \le \xi \le 1$ with the weight $\rho(\xi)$ (see (43)). The weights λ_i called the Christoffel coefficients are calculated by

$$\frac{1}{\lambda_i} = \frac{a_N P_N(\xi_i)}{a_{N+1} d_N^2} \frac{dP_{N+1}(\xi_i)}{d\xi} = \sum_{n=0}^{N} \frac{P_n^2(\xi_i)}{d_n^2}, \qquad i = 0,1,..., N,$$

where a_N is the leading coefficient of the polynomial P_N.

Consider expansion (43). The Lebesgue flux $\psi_g(m)$ and its principal part $\psi_{Ng}(m)$ have identical moments $\Psi_g^{(n)}$, $0 \le n \le N$. Since $Y(\xi) = P_n(\xi)\psi_{Ng}(m)$ for $0 \le n \le N$ is a polynomial of degree not higher than $2N$, its substitution into (51) gives an exact formula for the calculation of moments

$$\Psi_g^{(n)}(\mathbf{\Omega}, \mathbf{r}, t) = \int_{-1}^{1} \rho(\xi) P_n(\xi) \psi_g(m(\xi), \mathbf{\Omega}, \mathbf{r}, t)\, d\xi = \sum_{i=0}^{N} \lambda_i P_n(\xi_i) \psi_g(m_i, \mathbf{\Omega}, \mathbf{r}, t). \quad (52)$$

To use (52), the solution $\psi_g(m_i, \mathbf{\Omega}, \mathbf{r}, t)$ to the neutron transport equation (36) is sought on the optimal grid $m_i = m(\xi_i)$ $(i = 0,1,..., N)$ or on a grid in which the optimal grid is embedded, or on a close grid that admits high-order interpolation. The neutron arrival rate $q_g^s(\psi)$ can be calculated by formula (49) (instead of (33)).

To find the nodes m_i of the optimal grid, we invert (44) and (46):

$$S_i(m_i) L_g = \ln \frac{2}{[1-\xi_i] e^{-S_{0g} L_g} + [1+\xi_i] e^{-S_g L_g}} \quad \text{is the basis of exponential functions,} \quad (53)$$

$$S_i(m_i) = \frac{[1+\xi_i] S_g + [1-\xi_i] S_{0g} + 2 S_{0g} S_g L_g}{2 + [1-\xi_i] S_g L_g + [1+\xi_i] S_{0g} L_g} \quad \text{is the basis of rational functions.}$$

Example. Suppose that the Chebyshev polynomials that are orthogonal on the interval $-1 \le \xi \le 1$ with the weight $\rho(\xi) = [1-\xi^2]^{-1/2}$ are used to orthogonalize the ba-

sis functions of the neutron spectrum (44), (46) in expansion (43):

$$P_n(\xi) = \cos(n \arccos \xi), \qquad \int_{-1}^{1} \frac{P_n(\xi) P_m(\xi)}{\sqrt{1-\xi^2}} d\xi = d_n^2 \, \delta_{nm}, \qquad d_n^2 = \frac{\pi}{2}[1 + \delta_{n0}].$$

Then, the nodes, the values of polynomials, and the weights in the quadratic formula (51) are

$$\xi_i = \cos \frac{\pi[2i+1]}{2[N+1]}, \qquad P_n(\xi_i) = \cos \frac{\pi n[2i+1]}{2[N+1]}, \qquad \lambda_i = \frac{\pi}{N+1}, \qquad 0 \le i, n \le N .$$

The formula for calculating the Lebesgue moments (52) takes the form

$$\Psi_g^{(n)}(\mathbf{\Omega}, \mathbf{r}, t) = \frac{\pi}{N+1} \sum_{i=0}^{N} \cos \frac{\pi n[2i+1]}{2[N+1]} \psi_g(m_i, \mathbf{\Omega}, \mathbf{r}, t) .$$

Equations (53) for finding the optimal grid nodes m_i are

$$S_i(m_i) L_g = -\ln\left(e^{-S_{0g} L_g} \sin^2 \frac{\pi[2i+1]}{4[N+1]} + e^{-S_g L_g} \cos^2 \frac{\pi[2i+1]}{4[N+1]} \right), \quad \text{basis of exponential functions,}$$

$$S_i(m_i) = \frac{S_g \cos^2 \frac{\pi[2i+1]}{4[N+1]} + S_{0g} \sin^2 \frac{\pi[2i+1]}{4[N+1]} + S_{0g} S_g L_g}{1 + S_g L_g \sin^2 \frac{\pi[2i+1]}{4[N+1]} + S_{0g} L_g \cos^2 \frac{\pi[2i+1]}{4[N+1]}}, \quad \text{basis of rational functions.}$$

These equations are solved for m_i ($i = 0,1,...,N$) at the stage of data preparation for the transport calculation together with the preparation of the Lebesgue macroscopic sections. The macroscopic cross sections can be calculated directly from the nuclear data library files or from the files of the Lebesgue cross section library.

Discussion

The equations of the multigroup approximation in the resonance region have only a qualitative correspondence to the original neutron transport equations (1)–(3), (7) . Indeed, consider the «exact» group constants averaged with the neutron flux weight using the Riemann scheme. The exact group constants depend on the direction of the neutron flight $\mathbf{\Omega}$. Their variations with respect to $\mathbf{\Omega}$ can be as high as tens or even

hundreds of percent, especially in the vicinity of material interfaces, where the neutron flux (21) varies strongly in both arguments E, $\mathbf{\Omega}$ (e.g., see [50]). The group constants of the multigroup approximation are independent of $\mathbf{\Omega}$ (see [37], [38], [41]). This is because the true neutron flux is replaced by the approximate isotropic weighting spectrum due to averaging.

The equations of the Lebesgue averaging method and the Lebesgue moment method approximate the original neutron transport equations to high accuracy. The Lebesgue neutron cross sections lose little information about the neutron cross sections and have accuracy comparable to the accuracy of data in nuclear data libraries. Like the original neutron cross sections, they are independent of the direction of the neutron flight $\mathbf{\Omega}$.

The averaged equations (36)–(39), (49), and (52) describe such effects in neutron spectra as interference of potential and resonance scattering, increase of the fission neutron multiplicity $\nu^f(E)$ in the core of resonances, the inversion of narrow and wide resonances, the block-effect (decrease in the resonance self-shielding of the neutron field near the interfaces of different materials), and the spatial heterogeneity of fuel burn-up and nuclide formation in nuclear reactors. The ability to approximately reconstruct the neutron spectra makes the Lebesgue method a good tool for the numerical support of research in the field of neutron diagnostics of objects.

The cross sections of equations and the neutron flux are highly monotone in the spectrum variable (g,m). The equations have a structure similar to the structure of the original equations in the *E*-space. Therefore, they can be solved numerically on a coarse grid using the known methods for calculating the spatial–angular distribution of neutrons, direct methods for solving the transport equation (Sn-method, method of characteristics), the method of spherical harmonics, and the Monte Carlo method.

The zero-order moment method is a multigroup approximation in which groups are resonance carriers. This ensures a smooth transition from the moment method $N > 0$ to the multigroup approximation when passing through the boundary of the resonance region into the region of slow cross section variation.

The transitions of neutron down and up in the measure *m* within the resonance carrier ($h = g$) and transitions down and up in the carrier index ($h \neq g$) require iterations. These iterations are similar to the iterations that are performed in the thermalization region of neutrons when the multigroup approximation is used. In the Lebesgue moment method, these iterations are reduced to calculating the Lebesgue moments.

The implementation of the proposed method requires writing a computer code for preparing the Lebesgue neutron cross sections. The cross sections are calculated by averaging the microscopic cross sections using a slightly more complicated algorithm than the group averaging algorithm.

The effect of resonance inversion is inherent not only in neutron spectra but also in wide spectral distributions of photons, including the intrinsic thermal radiation photons emitted by an inhomogeneously heated gas or plasma. The radiation spec-

trum in the line cores is formed by local heat sources, and in the line wings it is formed by the hottest sources of the object due to the strong dependence of the intensity of the sources on temperature $\sim T^4$. Even small temperature variations, for example by a factor of three, result in strong inversion of the lines.

At the time of writing, the effectiveness of the Lebesgue moment method in a numerical simulation of neutron transport problems has not yet been tested. The results of test computations on the benchmark problems of thermal radiation transfer in a hot gas can be found in [13].

The solution of test problems, including those for heterogeneous objects consisting of different materials, showed the following. In the case of grouping the basis functions in the Chebyshev polynomials, expansion (43) rapidly converges to the exact solution; it is sufficient to use the terms of order $N = 4 \div 6$. In the case of grouping in the Legendre polynomials, this expansion converges somewhat slower—with $N = 6 \div 8$. To analyze the convergence rate, we analyzed, in addition to the error, the distribution of "energy" over the expansion harmonics

$$\int_{-1}^{1} \psi_g^2(m(\xi), \mathbf{\Omega}, \mathbf{r}, t) d\xi \approx \int_{-1}^{1} \psi_{Ng}^2(m(\xi), \mathbf{\Omega}, \mathbf{r}, t) d\xi = \sum_{n=0}^{N} \frac{1}{d_n^2} [\Psi_g^{(n)}(\mathbf{\Omega}, \mathbf{r}, t)]^2 .$$

A good property of the method is that the convergence rate weakly depends on changes of the scale parameter of the basis functions L_g (the characteristic size of the object) in a wide range. This parameter was decreased and increased relative to an optimal value by a factor of up to ten. The error and «energy» of the expansion tail began to grow only when approaching the boundaries of this range.

Assume that, in order to separate the resonance structure of neutron cross sections, it is sufficient to divide the resonance region into $30 \div 200$ resonance carriers, depending on the type of the problem and the accuracy requirements. The use of 4–6 Lebesgue moments in each carrier gives 100–1000 grid points of the spectrum (g, m). As a result of solving the averaged transport equations, the accuracy of the description of the neutron spectrum can be comparable with the accuracy of the detailed Monte Carlo calculations and the accuracy of nuclear data.

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