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Recommended form of bibliographic references: Ovchinnikov M. Y., Hao-Chi Chang, Mirer S. A., Tkachev S.S., Roldugin D.S. Dynamical model of a satellite with controllable solar array // Keldysh Institute Preprints. 2015. No. 26. 19 p. URL: http://library.keldysh.ru/preprint.asp?id=2015-26&lg=e

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Dynamical Model of a Satellite with Controllable Solar Array

Moscow — 2014

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Динамическая модель спутника с управляемой солнечной панелью

Рассматривается спутник, имеющий одну солнечную панель, закрепленную в трехстепенном управляемом шарнире. Спутник считается твердым телом, а панель состоит из трех частей, соединенных шарнирами. В работе получены уравнения движения такой системы.

Ключевые слова: сложная структура, солнечная панель, трехстепенной подвес, динамические уравнения

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Dynamical model of a satellite with controllable solar array

Satellite with 3DOF solar panel is considered. Satellite is a rigid body while solar panel consists of three parts connected by 1 DOF hinges. The work is aimed at the satellite-panel system attitude dynamical model construction.

Key words: complex structure, solar panel, 3DOF hinge, equations of motion

The work was supported by the contract NSPO-S-103132, RFBR grants №№ 13-01-00665, 14-01-31313, 15-31-20058.

Introduction

Present paper deals with elaboration of dynamical model of a satellite bus (SB) with controllable solar array (SA). The model is designed for the SA deployment stage study. SA should be deployed into operating position after the separation from a launch vehicle. SA is connected to SB using 3 DOF hinge which is approximated by three independent 1 DOF pointlike hinges. Each hinge is controllable. SA itself is modelled as comprising of three parts with 1 DOF each. This is aimed at rough flexibility approximation. SA parts hinges are also controllable, though the torque implemented is actually due to the flexibility (damping-spring torques). SB is controlled using reaction wheels with friction. The applied approach is based on d'Alambert principle [1] and demonstrates high efficiency in studying of multibody systems. Method of motion equations derivation, as well as variables and assumptions are the same as in [2].

1. Equations of motion

Dynamical system consists of a satellite bus with center of mass O_1 and solar array (see Fig. 1). SA itself consists of three parts. Each part is considered to be a rigid body with center of mass O_2, O_3, O_4 consequently. They are connected using 1 DOF hinges P_4, P_5. SA is connected to SB with 3 DOF hinge. It is modelled as three 1 DOF hinges P_1, P_2, P_3. These are connected with weightless rods P_1P_2 and P_2P_3 (their lengths considered being zero).

Each mechanical system may be interpreted as a set of material points. Each point has a mass m_μ and radius vector $\mathbf{r}_\mu$ relative to any point C in inertial space. This point may be the Earth center, for example. Each point satisfies equation

$$m_\mu \ddot{\mathbf{r}}_\mu = \mathbf{F}_\mu + \mathbf{R}_\mu$$

where $\mathbf{F}_\mu$ and $\mathbf{R}_\mu$ are resultant factors of active (external) forces and reactions affecting the point. Let the constraints imposed on the system be ideal, i.e. elementary work of reaction force on virtual displacement is equal to zero. Then at any virtual displacement $\delta\mathbf{r}_\mu$ compatible with the constraints the following relation holds

$$\sum_\mu \mathbf{R}_\mu \delta\mathbf{r}_\mu = 0 .$$

This leads to general dynamics equation

$$\sum_\mu \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu \right) \delta\mathbf{r}_\mu = 0$$

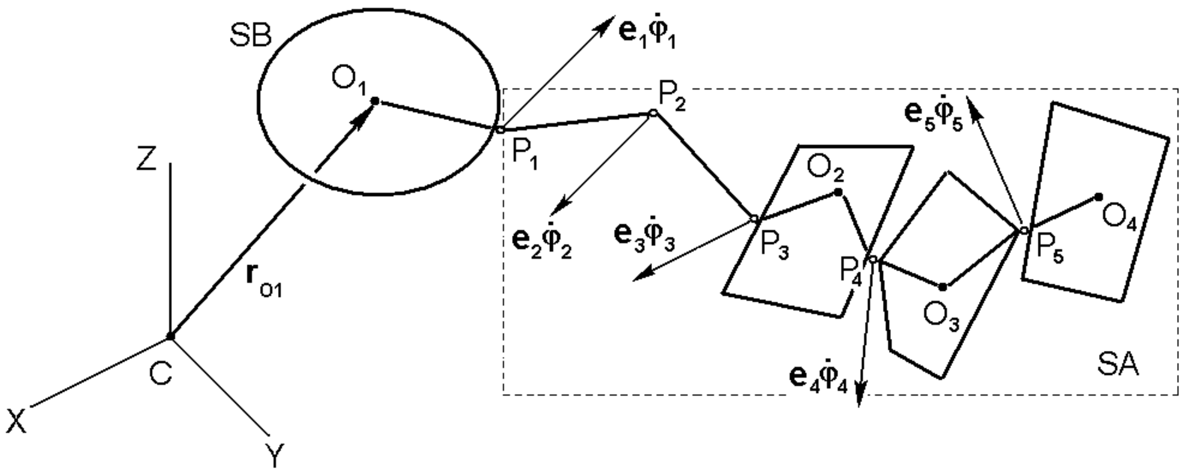

Fig. 1. General SB and SA view

Sum separately over points of the first body (SB) $\left(\sum^{(1)}\right)$ and three parts of SA $\left(\sum^{(2)}\right)$, $\left(\sum^{(3)}\right)$, $\left(\sum^{(4)}\right)$. The last expression is rewritten as

$$\sum_{i=1}^{4} \left[\sum_\mu^{(\mathrm{i})} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu \right) \delta\mathbf{r}_\mu \right] = \sum_{i=1}^{5} M_{ui} \delta\varphi_i . \qquad (1)$$

Here $\delta\varphi_i$ is virtual change of the slewing angle in the hinge P_i; M_{ui} are control torques in the hinges ($i = 1, 2, 3, 4, 5$). Introduce following notations:

$m_{\mathrm{O}i} = \sum^{(i)} m_\mu$ for the bodies masses ($i = 1, 2, 3, 4$);

$\mathbf{r}_{Oi}$ for radius-vectors of the bodies centers of mass ($i = 1, 2, 3, 4$);

$\mathbf{a}_1 = \mathbf{O}_1\mathbf{P}_1$, $\mathbf{c}_2 = \mathbf{P}_3\mathbf{O}_2$, $\mathbf{a}_2 = \mathbf{O}_2\mathbf{P}_4$, $\mathbf{c}_3 = \mathbf{P}_4\mathbf{O}_3$, $\mathbf{a}_3 = \mathbf{O}_3\mathbf{P}_5$, $\mathbf{c}_4 = \mathbf{P}_5\mathbf{O}_4$;

$\mathbf{e}_i$ for unit vector along the hinge axis P_i ($i = 1, 2, 3, 4, 5$).

Geometry of notations is present in Fig. 2.

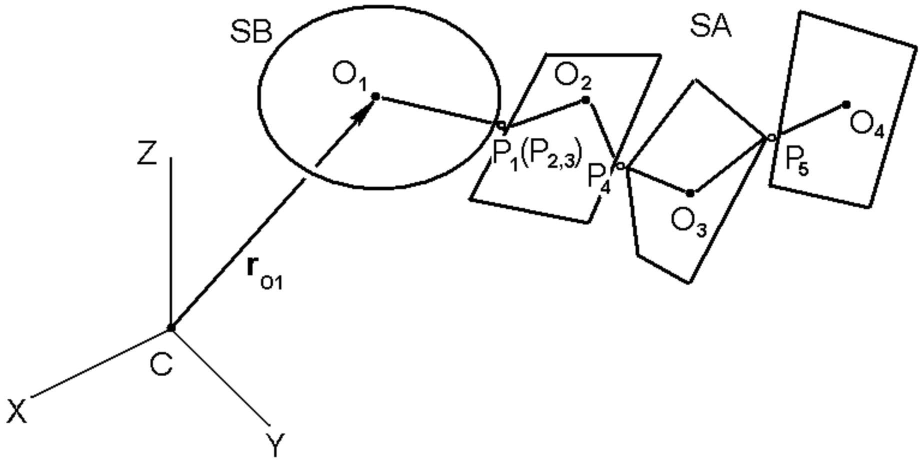

Fig. 2. Kinematics of a SB with three parts of SA

Then radius-vector and virtual displacement of the μ_{th} particle belonging to the first body (SB) are written as

$$\mathbf{r}_{\mu} = \mathbf{r}_{\mathrm{O1}} + \left(\mathbf{r}_{\mu} - \mathbf{r}_{\mathrm{O1}}\right), \qquad \delta\mathbf{r}_{\mu} = \delta\mathbf{r}_{\mathrm{O1}} + \delta\boldsymbol{\theta}_1 \times \left(\mathbf{r}_{\mu} - \mathbf{r}_{\mathrm{O1}}\right)$$

where $\delta\boldsymbol{\theta}_1$ is SB virtual rotation (Fig. 3).

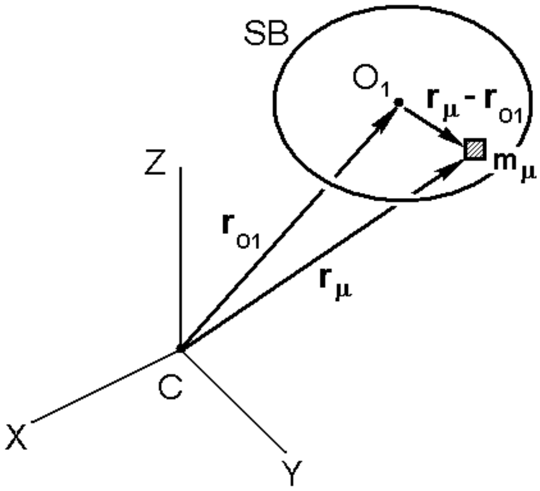

Fig. 3. SB material point (particle)

For each μ_{th} point of the first part of SA (Fig. 4)

$$\mathbf{r}_{\mu} = \mathbf{r}_{\mathrm{O1}} + \mathbf{a}_1 + \left(\mathbf{c}_2 + \mathbf{r}_{\mu} - \mathbf{r}_{\mathrm{O2}}\right),$$

$$\delta\mathbf{r}_{\mu} = \delta\mathbf{r}_{\mathrm{O1}} + \delta\boldsymbol{\theta}_1 \times \mathbf{a}_1 + \left(\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3\right) \times \left(\mathbf{c}_2 + \mathbf{r}_{\mu} - \mathbf{r}_{\mathrm{O2}}\right).$$

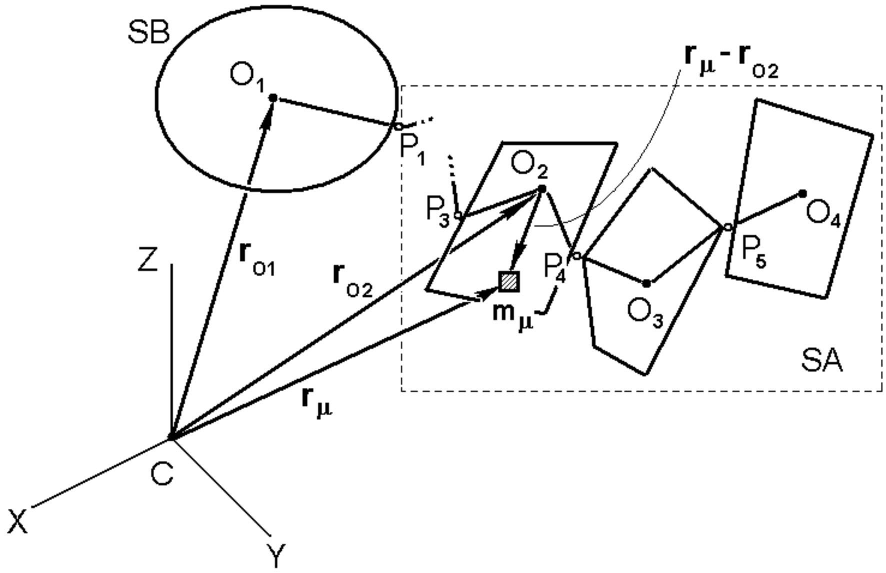

Fig. 4. SA material point

The motion of material point belonging to the second part of SA is described as

$$\mathbf{r}_\mu = \mathbf{r}_{O1} + \mathbf{a}_1 + (\mathbf{c}_2 + \mathbf{a}_2) + (\mathbf{c}_3 + \mathbf{r}_\mu - \mathbf{r}_{O3}),$$

$$\begin{aligned}\delta\mathbf{r}_\mu = \delta\mathbf{r}_{O1} + \delta\boldsymbol{\theta}_1 \times \mathbf{a}_1 + (\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3) \times (\mathbf{c}_2 + \mathbf{a}_2) + \\ + (\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3 + \mathbf{e}_4\delta\varphi_4) \times (\mathbf{c}_3 + \mathbf{r}_\mu - \mathbf{r}_{O3}).\end{aligned}$$

Finally, for the fourth part of SA point

$$\mathbf{r}_\mu = \mathbf{r}_{O1} + \mathbf{a}_1 + (\mathbf{c}_2 + \mathbf{a}_2) + (\mathbf{c}_3 + \mathbf{a}_3) + (\mathbf{c}_4 + \mathbf{r}_\mu - \mathbf{r}_{O4}),$$

$$\begin{aligned}\delta\mathbf{r}_\mu = \delta\mathbf{r}_{O1} + \delta\boldsymbol{\theta}_1 \times \mathbf{a}_1 + (\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3) \times (\mathbf{c}_2 + \mathbf{a}_2) + \\ + (\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3 + \mathbf{e}_4\delta\varphi_4) \times (\mathbf{c}_3 + \mathbf{a}_3) + \\ + (\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3 + \mathbf{e}_4\delta\varphi_4 + \mathbf{e}_5\delta\varphi_5) \times (\mathbf{c}_4 + \mathbf{r}_\mu - \mathbf{r}_{O4}).\end{aligned}$$

Substituting virtual displacements $\delta\mathbf{r}_\mu$ into (1) leads to

$$\begin{aligned}
&\sum_{\mu}^{(1)}\left(m_\mu\ddot{\mathbf{r}}_\mu-\mathbf{F}_\mu\right)\left[\delta\mathbf{r}_{O1}+\delta\boldsymbol{\theta}_1\times\left(\mathbf{r}_\mu-\mathbf{r}_{O1}\right)\right]+\\
&+\sum_{\mu}^{(2)}\left(m_\mu\ddot{\mathbf{r}}_\mu-\mathbf{F}_\mu\right)\Big[\delta\mathbf{r}_{O1}+\delta\boldsymbol{\theta}_1\times\left(\mathbf{a}_1+\mathbf{c}_2+\mathbf{r}_\mu-\mathbf{r}_{O2}\right)+\\
&\qquad+\left(\mathbf{e}_1\delta\varphi_1+\mathbf{e}_2\delta\varphi_2+\mathbf{e}_3\delta\varphi_3\right)\times\left(\mathbf{c}_2+\mathbf{r}_\mu-\mathbf{r}_{O2}\right)\Big]+\\
&+\sum_{\mu}^{(3)}\left(m_\mu\ddot{\mathbf{r}}_\mu-\mathbf{F}_\mu\right)\Big[\delta\mathbf{r}_{O1}+\delta\boldsymbol{\theta}_1\times\left(\mathbf{a}_1+\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{r}_\mu-\mathbf{r}_{O3}\right)+\\
&\qquad+\left(\mathbf{e}_1\delta\varphi_1+\mathbf{e}_2\delta\varphi_2+\mathbf{e}_3\delta\varphi_3\right)\times\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{r}_\mu-\mathbf{r}_{O3}\right)+\mathbf{e}_3\times\left(\mathbf{c}_3+\mathbf{r}_\mu-\mathbf{r}_{O3}\right)\delta\varphi_4\Big]+\\
&+\sum_{\mu}^{(4)}\left(m_\mu\ddot{\mathbf{r}}_\mu-\mathbf{F}_\mu\right)\Big[\delta\mathbf{r}_{O1}+\delta\boldsymbol{\theta}_1\times\left(\mathbf{a}_1+\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4+\mathbf{r}_\mu-\mathbf{r}_{O4}\right)+\\
&\qquad+\left(\mathbf{e}_1\delta\varphi_1+\mathbf{e}_2\delta\varphi_2+\mathbf{e}_3\delta\varphi_3\right)\times\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4+\mathbf{r}_\mu-\mathbf{r}_{O4}\right)+\\
&\qquad+\mathbf{e}_4\times\left(\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4+\mathbf{r}_\mu-\mathbf{r}_{O4}\right)\delta\varphi_4+\mathbf{e}_5\times\left(\mathbf{c}_4+\mathbf{r}_\mu-\mathbf{r}_{O4}\right)\delta\varphi_5\Big]=\sum_{i=1}^{5}M_{ui}\delta\varphi_i\,.
\end{aligned}\tag{2}$$

Denote

$\mathbf{F}_i=\sum^{(i)}\mathbf{F}_\mu$ as resultant vectors of external forces applied to the bodies $(i=1,2,3,4)$;

$\mathbf{M}_i=\sum^{(i)}\left(\mathbf{r}_\mu-\mathbf{r}_{Oi}\right)\times\mathbf{F}_\mu$ as resultant torque of external forces about the center of mass of corresponding body $(i=1,2,3,4)$;

$\mathbf{K}_i=\sum^{(i)}m_\mu\left(\mathbf{r}_\mu-\mathbf{r}_{Oi}\right)\times\left(\dot{\mathbf{r}}_\mu-\dot{\mathbf{r}}_{Oi}\right)$ as the body angular moments about the center of mass of corresponding body $(i=1,2,3,4)$.

Taking into account

$$\sum^{(i)}m_\mu\left(\mathbf{r}_\mu-\mathbf{r}_{Oi}\right)=0$$

the following relations are justified

$$\sum^{(\mathrm{i})}m_\mu\ddot{\mathbf{r}}_\mu=m_{O1}\ddot{\mathbf{r}}_{O1},$$

$$\sum^{(i)}\mathbf{F}_\mu\left[\delta\boldsymbol{\theta}_1\times\left(\mathbf{r}_\mu-\mathbf{r}_{Oi}\right)\right]=\sum^{(i)}\left[\left(\mathbf{r}_\mu-\mathbf{r}_{Oi}\right)\times\mathbf{F}_\mu\right]\delta\boldsymbol{\theta}_1=\mathbf{M}_i\delta\boldsymbol{\theta}_1,$$

$$\sum{}^{(i)} m_\mu \ddot{\mathbf{r}}_\mu \left[\delta\boldsymbol{\theta}_1 \times \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right)\right] = \left[\sum{}^{(i)} m_\mu \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right) \times \ddot{\mathbf{r}}_\mu\right]\delta\boldsymbol{\theta}_1 =$$
$$= \left\{\frac{d}{dt}\left[\sum{}^{(i)} m_\mu \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right) \times \dot{\mathbf{r}}_\mu\right]\right\}\delta\boldsymbol{\theta}_1 =$$
$$= \left\{\frac{d}{dt}\left[\sum{}^{(i)} m_\mu \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right) \times \left(\dot{\mathbf{r}}_\mu - \dot{\mathbf{r}}_{\mathrm{O}i}\right)\right]\right\}\delta\boldsymbol{\theta}_1 = \dot{\mathbf{K}}_i \delta\boldsymbol{\theta}_1 .$$

For example,

$$\sum{}^{(2)} \mathbf{F}_\mu \left[\mathbf{e}_1 \times \left(\mathbf{c}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}2}\right)\right]\delta\varphi_1 = \sum{}^{(2)} \left[\left(\mathbf{c}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}2}\right) \times \mathbf{F}_\mu\right]\mathbf{e}_1 \delta\varphi_1 =$$
$$= \left(\mathbf{c}_2 \times \mathbf{F}_2 + \mathbf{M}_2\right)\mathbf{e}_1 \delta\varphi_1 ,$$
$$\sum{}^{(2)} m_\mu \ddot{\mathbf{r}}_\mu \left[\mathbf{e}_1 \times \left(\mathbf{c}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}2}\right)\right]\delta\varphi_1 = \sum{}^{(2)} \left[\left(\mathbf{c}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}2}\right) \times m_\mu \ddot{\mathbf{r}}_\mu\right]\mathbf{e}_1 \delta\varphi_1 =$$
$$= \left(\mathbf{c}_2 \times m_{O2}\ddot{\mathbf{r}}_{O2} + \dot{\mathbf{K}}_2\right)\mathbf{e}_1 \delta\varphi_1 .$$

Sum over all material points for different bodies leads to

$$\sum{}^{(1)} = \left(m_{O1}\ddot{\mathbf{r}}_{O1} - \mathbf{F}_1\right)\delta\mathbf{r}_{\mathrm{O}1} + \left(\dot{\mathbf{K}}_1 - \mathbf{M}_1\right)\delta\boldsymbol{\theta}_1 ;$$

$$\sum{}^{(2)} = \left(m_{O2}\ddot{\mathbf{r}}_{O2} - \mathbf{F}_2\right)\delta\mathbf{r}_{\mathrm{O}1} + \left[\dot{\mathbf{K}}_2 - \mathbf{M}_2 + \left(\mathbf{a}_1 + \mathbf{c}_2\right) \times \left(m_{O2}\ddot{\mathbf{r}}_{O2} - \mathbf{F}_2\right)\right]\delta\boldsymbol{\theta}_1 +$$
$$+ \left[\dot{\mathbf{K}}_2 - \mathbf{M}_2 + \mathbf{c}_2 \times \left(m_{O2}\ddot{\mathbf{r}}_{O2} - \mathbf{F}_2\right)\right]\left(\mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3\right);$$

$$\sum{}^{(3)} = \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right)\delta\mathbf{r}_{\mathrm{O}1} + \left[\dot{\mathbf{K}}_3 - \mathbf{M}_3 + \left(\mathbf{a}_1 + \mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3\right) \times \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right)\right]\delta\boldsymbol{\theta}_1 +$$
$$+ \left[\dot{\mathbf{K}}_3 - \mathbf{M}_3 + \left(\mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3\right) \times \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right)\right]\left(\mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3\right) +$$
$$+ \left[\dot{\mathbf{K}}_3 - \mathbf{M}_3 + \mathbf{c}_3 \times \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right)\right]\mathbf{e}_4\delta\varphi_4 ;$$

$$\sum{}^{(4)} = \left(m_{O4}\ddot{\mathbf{r}}_{O4} - \mathbf{F}_4\right)\delta\mathbf{r}_{\mathrm{O}1} +$$
$$+ \left[\dot{\mathbf{K}}_4 - \mathbf{M}_4 + \left(\mathbf{a}_1 + \mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3 + \mathbf{a}_3 + \mathbf{c}_4\right) \times \left(m_{O4}\ddot{\mathbf{r}}_{O4} - \mathbf{F}_4\right)\right]\delta\boldsymbol{\theta}_1 +$$
$$+ \left[\dot{\mathbf{K}}_4 - \mathbf{M}_4 + \left(\mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3 + \mathbf{a}_3 + \mathbf{c}_4\right) \times \left(m_{O4}\ddot{\mathbf{r}}_{O4} - \mathbf{F}_4\right)\right]\left(\mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3\right) +$$
$$+ \left[\dot{\mathbf{K}}_4 - \mathbf{M}_4 + \left(\mathbf{c}_3 + \mathbf{a}_3 + \mathbf{c}_4\right) \times \left(m_{O4}\ddot{\mathbf{r}}_{O4} - \mathbf{F}_4\right)\right]\mathbf{e}_4\delta\varphi_4 +$$
$$+ \left[\dot{\mathbf{K}}_4 - \mathbf{M}_4 + \mathbf{c}_4 \times \left(m_{O4}\ddot{\mathbf{r}}_{O4} - \mathbf{F}_4\right)\right]\mathbf{e}_5\delta\varphi_5 .$$

Finally (2) is rewritten as

$$\sum_{i=1}^{4}\left(m_{Oi}\ddot{\mathbf{r}}_{Oi} - \mathbf{F}_i\right)\delta\mathbf{r}_{\mathrm{O}1} + \left[\sum_{i=1}^{4}\left(\dot{\mathbf{K}}_i - \mathbf{M}_i\right) + \left(\mathbf{a}_1 + \mathbf{c}_2\right) \times \left(m_{O2}\ddot{\mathbf{r}}_{O2} - \mathbf{F}_2\right) +\right.$$
$$+ \left(\mathbf{a}_1 + \mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3\right) \times \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right) + \left(\mathbf{a}_1 + \mathbf{c}_2 + \mathbf{a}_2 + \mathbf{c}_3\right) \times \left(m_{O3}\ddot{\mathbf{r}}_{O3} - \mathbf{F}_3\right) +$$

$$+\left[\sum_{i=2}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_2\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3\right)\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\right.$$
$$\left.+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\left(\mathbf{e}_1\delta\varphi_1+\mathbf{e}_2\delta\varphi_2+\mathbf{e}_3\delta\varphi_3\right)+ \tag{3}$$
$$+\left[\sum_{i=3}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_3\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\left(\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_4\delta\varphi_4+$$
$$+\left[\dot{\mathbf{K}}_4-\mathbf{M}_4+\mathbf{c}_4\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_5\delta\varphi_5=\sum_{i=1}^{5}M_{ui}\delta\varphi_i\,.$$

The values $\delta\mathbf{r}_{O1}$, $\delta\boldsymbol{\theta}_1$, $\delta\varphi_1$, $\delta\varphi_2$, $\delta\varphi_3$, $\delta\varphi_4$, $\delta\varphi_5$ are independent. Therefore the expression (3) is valid only if

$$\sum_{i=1}^{4}\left(m_{Oi}\ddot{\mathbf{r}}_{Oi}-\mathbf{F}_i\right)=0, \tag{4}$$

$$\sum_{i=1}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\left(\mathbf{a}_1+\mathbf{c}_2\right)\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)+\left(\mathbf{a}_1+\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3\right)\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+$$
$$+\left(\mathbf{a}_1+\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)=0, \tag{5}$$

$$\left[\sum_{i=2}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_2\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3\right)\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\right.$$
$$\left.+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_1=M_{u1}, \tag{6}$$

$$\left[\sum_{i=2}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_2\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3\right)\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\right.$$
$$\left.+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_2=M_{u2}, \tag{7}$$

$$\left[\sum_{i=2}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_2\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3\right)\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\right.$$
$$\left.+\left(\mathbf{c}_2+\mathbf{a}_2+\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_3=M_{u3}, \tag{8}$$

$$\left[\sum_{i=3}^{4}\left(\dot{\mathbf{K}}_i-\mathbf{M}_i\right)+\mathbf{c}_3\times\left(m_{O3}\ddot{\mathbf{r}}_{O3}-\mathbf{F}_3\right)+\left(\mathbf{c}_3+\mathbf{a}_3+\mathbf{c}_4\right)\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_4=M_{u4}, \tag{9}$$

$$\left[\dot{\mathbf{K}}_4-\mathbf{M}_4+\mathbf{c}_4\times\left(m_{O4}\ddot{\mathbf{r}}_{O4}-\mathbf{F}_4\right)\right]\mathbf{e}_5=M_{u5}. \tag{10}$$

SB's angular momentum is

$$\mathbf{K}_1=\mathbf{I}_1\boldsymbol{\omega}_1,$$

therefore

$$\dot{\mathbf{K}}_1=\mathbf{I}_1\dot{\boldsymbol{\omega}}_1+\boldsymbol{\omega}_1\times\mathbf{I}_1\boldsymbol{\omega}_1$$

where $\mathbf{I}_1$ is SB's inertia tensor. Similarly SA's part angular momentum derivative has the form

$$\dot{\mathbf{K}}_i = \mathbf{I}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i, \qquad i = 2, 3, 4$$

where $\mathbf{I}_i$ is inertia tensor of corresponding SA's part. SA's parts velocities are

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2 + \mathbf{e}_3\dot{\varphi}_3, \quad \boldsymbol{\omega}_3 = \boldsymbol{\omega}_2 + \mathbf{e}_4\dot{\varphi}_4, \quad \boldsymbol{\omega}_4 = \boldsymbol{\omega}_3 + \mathbf{e}_5\dot{\varphi}_5. \tag{11}$$

Further the total system center of mass $\mathbf{r}_\mathrm{O}$ is used instead of SB center of mass $\mathbf{r}_{\mathrm{O}1}$. Above equations may be rewritten taking into account

$$\mathbf{r}_{\mathrm{O}2} = \mathbf{r}_{\mathrm{O}1} + \mathbf{a}_1 + \mathbf{c}_2, \quad \mathbf{r}_{\mathrm{O}3} = \mathbf{r}_{\mathrm{O}2} + \mathbf{a}_2 + \mathbf{c}_3, \quad \mathbf{r}_{\mathrm{O}4} = \mathbf{r}_{\mathrm{O}3} + \mathbf{a}_3 + \mathbf{c}_4, \tag{12}$$

$$m\mathbf{r}_\mathrm{O} = \sum_{i=1}^{4} m_{\mathrm{Oi}}\mathbf{r}_{\mathrm{Oi}}$$

where $m = \sum_{i=1}^{4} m_{\mathrm{Oi}}$. Clearly,

$$\mathbf{r}_{\mathrm{O}1} = \mathbf{r}_\mathrm{O} - \frac{m_{\mathrm{O}2} + m_{\mathrm{O}3} + m_{\mathrm{O}4}}{m}(\mathbf{a}_1 + \mathbf{c}_2) - \frac{m_{\mathrm{O}3} + m_{\mathrm{O}4}}{m}(\mathbf{a}_2 + \mathbf{c}_3) - \frac{m_{\mathrm{O}4}}{m}(\mathbf{a}_3 + \mathbf{c}_4). \tag{13}$$

Equations of motion (4)-(10) should be complemented with kinematic relations. Derivatives of (12) and (13) are

$$\begin{aligned}
&\ddot{\mathbf{r}}_{\mathrm{O}1} = \ddot{\mathbf{r}}_\mathrm{O} - \frac{m_{\mathrm{O}2} + m_{\mathrm{O}3} + m_{\mathrm{O}4}}{m}(\ddot{\mathbf{a}}_1 + \ddot{\mathbf{c}}_2) - \frac{m_{\mathrm{O}3} + m_{\mathrm{O}4}}{m}(\ddot{\mathbf{a}}_2 + \ddot{\mathbf{c}}_3) - \frac{m_{\mathrm{O}4}}{m}(\ddot{\mathbf{a}}_3 + \ddot{\mathbf{c}}_4), \\
&\ddot{\mathbf{r}}_{\mathrm{O}2} = \ddot{\mathbf{r}}_{\mathrm{O}1} + \ddot{\mathbf{a}}_1 + \ddot{\mathbf{c}}_2, \quad \ddot{\mathbf{r}}_{\mathrm{O}3} = \ddot{\mathbf{r}}_{\mathrm{O}2} + \ddot{\mathbf{a}}_2 + \ddot{\mathbf{c}}_3, \quad \ddot{\mathbf{r}}_{\mathrm{O}4} = \ddot{\mathbf{r}}_{\mathrm{O}3} + \ddot{\mathbf{a}}_3 + \ddot{\mathbf{c}}_4
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
&\ddot{\mathbf{a}}_i = \dot{\boldsymbol{\omega}}_i \times \mathbf{a}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{a}_i), \quad i = 1, 2, 3; \\
&\ddot{\mathbf{c}}_i = \dot{\boldsymbol{\omega}}_i \times \mathbf{c}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{c}_i), \quad i = 2, 3, 4;
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}_2 = \dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\ddot{\varphi}_1 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\ddot{\varphi}_2 + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1) \times \mathbf{e}_2\dot{\varphi}_2 + \mathbf{e}_3\ddot{\varphi}_3 + \\
&\qquad + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2) \times \mathbf{e}_3\dot{\varphi}_3, \\
&\dot{\boldsymbol{\omega}}_3 = \dot{\boldsymbol{\omega}}_2 + \mathbf{e}_4\ddot{\varphi}_4 + \boldsymbol{\omega}_2 \times \mathbf{e}_4\dot{\varphi}_4, \\
&\dot{\boldsymbol{\omega}}_4 = \dot{\boldsymbol{\omega}}_3 + \mathbf{e}_5\ddot{\varphi}_5 + \boldsymbol{\omega}_3 \times \mathbf{e}_5\dot{\varphi}_5.
\end{aligned} \tag{16}$$

Equation (4) becomes

$$m\ddot{\mathbf{r}}_{\mathrm{O}} = \sum_{i=1}^{4} \mathbf{F}_i$$

where

$$\mathbf{F}_i = \sum{}^{(i)}\mathbf{F}_\mu = -fM\sum{}^{(i)} m_\mu \frac{\mathbf{r}_\mu}{r_\mu^3} = -fM\sum{}^{(i)} m_\mu \frac{\mathbf{r}_{\mathrm{O}} + \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}}\right)}{\left|\mathbf{r}_{\mathrm{O}} + \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}}\right)\right|^3} =$$

$$= -fM\frac{1}{r_{\mathrm{O}}^2}\sum{}^{(i)} m_\mu \frac{\dfrac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}} + \dfrac{\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}}}{\left|\dfrac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}} + \dfrac{\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}}\right|^3} \approx -fMm_{\mathrm{Oi}}\frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3}.$$

The latter expression is valid since the SB and SA dimensions are negligible in comparison with the radius-vector of the system, $\left|\mathbf{r}_\mu - \mathbf{r}_O\right| \ll r_O$. Gravitational forces $\mathbf{F}_i$ may be substituted with common expressions

$$\mathbf{F}_i = -\mu m_{\mathrm{O}i}\frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3}$$

where $\left|\mathbf{r}_O\right| = r_O$, $\mu = fM$ is the Earth gravitational parameter, f is the gravitational constant, M is the Earth mass. Hence equation (4) takes the final form

$$\ddot{\mathbf{r}}_{\mathrm{O}} + \mu\frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3} = 0. \qquad (17)$$

It means that the system's center of mass moves along keplerian orbit. In case of perturbed motion corresponding equation becomes

$$\ddot{\mathbf{r}}_{\mathrm{O}} + \mu\frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3} = \mathbf{F}_{per}. \qquad (18)$$

Equations (5)-(10) are not resolved with respect to highest derivatives. At the same time, to implement convenient numerical integration methods such a resolving is needed. Designate $\dot{\varphi}_i = \psi_i$, $i = 1,\ldots 5$. Substituting (14)-(16) into (5)-(10) results in the following form of motion equations:

$$\mathbf{S}\begin{pmatrix}\dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{\psi}}\end{pmatrix} = \begin{pmatrix}\mathbf{N}_1 \\ \mathbf{N}_2\end{pmatrix}. \qquad (19)$$

Here

$$\mathbf{S}=\begin{pmatrix} \mathbf{S}_{1-3,1-3} & \mathbf{S}_{1-3,4} & \mathbf{S}_{1-3,5} & \mathbf{S}_{1-3,6} & \mathbf{S}_{1-3,7} & \mathbf{S}_{1-3,8} \\ \mathbf{S}_{4,1-3} & S_{4,4} & S_{4,5} & S_{4,6} & S_{4,7} & S_{4,8} \\ \mathbf{S}_{5,1-3} & S_{5,4} & S_{5,5} & S_{5,6} & S_{5,7} & S_{5,8} \\ \mathbf{S}_{6,1-3} & S_{6,4} & S_{6,5} & S_{6,6} & S_{6,7} & S_{6,8} \\ \mathbf{S}_{7,1-3} & S_{7,4} & S_{7,5} & S_{7,6} & S_{7,7} & S_{7,8} \\ \mathbf{S}_{8,1-3} & S_{8,4} & S_{8,5} & S_{8,6} & S_{8,7} & S_{8,8} \end{pmatrix},\ \dot{\boldsymbol{\psi}}=\begin{pmatrix}\dot{\psi}_1\\ \dot{\psi}_2\\ \dot{\psi}_3\\ \dot{\psi}_4\\ \dot{\psi}_5\end{pmatrix}.$$

Designate radius-vectors between corresponding bodies (SB and SA's parts) of the system,

$$\mathbf{d}_{1,2}=\mathbf{a}_1+\mathbf{c}_2,\qquad \mathbf{d}_{1,3}=\mathbf{d}_{1,2}+\mathbf{a}_2+\mathbf{c}_3,\qquad \mathbf{d}_{1,4}=\mathbf{d}_{1,3}+\mathbf{a}_3+\mathbf{c}_4,$$

$$\mathbf{d}_{2,3}=\mathbf{a}_2+\mathbf{c}_3,\qquad \mathbf{d}_{2,4}=\mathbf{d}_{2,3}+\mathbf{a}_3+\mathbf{c}_4,\qquad \mathbf{d}_{3,4}=\mathbf{a}_3+\mathbf{c}_4,$$

and mass-related expressions,

$$\begin{aligned} m_{1,2}&=\frac{m_1m_2}{m},\quad m_{1,3}=\frac{m_1m_3}{m},\quad m_{1,4}=\frac{m_1m_4}{m},\\ m_{2,3}&=\frac{m_2m_3}{m},\quad m_{2,4}=\frac{m_2m_4}{m},\quad m_{3,4}=\frac{m_3m_4}{m} \end{aligned}.$$

Finally, triple vector product $\mathbf{a}\times(\mathbf{y}\times\mathbf{b})$ will be written as $\mathbf{K}(\mathbf{a},\mathbf{b})\mathbf{y}$ where

$$\mathbf{K}(\mathbf{a},\,\mathbf{b})=\begin{pmatrix} a_2b_2+a_3b_3 & -a_2b_1 & -a_3b_1 \\ -a_1b_2 & a_1b_1+a_3b_3 & -a_3b_2 \\ -a_1b_3 & -a_2b_3 & a_1b_1+a_2b_2 \end{pmatrix}.$$

As a result the elements of matrix $\mathbf{S}$ can be rewritten in rather compact form

$$\begin{aligned} \mathbf{S}_{1-3,1-3}=&\ \mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4+m_{1,2}\mathbf{K}(\mathbf{d}_{1,2},\mathbf{d}_{1,2})+m_{1,3}\mathbf{K}(\mathbf{d}_{1,3},\mathbf{d}_{1,3})+m_{1,4}\mathbf{K}(\mathbf{d}_{1,4},\mathbf{d}_{1,4})+\\ &+m_{2,3}\mathbf{K}(\mathbf{d}_{2,3},\mathbf{d}_{2,3})+m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{d}_{2,4})+m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{d}_{3,4}),\\ \mathbf{S}_{1-3,4}=&\Big[\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4+m_{1,2}\mathbf{K}(\mathbf{d}_{1,2},\mathbf{c}_2)+m_{1,3}\mathbf{K}(\mathbf{d}_{1,3},\mathbf{c}_2+\mathbf{d}_{2,3})+m_{1,4}\mathbf{K}(\mathbf{d}_{1,4},\mathbf{c}_2+\mathbf{d}_{2,4})+\\ &+m_{2,3}\mathbf{K}(\mathbf{d}_{2,3},\mathbf{d}_{2,3})+m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{d}_{2,4})+m_{3,4}K(\mathbf{d}_{3,4},\mathbf{d}_{3,4})\Big]\mathbf{e}_1=\mathbf{A}_{1-3,4}\mathbf{e}_1, \end{aligned}$$

$$\begin{aligned} \mathbf{S}_{1-3,5}&=\mathbf{A}_{1-3,4}\mathbf{e}_2,\\ \mathbf{S}_{1-3,6}&=\mathbf{A}_{1-3,4}\mathbf{e}_3, \end{aligned}$$

$$\mathbf{S}_{1-3,7} = \Big[\mathbf{I}_3 + \mathbf{I}_4 + m_{1,3}\mathbf{K}(\mathbf{d}_{1,3},\mathbf{c}_3) + m_{1,4}\mathbf{K}(\mathbf{d}_{1,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{2,3}\mathbf{K}(\mathbf{d}_{2,3},\mathbf{c}_3) +$$
$$+ m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{d}_{3,4})\Big] = \mathbf{A}_{1-3,7}\mathbf{e}_4,$$

$$\mathbf{S}_{1-3,8} = \Big[\mathbf{I}_4 + m_{1,4}\mathbf{K}(\mathbf{d}_{1,4},\mathbf{c}_4) + m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{c}_4) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{c}_4)\Big]\mathbf{e}_5 = \mathbf{A}_{1-3,8}\mathbf{e}_5,$$
$$\mathbf{S}_{4,1-3} = \mathbf{S}^T_{1-3,4},$$

$$S_{4,4} = \mathbf{e}_1^T\Big[\Big(\mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + m_{1,2}\mathbf{K}(\mathbf{c}_2,\mathbf{c}_2) + m_{1,3}\mathbf{K}(\mathbf{c}_2 + \mathbf{d}_{2,3},\mathbf{c}_2 + \mathbf{d}_{2,3}) +$$
$$+ m_{1,4}\mathbf{K}(\mathbf{c}_2 + \mathbf{d}_{2,4},\mathbf{c}_2 + \mathbf{d}_{2,4}) + m_{2,3}\mathbf{K}(\mathbf{d}_{2,3},\mathbf{d}_{2,3}) + m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{d}_{2,4}) +$$
$$+ m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{d}_{3,4})\Big)\mathbf{e}_1\Big] = \mathbf{e}_1^T\mathbf{A}_{4,4}\mathbf{e}_1,$$

$$S_{4,5} = \mathbf{e}_1^T\mathbf{A}_{4,4}\mathbf{e}_2,$$

$$S_{4,6} = \mathbf{e}_1^T\mathbf{A}_{4,4}\mathbf{e}_3,$$

$$S_{4,7} = \mathbf{e}_1^T\Big[\Big(\mathbf{I}_3 + \mathbf{I}_4 + m_{1,3}\mathbf{K}(\mathbf{c}_2 + \mathbf{d}_{2,3},\mathbf{c}_3) + m_{1,4}\mathbf{K}(\mathbf{c}_2 + \mathbf{d}_{2,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{2,3}\mathbf{K}(\mathbf{d}_{2,3},\mathbf{c}_3) +$$
$$+ m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{d}_{3,4})\Big)\mathbf{e}_4\Big] = \mathbf{e}_1^T\mathbf{A}_{4,7}\mathbf{e}_4,$$

$$S_{4,8} = \mathbf{e}_1^T\Big[\Big(\mathbf{I}_4 + m_{1,4}\mathbf{K}(\mathbf{c}_2 + \mathbf{d}_{2,4},\mathbf{c}_4) + m_{2,4}\mathbf{K}(\mathbf{d}_{2,4},\mathbf{c}_4) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{c}_4)\Big)\mathbf{e}_5\Big] = \mathbf{e}_1^T\mathbf{A}_{4,8}\mathbf{e}_5,$$

$$\mathbf{S}_{5,1-3} = \mathbf{S}^T_{1-3,5}, \quad S_{5,4} = S_{4,5}, \quad S_{5,5} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_2,$$
$$S_{5,6} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_3, \quad S_{5,7} = \mathbf{e}_2^T\mathbf{A}_{4,7}\mathbf{e}_4, \quad S_{5,8} = \mathbf{e}_2^T\mathbf{A}_{4,8}\mathbf{e}_5,$$

$$\mathbf{S}_{5,1-3} = \mathbf{S}^T_{1-3,5}, \quad S_{5,4} = S_{4,5}, \quad S_{5,5} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_2,$$
$$S_{5,6} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_3, \quad S_{5,7} = \mathbf{e}_2^T\mathbf{A}_{4,7}\mathbf{e}_4, \quad S_{5,8} = \mathbf{e}_2^T\mathbf{A}_{4,8}\mathbf{e}_5,$$

$$\mathbf{S}_{5,1-3} = \mathbf{S}^T_{1-3,5}, \quad S_{5,4} = S_{4,5}, \quad S_{5,5} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_2,$$
$$S_{5,6} = \mathbf{e}_2^T\mathbf{A}_{4,4}\mathbf{e}_3, \quad S_{5,7} = \mathbf{e}_2^T\mathbf{A}_{4,7}\mathbf{e}_4, \quad S_{5,8} = \mathbf{e}_2^T\mathbf{A}_{4,8}\mathbf{e}_5,$$

$$\mathbf{S}_{6,1-3} = \mathbf{S}^T_{1-3,6}, \quad S_{6,4} = S_{4,6}, \quad S_{6,5} = \mathbf{e}_3^T\mathbf{A}_{4,4}\mathbf{e}_2,$$
$$S_{6,6} = \mathbf{e}_3^T\mathbf{A}_{4,4}\mathbf{e}_3, \quad S_{6,7} = \mathbf{e}_3^T\mathbf{A}_{4,7}\mathbf{e}_4, \quad S_{6,8} = \mathbf{e}_3^T\mathbf{A}_{4,8}\mathbf{e}_5,$$

$$S_{7,1-3} = S^T_{1-3,7}, \quad S_{7,4} = S_{4,7}, \quad S_{7,5} = S_{5,7}, \quad S_{7,6} = S_{6,7},$$

$$S_{7,7} = \mathbf{e}_4^T\Big[\Big(\mathbf{I}_3 + \mathbf{I}_4 + m_{1,3}\mathbf{K}(\mathbf{c}_3,\mathbf{c}_3) + m_{1,4}\mathbf{K}(\mathbf{c}_3 + \mathbf{d}_{3,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{2,3}\mathbf{K}(\mathbf{c}_3,\mathbf{c}_3) +$$
$$+ m_{2,4}\mathbf{K}(\mathbf{c}_3 + \mathbf{d}_{3,4},\mathbf{c}_3 + \mathbf{d}_{3,4}) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{d}_{3,4})\Big)\mathbf{e}_4\Big],$$

$$S_{7,8} = \mathbf{e}_4^T\Big[\Big(\mathbf{I}_4 + m_{1,4}\mathbf{K}(\mathbf{c}_3 + \mathbf{d}_{3,4},\mathbf{c}_4) + m_{2,4}\mathbf{K}(\mathbf{c}_3 + \mathbf{d}_{3,4},\mathbf{c}_4) + m_{3,4}\mathbf{K}(\mathbf{d}_{3,4},\mathbf{c}_4)\Big)\mathbf{e}_5\Big] = \mathbf{e}_4^T\mathbf{A}_{7,8}\mathbf{e}_5,$$

$$\mathbf{S}_{8,1-3} = \mathbf{S}_{1-3,8}^T, \quad S_{8,4} = S_{4,8}, \quad S_{8,5} = S_{5,8}, \quad S_{8,6} = S_{6,8}, \quad S_{8,7} = S_{7,8},$$

$$S_{8,8} = \mathbf{e}_5^T \left[\left(\mathbf{I}_4 + m_{1,4}\mathbf{K}(\mathbf{c}_4, \mathbf{c}_4) + m_{2,4}\mathbf{K}(\mathbf{c}_4, \mathbf{c}_4) + m_{3,4}\mathbf{K}(\mathbf{c}_4, \mathbf{c}_4) \right) \mathbf{e}_5 \right].$$

Right-hand side parts of equations (19) can be written as

$$\mathbf{N}_1 = \sum_{i=1}^{4} \mathbf{M}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i - \mathbf{I}_2 \mathbf{f}_2 - \mathbf{I}_3 \mathbf{f}_3 - \mathbf{I}_4 \mathbf{f}_4 - m_{1,2}\mathbf{d}_{1,2} \times \mathbf{j}_{1,2} - m_{1,3}\mathbf{d}_{1,3} \times \mathbf{j}_{1,3} - m_{1,4}\mathbf{d}_{1,4} \times \mathbf{j}_{1,4} -$$
$$- m_{2,3}\mathbf{d}_{2,3} \times \mathbf{j}_{2,3} - m_{2,4}\mathbf{d}_{2,4} \times \mathbf{j}_{2,4} - m_{3,4}\mathbf{d}_{3,4} \times \mathbf{j}_{3,4},$$

$$\mathbf{N}_2 = \left(n_1, n_2, n_3, n_4, n_5 \right)^T$$

where

$$\mathbf{f}_2 = \boldsymbol{\omega}_1 \times \mathbf{e}_1 \psi_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1) \times \mathbf{e}_2 \psi_2 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1 + \mathbf{e}_2 \psi_2) \times \mathbf{e}_3 \psi_3,$$
$$\mathbf{f}_3 = \boldsymbol{\omega}_1 \times \mathbf{e}_1 \psi_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1) \times \mathbf{e}_2 \psi_2 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1 + \mathbf{e}_2 \psi_2) \times \mathbf{e}_3 \psi_3 + \boldsymbol{\omega}_2 \times \mathbf{e}_4 \psi_4,$$
$$\mathbf{f}_4 = \boldsymbol{\omega}_1 \times \mathbf{e}_1 \psi_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1) \times \mathbf{e}_2 \psi_2 + (\boldsymbol{\omega}_1 + \mathbf{e}_1 \psi_1 + \mathbf{e}_2 \psi_2) \times \mathbf{e}_3 \psi_3 + \boldsymbol{\omega}_2 \times \mathbf{e}_4 \psi_4 + \boldsymbol{\omega}_3 \times \mathbf{e}_5 \psi_5,$$

$$\mathbf{g}_1 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_1 \times \mathbf{a}_1,$$
$$\mathbf{g}_i = \mathbf{f}_i \times \mathbf{a}_i + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_i \times \mathbf{a}_i \quad (i = 2,3),$$
$$\mathbf{h}_i = \mathbf{f}_i \times \mathbf{c}_i + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_i \times \mathbf{c}_i \quad (i = 2,3,4),$$

$$\mathbf{j}_{1,2} = \mathbf{g}_1 + \mathbf{h}_2, \quad \mathbf{j}_{1,3} = \mathbf{j}_{1,2} + \mathbf{g}_2 + \mathbf{h}_3, \quad \mathbf{j}_{1,4} = \mathbf{j}_{1,3} + \mathbf{g}_3 + \mathbf{h}_4,$$
$$\mathbf{j}_{2,3} = \mathbf{g}_2 + \mathbf{h}_3, \quad \mathbf{j}_{2,4} = \mathbf{j}_{2,3} + \mathbf{g}_3 + \mathbf{h}_4, \quad \mathbf{j}_{3,4} = \mathbf{g}_3 + \mathbf{h}_4;$$

$$n_1 = M_{u1} + \mathbf{e}_1^T \left[\sum_{i=2}^{4} \left(\mathbf{M}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i - \mathbf{I}_i \mathbf{f}_i \right) - m_{1,2}\mathbf{c}_2 \times \mathbf{j}_{1,2} - m_{1,3}(\mathbf{c}_2 + \mathbf{d}_{2,3}) \times \mathbf{j}_{1,3} - \right.$$
$$\left. - m_{1,4}(\mathbf{c}_2 + \mathbf{d}_{2,4}) \times \mathbf{j}_{1,4} - m_{2,3}\mathbf{d}_{2,3} \times \mathbf{j}_{2,3} - m_{2,4}\mathbf{d}_{2,4} \times \mathbf{j}_{2,4} - m_{3,4}\mathbf{d}_{3,4} \times \mathbf{j}_{3,4} \right],$$

$$n_2 = M_{u2} + \mathbf{e}_2^T \left[\sum_{i=2}^{4} \left(\mathbf{M}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i - \mathbf{I}_i \mathbf{f}_i \right) - m_{1,2}\mathbf{c}_2 \times \mathbf{j}_{1,2} - m_{1,3}(\mathbf{c}_2 + \mathbf{d}_{2,3}) \times \mathbf{j}_{1,3} - \right.$$
$$\left. - m_{1,4}(\mathbf{c}_2 + \mathbf{d}_{2,4}) \times \mathbf{j}_{1,4} - m_{2,3}\mathbf{d}_{2,3} \times \mathbf{j}_{2,3} - m_{2,4}\mathbf{d}_{2,4} \times \mathbf{j}_{2,4} - m_{3,4}\mathbf{d}_{3,4} \times \mathbf{j}_{3,4} \right],$$

$$n_3 = M_{u3} + \mathbf{e}_3^T \left[\sum_{i=2}^{4} \left(\mathbf{M}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i - \mathbf{I}_i \mathbf{f}_i \right) - m_{1,2}\mathbf{c}_2 \times \mathbf{j}_{1,2} - m_{1,3}(\mathbf{c}_2 + \mathbf{d}_{2,3}) \times \mathbf{j}_{1,3} - \right.$$
$$\left. - m_{1,4}(\mathbf{c}_2 + \mathbf{d}_{2,4}) \times \mathbf{j}_{1,4} - m_{2,3}\mathbf{d}_{2,3} \times \mathbf{j}_{2,3} - m_{2,4}\mathbf{d}_{2,4} \times \mathbf{j}_{2,4} - m_{3,4}\mathbf{d}_{3,4} \times \mathbf{j}_{3,4} \right],$$

$$n_4 = M_{u4} + \mathbf{e}_4^T \left[\sum_{i=3}^{4} \left(\mathbf{M}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i - \mathbf{I}_i \mathbf{f}_i \right) - m_{1,3}\mathbf{c}_3 \times \mathbf{j}_{1,3} - m_{1,4}(\mathbf{c}_3 + \mathbf{d}_{3,4}) \times \mathbf{j}_{1,4} - \right.$$
$$\left. - m_{2,3}\mathbf{c}_3 \times \mathbf{j}_{2,3} - m_{2,4}(\mathbf{c}_3 + \mathbf{d}_{2,4}) \times \mathbf{j}_{2,4} - m_{3,4}\mathbf{d}_{3,4} \times \mathbf{j}_{3,4} \right],$$

$$n_5 = M_{u5} + \\ + \mathbf{e}_5^T \left[\mathbf{M}_4 - \boldsymbol{\omega}_4 \times \mathbf{I}_4 \boldsymbol{\omega}_4 - \mathbf{I}_4 \mathbf{f}_4 - m_{1,4} \mathbf{c}_4 \times \mathbf{j}_{1,4} - m_{2,4} \mathbf{c}_4 \times \mathbf{j}_{2,4} - m_{3,4} \mathbf{c}_4 \times \mathbf{j}_{3,4} \right].$$

It should be noted that matrix S and right-hand side parts in (19) include vectors given in different reference frames, it is necessary to reduce all vectors to the same reference frame which is more convenient. The following frames are used to describe SB motion and represent different SB-SA system parameters and state vector parts:

$CXYZ$ is any inertial reference frame located in the Earth center;

$Oxyz$ is orbital reference frame with origin at the system center of mass;

$O_1x_1y_1z_1$ is SB-fixed reference frame with axes directed along principal axes of inertia (further this reference frame is chosen as a basic one);

$O_ix_iy_iz_i$ ($i = 2,3,4$) are SA parts-fixed frames;

$P_i\xi_i\eta_i\zeta_i$ ($i = 1,2,3,4,5$) are reference frames associated with i th hinge and its axis $\mathbf{e}_i$: $P_1\xi_1\eta_1\zeta_1$ is related to connection P_1P_2; $P_2\xi_2\eta_2\zeta_2$ is related to connection P_2P_3; $P_3\xi_3\eta_3\zeta_3$ is related to first SA part; $P_4\xi_4\eta_4\zeta_4$ is related to second SA part; $P_5\xi_5\eta_5\zeta_5$ is related to third SA part.

The following relations highlight transition rules and corresponding matrices of rotation:

$$(x, y, z)^T = \mathbf{C}(X, Y, Z)^T;$$
$$(x, y, z)^T = \mathbf{A}_i (x_i, y_i, z_i)^T, \quad i = 1,2,3,4;$$
$$(\xi_1, \eta_1, \zeta_1)^T = \mathbf{B}_1 (x_1, y_1, z_1)^T;$$
$$(\xi_i, \eta_i, \zeta_i)^T = \mathbf{B}_i (\xi_{i-1}, \eta_{i-1}, \zeta_{i-1})^T, \quad i = 2,3;$$
$$(\xi_i, \eta_i, \zeta_i)^T = \mathbf{B}_i (x_{i-2}, y_{i-2}, z_{i-2})^T, \quad i = 4,5;$$
$$(x_i, y_i, z_i)^T = \mathbf{D}_i (\xi_{i+1}, \eta_{i+1}, \zeta_{i+1})^T, \quad i = 2,3,4.$$

All vectors involved in equations of motion are grouped by reference frames where they are initially given (See Table 1 below).

Table 1

Vectors	Reference frame	Vectors	Reference frame
$\mathbf{r}_O$	$CXYZ$	$\mathbf{I}_2, \mathbf{c}_2, \mathbf{a}_2, \mathbf{M}_2$	$O_2x_2y_2z_2$
$\boldsymbol{\omega}_1, \mathbf{I}_1, \mathbf{a}_1, \mathbf{M}_1$	$O_1x_1y_1z_1$	$\mathbf{e}_4$	$P_4\xi_4\eta_4\zeta_4$
$\mathbf{e}_1$	$P_1\xi_1\eta_1\zeta_1$	$\mathbf{I}_3, \mathbf{c}_3, \mathbf{a}_3, \mathbf{M}_3$	$O_3x_3y_3z_3$
$\mathbf{e}_2$	$P_2\xi_2\eta_2\zeta_2$	$\mathbf{e}_5$	$P_5\xi_5\eta_5\zeta_5$
$\mathbf{e}_3$	$P_3\xi_3\eta_3\zeta_3$	$\mathbf{I}_4, \mathbf{c}_4, \mathbf{M}_4$	$O_4x_4y_4z_4$

All vectors except $\mathbf{r}_O$ should be transformed to $O_1x_1y_1z_1$. Table 2 summarizes resulting matrices necessary to rotate vector from specific frame to $O_1x_1y_1z_1$. Note that rotation matrices are orthogonal and matrix inverse may be substituted with matrix transpose.

Table 2

$P_1\xi_1\eta_1\zeta_1$	$\Rightarrow O_1x_1y_1z_1$	$\mathbf{B}_1^T$
$P_2\xi_2\eta_2\zeta_2$		$\mathbf{B}_1^T\mathbf{B}_2^T$
$P_3\xi_3\eta_3\zeta_3$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T$
$O_2x_2y_2z_2$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}_2^T$
$P_4\xi_4\eta_4\zeta_4$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}_2^T\mathbf{B}_4^T$
$O_3x_3y_3z_3$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}_2^T\mathbf{B}_4^T\mathbf{D}_3^T$
$P_5\xi_5\eta_5\zeta_5$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}_2^T\mathbf{B}_4^T\mathbf{D}_3^T\mathbf{B}_5^T$
$O_4x_4y_4z_4$		$\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}_2^T\mathbf{B}_4^T\mathbf{D}_3^T\mathbf{B}_5^T\mathbf{D}_4^T$

Gravitational torque can be calculated directly in reference frame $O_1x_1y_1z_1$ using relation

$$\mathbf{M}_{iG} = -3\frac{\mu}{r_O^3}\mathbf{E}\times\mathbf{I}_i\mathbf{E}$$

where $\mathbf{E}$ is a local vertical in $O_1x_1y_1z_1$ frame.

Finally summarizing dynamical equations and kinematic relations we get complete equations of motion in the form

$$\mathbf{S}\begin{pmatrix}\dot{\boldsymbol{\omega}}_1\\ \dot{\boldsymbol{\psi}}\end{pmatrix} = \begin{pmatrix}\mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}; \qquad (20)$$

$$\dot{\varphi}_i = \psi_i \ (i = 1,...,5);$$

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{\Omega}\mathbf{q}.$$

Here $\mathbf{q}$ is attitude quaternion representing rotation from $CXYZ$ to $O_1x_1y_1z_1$, matrix $\mathbf{\Omega}$ has the form

$$\mathbf{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}$$

2. Torques

Gravitational torque is the main disturbing one. Control torque is implemented using reaction wheels. Apart from control torque imposed by reaction wheels disturbing torques arising in hinges should be taken into account. Reaction wheels impose control torque

$$\mathbf{M}_{rw} = -\dot{\mathbf{H}} - \boldsymbol{\omega}_1 \times \mathbf{H}$$

where $\mathbf{H}$ is angular momentum of reaction wheels in frame $O_1x_1y_1z_1$.

Each reaction wheel angular momentum magnitude may be written as $H = I\Omega$ where I is wheels' inertia moment, Ω is angular velocity of a wheel. Reaction wheel angular velocity Ω is affected by friction in framing and becomes[3]

$$I\dot{\Omega} = \tau - \left(\alpha_0 + \alpha_1 e^{-\left(\Omega/\Omega_0\right)^2} \right) \operatorname{sgn}\Omega - \alpha_2 \Omega .$$

Here α_0 is Coulomb friction factor, α_2 is viscous friction factor, and α_1 is Stribeck effect factor (maximum Coulomb friction slowly degrades), Ω_0 is Stribeck velocity.

Dissipative and tension torques are

$$M_{ui} = -\delta_i \psi_i - \lambda_i \varphi_i , \quad (i = 1,...,5).$$

Here δ_i is a dissipation factor, λ_i is a tension factor. Listed above torques are used in equations of motion (20) to obtain final SB-SA system motion representation.

Remark. Equations (20) do not depend directly on the center of mass position but the torques entering into the equations may contain expressions depending on the position of the center of mass of SB and SA. Therefore, for some torques affecting the satellite the input parameters are supplemented by the orbital motion state vector. Equations of the satellite center of mass motion can be specified both as undisturbed motion equations (17) and equations (18).

3. Conclusion

The mathematical model for complex multibody system, a satellite with controllable three part solar panel, is obtained. The elaborated approach demonstrates high efficiency and can be applied in studying dynamics of the system during the stage of solar arrays deployment.

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