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Libration Point Orbits and Manifolds: Design and Station-Keeping

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Libration Point Orbits and Manifolds: Design and Station-Keeping

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Методы проектирования и поддержания орбит вокруг точек либрации

В работе дан предварительный обзор методов проектирования периодических и квазипериодических орбит вокруг коллинеарных точек либрации, а также связанных с ними инвариантных многообразий, в круговой ограниченной задаче трех тел. Представлены также работы по поддержанию движения вокруг точек либрации.

Ключевые слова: точка либрации, периодическая орбита, инвариантное многообразие, поддержание движения

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Libration point orbits and manifolds: design and station-keeping

The paper includes a literature review of the (quasi-)periodic orbits' design around collinear libration points, as well as associated invariant manifolds, in circular restricted three-body problem. An overview of methods for station-keeping around libration points is also presented.

Key words: libration point, periodic orbit, invariant manifold, station-keeping

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1 Introduction

Libration point missions are of special interest today. The dynamics around these equilibrium points provide families of periodic orbits which are convenient for placing spacecraft and are useful for astrophysics missions, solar observation, communication links, etc. There are a number of future-promising projects proposed by leading space agencies; some of them are Deep Space Climate Observatory (NASA), LISA Pathfinder (ESA/NASA), Spektr-RG (Roscosmos/ESA). The majority of them exploit the L_1/L_2 libration points of the Sun-Earth system, though after the recent success of the ARTEMIS missions, the collinear libration points of the Earth-Moon system have received a growing interest.

Taking into account the importance of the libration points' exploitation, very much was done to the design of the associated periodic orbits and invariant manifolds. In this work, an attempt was made to give a preliminary overview of methods and techniques concerning designing of these objects. For this purpose we place in Section 2 some information relating to collinear libration points and the phase space around them. Then we turn to the problem of finding periodic orbits and their invariant manifolds. Here, both pure numerical and semi-analytical results are mentioned. Finally, analytical results concerning the elliptical restricted three-body problem are mentioned. In the future, a comprehensive literature survey containing the description, comparison, and discussion of the methods will be prepared.

Station-keeping techniques for collinear libration point orbits are considered in Section 3. Indeed, as the collinear libration points and the associated (quasi-)periodical orbits inherit the unstable character in the circular restricted three-body problem, methods of the motion maintaining near the nominal orbits are required to be developed. This becomes essential when disturbances from bodies that are not included in the given system are taking into account (for example, solar radiation pressure and the Sun's gravity in the Earth-Moon system). In this review, single spacecraft station-keeping methods are mainly considered, though some works on formation flying are mentioned. Investigations relating to tethered systems' keeping as well as approaches utilizing solar sails are not considered.

2 Three-body dynamics: libration point orbits and manifolds

2.1 Introduction to the collinear libration points

The libration points are the equilibrium points of the circular restricted three-body problem (CR3BP). Euler and Lagrange proved the existence of five equilibrium points: three collinear points on the axis joining the center of the two primaries, generally noted L_1, L_2 and L_3, and two equilateral points noted L_4 and L_5.

Libration points offer many good properties. Spacecraft in the vicinity of these points is free of atmospheric drag, space debris, and atomic oxygen. In addition, space transportation between these points and the Earth is very convenient, making them ideal places to establish gateway stations for further deep-space exploration. With the many future missions planned for regions around libration points, the need for efficient approaches for trajectory design is apparent.

According to the CR3BP model, a spacecraft of negligible mass moves under the gravitational influence of two masses m_1 and m_2, referred to as the *primaries*, such that m_1 and m_2 move in circular orbits about their center of mass C. Here, the spacecraft is not restricted to move in the orbital plane of the primaries. To avoid ambiguity, let m_1 be greater than m_2. In particular, m_1 can represent the Sun while m_2 can represent the Earth.

It is convenient to write the spacecraft equations of motion in the standard for the CR3BP non-dimensional rotating (sometimes referred to as *synodic*) coordinate frame, see Fig. 1. The masses m_1 and m_2 are normalized so that $m_1 = 1 - \mu$ and $m_2 = \mu$, where $\mu = m_2/(m_1 + m_2)$ stands for the mass parameter of the system. The origin of the frame is chosen at C. If we normalize to one the angular velocity of the rotating frame and the distance between the primaries, the last would be at fixed positions along the x-axis at points $(-\mu, 0, 0)$ and $(1 - \mu, 0, 0)$, respectively. Note that the value of m_2 for the Sun-Earth system often additionally includes the mass of the Moon and therefore $\mu = 3.0393890 \times 10^{-6}$. As for the Earth-Moon system, $\mu = 1.2150668 \times 10^{-2}$. The spacecraft equations of motion can be expressed as follows:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= -U_x, \\ \ddot{y} + 2\dot{x} &= -U_y, \\ \ddot{z} &= -U_z, \end{aligned}$$

where

$$U(x, y, z) = -\frac{1}{2}\left(x^2 + y^2\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1-\mu),$$

is the so called *effective potential*; U_x, U_y and U_z are the partial derivatives of U with respect to the position variables. The distances between the spacecraft and the primaries are given by equalities

$$\begin{aligned} r_1^2 &= (x + \mu)^2 + y^2 + z^2, \\ r_2^2 &= (x - 1 + \mu)^2 + y^2 + z^2. \end{aligned}$$

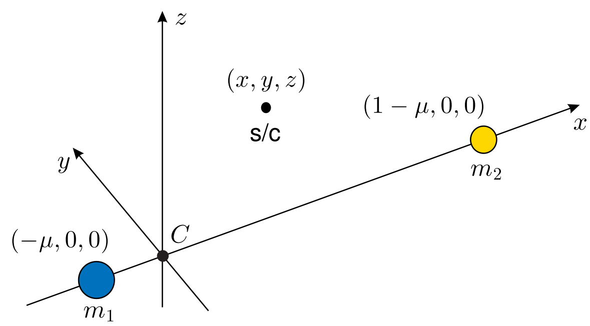

Fig. 1. Rotating frame of the circular restricted three-body problem

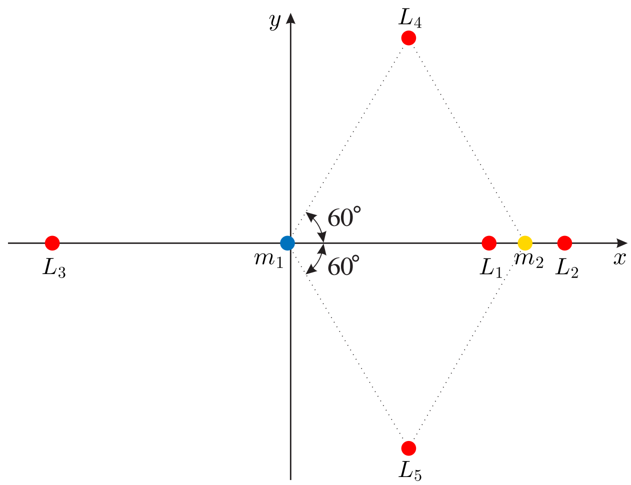

Fig. 2. Equilibrium points of the circular restricted three-body problem

The system has five relative equilibrium points, see Fig. 2; three of them lie at the x-axis and are referred to as *collinear libration points*. Usually denoted by L_1, L_2, and L_3, these points are proved to be unstable. Their x-coordinates for the Sun-Earth system are respectively equal to $x_{L1} = 0.9899871$, $x_{L2} = 1.0100740$, and $x_{L3} = -1.0000013$, and for the Earth-Moon system they are $x_{L1} = 0.8369147$, $x_{L2} = 1.1556825$, and $x_{L3} = -1.0050627$.

Recall that the collinear points are shown to be unstable in every system. Moreover, due to Moser's [102] generalization of a theorem of Lyapunov in the vicinity of the libration points, solutions of the nonlinear system have the same qualitative behavior as the solutions of the linearized system.

2.2 Phase space around the collinear libration points

It is known that the collinear libration points are of center $\times$ center $\times$ saddle type. Due to the center $\times$ center part, and according to Lyapunov's center theorem [99], each collinear libration point gives rise to two one-parameter families of periodic orbits, known as the *planar* (see Fig. 3 and Fig. 4) and the *vertical* Lyapunov orbits, see Fig. 5. Note that the planar orbits lie in the (x, y)-plane while vertical orbits are tangent to the $(z, \dot{z})$-direction when crossing the plane. Along the families of Lyapunov periodic orbits, as the energy increases, the linear stability of the orbits change, and there appear other families of periodic orbits. The family bifurcating from the planar Lyapunov one corresponds to the three-dimensional periodic orbits symmetric with respect to the (x, z)-plane, the so-called *halo orbits* [29]. At the bifurcation, the two families of orbits are born, known as the *northern* and *southern halo* families, see Fig. 5. Halo orbits around the L_1 point of the Sun-Earth system were used in the ISEE-3 and SOHO missions. For the computation of periodic orbits, one can use differential correction techniques and continuation methods, see e.g. [86]. Analytical approximations are also exist, see e.g. [86, 46].

Note that besides Lyapunov and halo families there are other families of periodic orbits in the CR3BP. For example, so called *axial* orbits are bifurcating from vertical Lyapunov orbits and connect vertical and planar Lyapunov orbits, see Fig. 6. These families are all well-documented in the literature, but their names are sometimes different. For example, the Halo, Axial, and Vertical orbits are known as type "A", "B", and "C", respectively, in Goudas [51] and Hénon [61]. Farquhar [29] coined the name "Halo" for that family. The term "Axial" comes from Doedel et al. [22], as these orbits intersect x-axis at two points. Doedel et al. [21] used the term "Y" for "Yellow".

See reference [22] for the detailed computational results for the families of the periodic orbits that emanate from the five libration points in the CR3BP, as well as for various secondary bifurcating families. This vast overview covers all values of the mass-ratio parameter, and includes many known families that have been studied in the past.

According to works [98, 85, 125], there also exit homoclinic and heteroclinic connections between periodic orbits around libration points. The computation of these trajectories is described in subsection 2.4.

In addition, the four-dimensional center manifold around collinear libration points is occupied by quasi-periodic orbits of two different families: the *Lissajous family* around the vertical Lyapunov orbits [48], and the *quasi-halos* around the halo orbits [49].

There are also so called *stable* and *unstable manifolds* associated with periodic orbits around libration points. For chaotic systems such stable and unstable

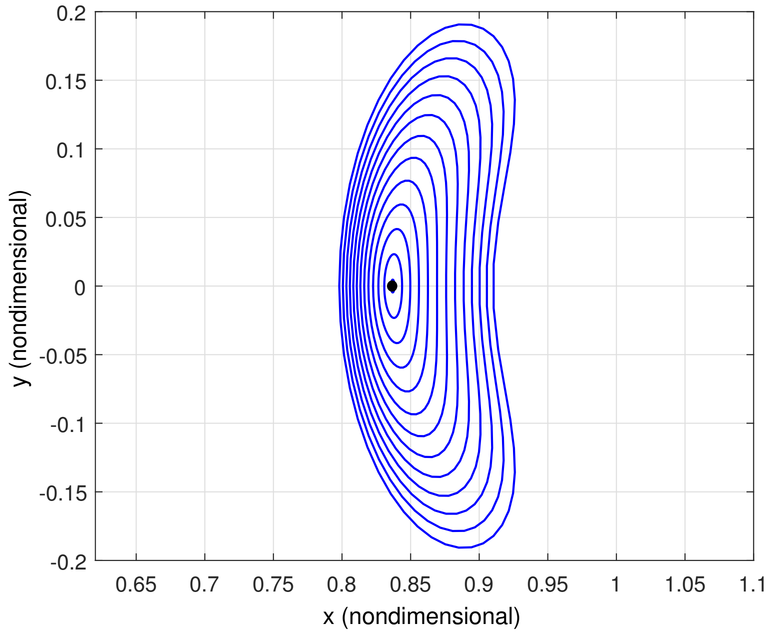

Fig. 3. Family of Earth–Moon L_1 planar Lyapunov orbits

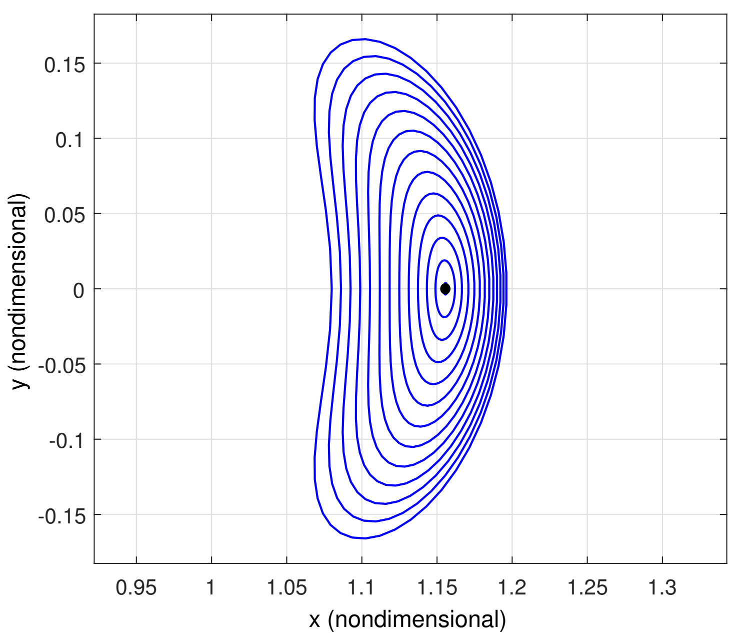

Fig. 4. Family of Earth–Moon L_2 planar Lyapunov orbits

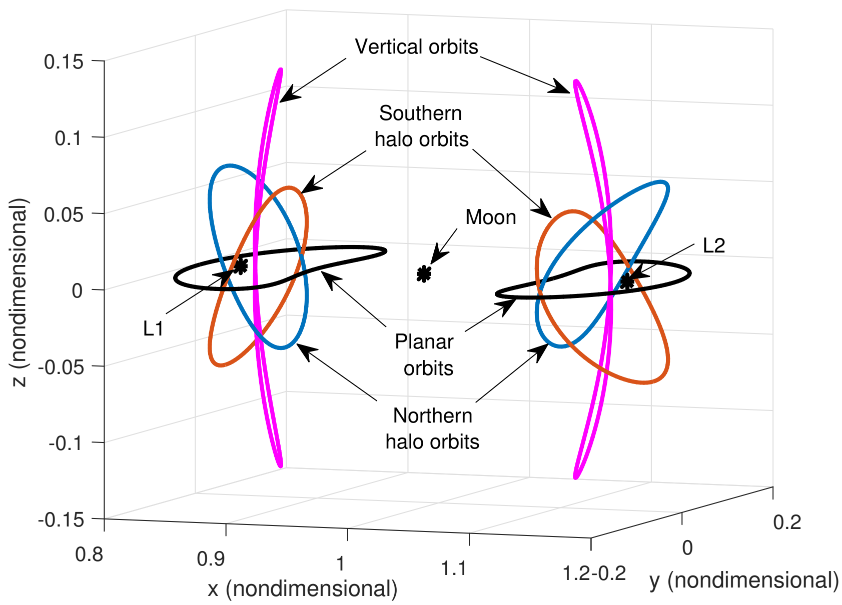

Fig. 5. Sample planar (black), northern halo (blue), southern halo (red), and vertical orbits (magenta) in the Earth-Moon system

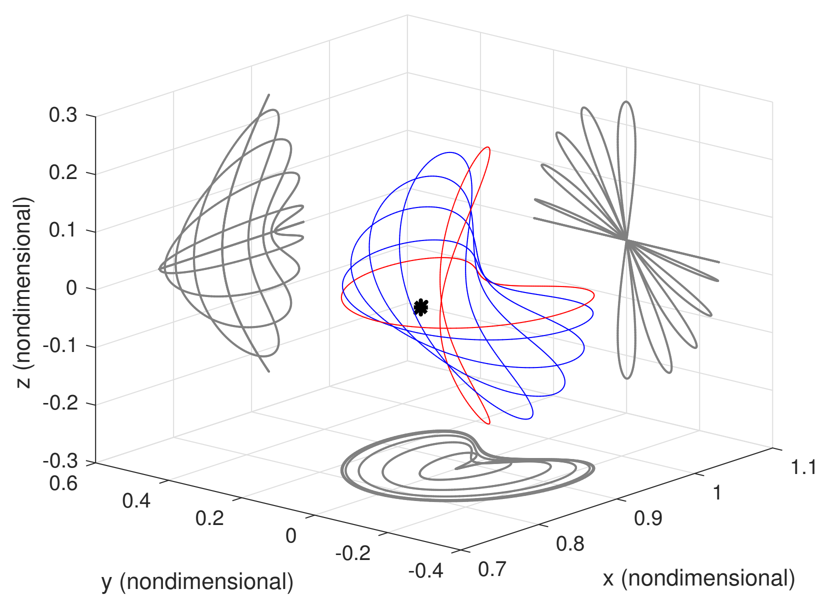

Fig. 6. Family of the axial orbits (blue); they connect the planar and the vertical families (red); the gray curves define projections onto the coordinate planes

manifolds organize the system's motion. This is one reason invariant solutions are of particular interest for the design of spacecraft trajectories. The libration points and the periodic orbits about them can have favorable properties for meeting science objectives and communication requirements. In addition, once a spacecraft is on a periodic orbit's stable manifold, it asymptotically approaches the orbit with no fuel cost. Likewise, it can depart the periodic orbit along the unstable manifold for free. This dynamical property is beneficial for the design of low-energy transfers.

To find the directions along the manifolds at some point of a periodic orbit $\mathbf{x}(t) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$, a state transition matrix $\mathbf{\Phi}(t)$ at point $\mathbf{x}(t)$ may be considered. Note that the matrix function $\mathbf{\Phi}(t)$ satisfies the equation

$$\dot{\mathbf{\Phi}}(t) = \mathbf{D}(\mathbf{x}(t))\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I}_{6\times 6},$$

where $\mathbf{I}_{6\times 6}$ is the identity 6×6 matrix and the Jacobian matrix $\mathbf{D}(\mathbf{x}(t))$ evaluated at $\mathbf{x}(t)$ is

$$\mathbf{D}(\mathbf{x}(t)) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & -U_{xz} & 0 & 2 & 0 \\ -U_{xy} & -U_{yy} & -U_{yz} & -2 & 0 & 0 \\ -U_{xz} & -U_{yz} & -U_{zz} & 0 & 0 & 0 \end{pmatrix}_{\mathbf{x}(t)}$$

Let T be a period of $\mathbf{x}(t)$, then $\mathbf{\Phi}(T)$ is called the *monodromy matrix*. For the unstable orbits in the vicinity of the libration points this matrix has six eigenvalues

$$\lambda_1 > 1, \ \lambda_2 = \lambda_1^{-1} < 1, \ \lambda_3 = \lambda_4 = 1, \ \lambda_5 = \lambda_6^*, \ |\lambda_5| = |\lambda_6| = 1,$$

where λ_5 and λ_6 are complex conjugates. Stable (and unstable) eigenspaces, E^s (E^u) are spanned by the eigenvectors whose eigenvalues less than one (modulus greater than one). Local stable and unstable manifolds are tangent to the eigenspaces at the fixed point and of the same dimension. Thus, for a fixed point $\mathbf{x}_0 = \mathbf{x}_t$ at the periodic orbit, the one-dimensional stable (unstable) manifold is approximated by the eigenvector $\mathbf{u}_s$ ($\mathbf{u}_u$) associated with the eigenvalue λ_2 (λ_1). It means that the corresponding eigenvector serves as the local approximation to the (non-local) stable or unstable manifold. Now consider the phase vectors

$$\mathbf{x}_s = \mathbf{x}_0 + \varepsilon \mathbf{u}_s,$$
$$\mathbf{x}_u = \mathbf{x}_0 + \varepsilon \mathbf{u}_u,$$

where ε is small. Integrating both forward and backward in time from $\mathbf{x}_s$ and $\mathbf{x}_u$ produces stable and unstable manifolds, each consisted of a single trajectory. In practice, if $\mathbf{u}_s$ and $\mathbf{u}_u$ are normalized (to one) vectors, the value of ε may be chosen equal to 10^{-6} [43].

2.3 Periodic and quasi-periodic orbits

The problem of finding periodic and quasi-periodic orbits in the CR3BP has been of great interest since the nineteen century [19, 64]. Pioneering contributions to studying periodic orbits are due to Moulton et al. [103], Stroemgren [123], Bray and Goudas [9], and Broucke [13]. Halo, vertical and planar orbits were studied using various methods by Farquhar [28, 31], Hénon [61, 62, 63], Breakwell [11], Howell [70] and others. Farquhar and Kamel [31] used the method of Linstedt-Poincaré to generate analytical solutions in the vicinity of the Earth-Moon L_2 point, whereas Breakwell and Brown [11] focused on determining three-dimensional halo orbits in the Earth-Moon restricted three-body problem. During the same time period, Richardson and Cary developed a third-order approximation of quasi-periodic motion near the Sun-Earth L_1 and L_2 libration points via the method of multiple time scales [114]. Still later, Richardson [112, 113] derived the third-order expansions of collinear libration point halo orbits in the Sun-Earth system with application to the ISEE-3 mission. Higher order formulas for periodic and quasi-periodic formulas can be found in Gómez et al. [46, 44]. These analytical developments can be profitably employed to find approximate periodic orbits.

Further relevant contributions to finding periodic orbits are related to the researches of Howell and Pernicka who in the 1980s presented a numerical shooting approach for correcting approximate quasi-periodic orbits, though lacking control of orbit parameters [75]. Barden and Howell in [5, 7] applied multiple shooting method to find quasi-halos in application for formation flying. Gómez et al. [49] used Fourier analysis and presented two algorithms for the Lindstedt-Poincaré procedure to construct quasi-halos. Guibout and Scheeres [53, 54, 55] applied the generation functions to search of periodic orbits. Recently Archambeau et al. [3] computed families of vertical Lyapunov orbits and investigated their specific stability properties. Kolemen et al. [84] presented a novel fully-numerical and fast method for finding quasi-periodic orbits around libration points. The orbit generation problem can be reposed as an optimal control problem: Pontani and Conway [108] use the particle swarm optimization technique for that purpose. Martin et al. [96] generates periodic orbits solving the optimal problem with the method of direct collocation with non-linear programming.

2.4 Manifolds and connecting orbits

Invariant manifold theory provides a powerful tool for understanding dynamical behavior in the CR3BP and play important role in mission design [2]. The Genesis spacecraft mission, designed to collect samples of solar wind and return them to the Earth [92], is often considered as the first mission to use invariant manifolds for its planning, while other missions have used libration point techniques [23]. Having a precise idea of the geometry of invariant manifolds and their connections is desirable in the design of complex low-energy missions.

There is much literature on invariant manifolds and connecting orbits in the CR3BP; see for example Gómez et al. [45] and Koon et al. [85, 86]. Calleja et al. [14] demonstrate the effectiveness of boundary value formulations coupled to numerical continuation for the computation of stable and unstable manifolds in CR3BP. This reference serves also as a detailed overview of continuation techniques, as used to compute periodic orbits, invariant manifolds, and connecting orbits.

The existence of connecting orbits in the planar problem has been proved analytically in Llibre et al. [91], and by computer assisted methods in Wilczak and Zgliczyński [125, 126]. Furthermore, these orbits have been extensively studied numerically using initial-value techniques and semi-analytical tools; see Barrabés et al. [8], Canalias and Masdemont [16] and references therein. In the case of initial-value techniques the initial conditions are varied in order for an appropriately chosen end point condition to be satisfied. This approach is commonly referred to as a "shooting method" and, for a more stable version, "multiple shooting".

Initial-value techniques can also be very effective in the computation of invariant manifolds in the CR3BP. However, sensitive dependence on initial conditions may leave parts of the manifolds unexplored, unless very high accuracy is used.

Jorba and Olmedo [78, 80] use a Fourier series to describe an invariant curve representing the intersection of an invariant torus with a Poincaré section in a perturbed CR3BP. Gómez and Mondelo [50] develop the Fourier expansion associated with a curve lying on the surface of a two-dimensional torus in the CR3BP, and use an invariance condition based on multiple shooting. Olikara and Howell [105] provide the direct computation of two-dimensional invariant tori including a natural parameterization and a continuation scheme. The development of a computational scheme is based on a method developed by Schilder et al. [119]. Gómez and Mondelo [50] developed a scheme for computing two-dimensional quasi-periodic tori by using an invariant circle parameterized by Fourier coefficients of a stroboscopic map. Kolemen et al. [84] use a similar approach except with multiple Poincaré maps and directly parameterizing the invariant circle using states. Olikara and Scheeres [104] used similar concepts to these methods but with a more general formulation. An alternative approach presented by Schilder et al. computes a torus of a flow directly [119]. The flow approach has been applied to the CR3BP [105]; however, this requires dealing with a torus of dimension one larger than the torus of an associated map. Mondelo et al. [101] present a methodology for the fast automatic generation of quasi-periodic libration point trajectories. This is based on the computation of a mesh of orbits and further interpolating strategies.

For computing the Floquet stability of tori, Jorba introduces a numerical approach based on the eigenvalues of a large, dense matrix [78]. Jorba and Olmedo also have an efficient method that combines the torus and stability computations, but it is specific to non-autonomous systems with known forcing frequencies [80].

2.5 Semi-analytical results

For Hamiltonian systems such as the CR3BP, semi-analytical methods are available for computing (quasi-)periodic orbits and invariant manifolds. Two such methods are the Lindstedt-Poincaré method and normal form scheme (center manifold reduction).

The Lindstedt-Poincaré method finds semi-analytical expressions for orbits and manifolds in terms of suitable amplitudes and phases by series expansions (see Masdemont [97]). This technique takes in consideration high order terms of the equations of motion and produces a formal series expansion of the solution of the equations of motion with high degree of accuracy. Gómez et al. [49] used semi-analytical methods based on the Lindstedt-Poincaré procedure to find the quasi-periodic orbits around libration points, though series expansions have slow convergence. Then, Gómez and Mondelo [50, 100] designed a refined Fourier analysis to find the full families, but this method is very slow and thus must be implemented on a cluster of parallel computers. Although these expansions are in general divergent, some practical domains of convergence can be computed. These local methods offer a thorough view of the dynamics in the vicinity of the libration points, but both are limited by their regions of convergence. Specialized algebraic manipulators are often required as well, which can create difficulties in the implementation. Another efficient method for computing invariant manifolds were designed by Alessi et al. [1]. Methods [50, 1] are very precise in a neighborhood of the center of expansion, and rely on other methods to extend the manifolds outside these neighborhoods [44].

Jorba and Villanueva [81] used the normal form method to find the center manifold around libration points. This process is based on expanding the initial Hamiltonian around a given equilibrium point and performing a partial normal form scheme, uncoupling (up to a high order) the hyperbolic directions from the elliptic ones. While the Lindstedt-Poincaré method provides compact expressions written in the initial coordinates for all trajectories, the reduction to the center manifold provides an easy way of producing qualitative plots of the dynamics close to the point.

In Jorba and Masdemont [79], the Lindstedt-Poincaré method as well as normal form scheme are adopted to semi-analytically construct the high-order solutions about the dynamics in the center manifolds of the collinear libration points in

CR3BP. Considering the hyperbolic behaviors together with the center behaviors around the collinear libration points in CR3BP, Masdemont [97] expanded the invariant manifolds as power series of hyperbolic and center amplitudes. These series expansions could explicitly describe the general dynamics around collinear libration points of CR3BP. Taking into account the perturbation of the Solar gravity and lunar eccentricity, Farquhar and Kamel [31] analytically developed the third order solutions of quasi-periodic orbits around the collinear libration point L_2 of Earth-Moon system and discussed the relationship between the frequency and amplitude for large halo orbits. In the real Earth-Moon system, defined by the JPL ephemeris, the quasi-periodic motions around the collinear libration points are analytically studied by Hou and Liu [67].

McGhee [98] provided analytical proofs of the existence of homoclinic orbits to Lyapunov periodic orbits around the L_1 and L_2 points for some particularly shaped homoclinic orbits.

Often finding semi-analytical solutions is followed by their refinement with differential correction methods. In particular, the periodic orbits are refined by shooting method [70], and the quasi-periodic orbits are refined by double-loop shooting method [75] or Fourier series correction method, see Jorba [78], Kolemen et al. [83, 84], Gómez and Mondelo [50]. Though recently Ren and Shan [110] presented a numerical algorithm that can generate long-term libration points orbits in the CR3BP and the full solar system model without using of semi-analytical approximations as initial guesses.

2.6 Elliptical restricted three-body problem

Compared to CR3BP, the elliptical restricted three-body problem (ER3BP) could approximate the Solar system better. Due to the existence of eccentricity of the primaries, the equations of motion in ER3BP are non-autonomous. Fortunately, the equations of motion of ER3BP in the pulsating synodic reference frame have the same symmetries as the ones of CR3BP, meanwhile, the dynamical properties are similar to those of CR3BP. For example, libration points and corresponding bounded orbits (Lissajous and halo orbits) also exist, and are unstable in nature. However, the investigation about the dynamics around the collinear libration points in ER3BP is more complicated by analytical method. Hou and Liu [66] constructed the high-order analytical solutions of the center manifolds, such as Lissajous and halo orbits in ER3BP, by means of semi-analytical method, and discussed the applications in the Earth-Moon and Sun-Earth system.

Early analytical expansions included the eccentricity of the primary bodies [31, 114] and recently periodic orbits have been numerically investigated [15]. Unlike the CR3BP, these orbits are isolated since most periodic orbits that persist become quasi-periodic orbits with the in influence of eccentricity. Considering the unstable dynamics of the collinear libration points and associated center manifolds in ER3BP, the general solutions of the equations of motion around the collinear libration points in ER3BP consist of the hyperbolic component (saddle behavior) and center component (center behavior). Lei et. al [89] expanded the solutions of invariant manifolds associated with libration point orbits in ER3BP as formal series of the orbital eccentricity and four amplitudes, thereinto, two amplitudes correspond to the hyperbolic manifolds, and the remaining two amplitudes correspond to the center manifolds. The series expansions constructed can describe the general dynamics around the collinear libration points of ER3BP, and can be considered as an extension of the ones discussed in Jorba and Masdemont [79], Masdemont [97], and Hou and Liu [66].

3 Station-keeping techniques for libration point orbits

3.1 Introduction, Target Point and Floquet Mode approaches

The problem of station-keeping for libration point missions has received much attention even before the first libration point mission of ISEE-3 in 1982. Some preliminary investigation on the motion control near libration points was done in 60's – 70's by Colombo [18], Farquhar [27, 29, 30], Euler [26] and Breakwell [10, 12]. Trajectory control investigation for ISEE-3 mission was done by Erickson and Farquhar [25, 33, 32] and also appeared in [46, 44].

Existing methods can be categorized by an approach to developing station-keeping techniques. First, the *Target Point Approach* is to be noted. The method computes correction maneuvers by minimizing a weighted cost function that is defined in terms of delta-v as well as position and velocity deviations from a nominal orbit at specified times called "target points". The approach was firstly presented by Howell and Pernicka [76] and Howell and Gordon [73]. The *Floquet Mode Approach* incorporates invariant manifold theory and Floquet modes to compute the maneuvers. Thus, the study of the station-keeping problem is conducted in the dynamical systems theory point of view. Floquet modes associated with the monodromy matrix are used to determine the unstable component corresponding to the local error vector. The maneuver is then computed in such a way that the dominant unstable component of the error is eradicated. The method was developed by Simó et al. in [121, 122]. An overview and comparison of Target Point Approach and Floquet Mode Approach was done by Keeter in [82].

3.2 Linear Time-Invariant and Linear Time-Varying models

Alternative classification can be done based on the model used. Indeed, many of the station-keeping control methods are designed based on *Linear Time-Invariant (LTI)* model via local linearization at the libration points. Here, Breakwell et al. [10, 12] were first to introduce classical optimal control strategies for Halo orbit missions and Erickson and Glass [25] specially analyzed the ISEE-3 mission to make this approach into implementation. Hoffman [65] applied the concept of disturbance accommodation for Halo orbits control. Later, Cielaszyk and Wie [17] additionally treated nonlinearities as the trajectory-dependent, persistent disturbance inputs to apply similar approach. Di Giamberardino and Monaco [20] designed a nonlinear controller based on LTI model to solve the problem of reaching and tracking a prescribed quasi-Halo orbit about L_2 in the Earth-Moon system. Under suitable assumptions the control strategy achieves asymptotic tracking and asymptotic disturbance compensation.

Noting that the LTI model includes only 1st, 2nd, or 3rd order term of the gravitational force and effective only in the neighborhood of the libration point, *Linear Time-Varying (LTV)* model were employed to improve the modeling accuracy. Developed by Howell and Pernicka [76] the Target Point Approach and the nonlinear techniques via Floquet Mode Approach developed by the Barcolona Group [122, 46, 44] are based on LTV model. Gurfil and Kasdin [57], motivated by the problem of station-keeping for spacecraft formations, derived a time-varying continuous linear quadratic control law with linearized dynamics about an arbitrary reference trajectory about L_2 in the Sun-Earth system.

3.3 Other methods

Later, Kulkarni et al. [87, 88] extended the traditional H_∞ framework to periodic discrete LTV systems for stabilization of spacecraft flight in Halo orbits around L_1 of the Sun-Earth system. The work points out that Target Point Approach methods suffer from the disadvantage that the choice of the various parameters (the number and spacing of future target times, the particular values for elements of the various weighting matrices) can be determined only by trial and error, while for the H_∞ method, the choice of the various matrices is guided by mission requirements and can be chosen easily. Similar to the Target Point Approach, Floquet Mode Approach suffer from the disadvantage that a search for an optimal controller, which minimizes a linear combination of the cost and the deviation from the halo orbit, cannot be conducted in a systematic fashion. Rahmani et al. [109] solved the problem of Halo orbit control using the optimal control theory and the variation of the extreme technique to solve the resulting two point boundary value problem. This method benefits from control of spacecraft exactly on the nominal halo orbit and utilizes full nonlinear equations of motion and a numerical method to determine the required control accelerations. As concerns some of the recent station-keeping methods for single spacecraft, Bai and Junkins [4] proposed a modified Chebyshev-Picard iteration method for station-keeping of L_2 Halo orbits of the Earth-Moon system. To meet the requirements of robustness and low computation burden, Zhu et al. [132] applied a new nonlinear station-keeping control law based on active disturbance rejection control method.

3.4 Formation-keeping

A variety of methods was developed specially for spacecraft formations near libration points. An analysis of natural formation dynamics along the central manifold was done by Howell and Marchand [74]. Earlier, Howell and Barden [6, 5, 72] investigated formation flying in the perturbed Sun-Earth system. An investigation of natural and non-natural formations was done also by Marchand in her thesis [93].

Scheeres and Vinh [118] present control law that achieves bounded motion near the vicinity of a halo orbit, as determined in Hill's model. Although the latter approach is not suitable for precise formation-keeping, nor is it necessarily the optimal way to achieve boundedness, it does satisfy other goals that may be important for certain types of missions. In particular, the natural winding frequency of the spacecraft around the reference halo orbit is significantly increased. This is consistent with one of the stated requirements for the TPF mission, where the formation is required to achieve a particular rotation rate that is not consistent with the natural dynamics near this region of space.

Gurfil et al. [56] presented a novel nonlinear adaptive neural control methodology for deep-space formation flying in the Sun-Earth system. Authors note that future missions such as MAXIM Pathfinder raise stringent submillimeter relative position control accuracy specifications that do not permit the utilization of linearization techniques. Thus, the complete nonlinear models should be used, so that no approximation is involved in the design procedure. In that purpose the neural-network control methods is applied. In addition to controlling the formation on complex trajectories, this algorithm effectively compensates for deep-space disturbances such as solar radiation pressure (SRP) and fourth-body gravitation and allows to keep submillimeter relative position accuracy. Xin et al. [131, 130] used a suboptimal control technique (the θ – D technique) and applied it in [129] to control each individual spacecraft so as to keep a constant relative distance from the center of the virtual body followed a nominal orbit around the L_2 libration point in the Sun-Earth system. Wang et al. [124] presented a nonlinear controller based on polynomial eigenstructure assignment of Quasi-LTV model for the control of Sun-Earth L_2 point formation flying, though the system uncertainties were not considered. Within the framework of the ephemeris model, Hamilton and Folta [60, 34] consider linear optimal control for formation flight relative to Lissajous trajectories, as determined in the ephemeris model. However, the evolution of the controlled formation is approximated from a linear dynamic model relative to the integrated reference orbit. The analysis within the context of both the circular restricted three-body problem and the more complete ephemeris model was also done by Marchand and Howell [95]. Here the LQR and feedback linearization approach was employed for formation flight in the vicinity of libration points.

3.5 Station-keeping in the Earth-Moon system

Many investigations were applied to station-keeping in the Earth-Moon system. In this case the station-keeping is rather challenging than in the Sun-Earth system because of short time scales of divergence, effects of large orbital eccentricity of the Moon, and perturbations of the Sun. Due to the flight of ARTEMIS, the first Earth-Moon libration orbiter, several station-keeping strategies was recently investigated by Folta et al. [38, 39, 40, 37, 35, 128], and more earlier works by Folta et al. [36, 34]. Classical works based on the Floquet Mode Approach are presented in papers by Farquhar [30], Breakwell et al. [12], Simó et al. [122], Keeter [82], Gómez et al. [42]. Janes and Beckman [77] designed a global search station-keeping approach that maintains a spacecraft in orbit for the next 1-2 revolutions downstream. Gurfil and Meltzer [58] utilized a finite-horizon LQR scheme to track reference trajectories while rejecting persistent disturbances using the realistic ER3BP model. Grewbow et al. [52] used maneuvers to target back to a rigid baseline solution, or, at least, to target specific parameters downstream, but this general approach can result in higher station-keeping costs and can create challenges when transitioning into the next mission phase. Based on this work, Pavlak in his thesis [106] applied a control-point station-keeping algorithm to a variety of orbits in the vicinity of the L_2 libration point. Note that Floquet Mode Approach is useful to estimate the maneuver directions only for short-term station-keeping. Pavlak and Howell [107] establish a long-term station-keeping strategy based on multiple shooting method that does not require close tracking of a baseline trajectory but still meets a specific set of end-of-mission objectives, i.e., conditions at lunar arrival. Hou et al. [68] considered quasi-Floquet approach to cancel the unstable component of the motions around dynamical substitutes – quasi-periodic orbits caused by perturbations from the Moon's orbit eccentricity and the Sun. Ghorbani and Assadian [41] developed continuous and impulsive control strategies taking account of the gravitational perturbation of the

Sun and other planets and Moon's eccentricity. A sliding mode control technique was applied by Lian [90].

3.6 Stability discussions

As concerns discussions of stability and control for vehicles at both collinear and triangular libration point locations, Hoffman [65] and Farquhar [30] both provide analysis and discussion of stability and control in the Earth-Moon collinear L_1 and L_2 locations, respectively, within the context of classical control theory or linear approximations; Scheeres et al. offer a statistical analysis approach and investigated the generalized optimal placement of statistical control maneuvers [117, 111]. Gustafson and Scheeres [59] studied the optimal timing to update control-law with continuous control. Marchand and Howell [94] discuss stability including the eigenstructures near the Sun-Earth locations.

3.7 Missions

At last we give here some references concerning the station-keeping design of some successful and future missions around libration points: ISEE-3 [25, 33, 32, 46, 44], SOHO [115], WIND [120], MAP [116], Genesis [71, 127], ARTEMIS [35, 38], TPF [47], World Space Observatory/Ultraviolet [69]. Some information concerning different methods for calculating libration-point orbit station-keeping maneuvers for ISEE-3, SOHO and ACE missions can be found in [24].

Conclusion

In this overview, methods for designing the periodic orbits around collinear libration points as well as associated invariant manifolds are considered. First, basic information relating to collinear libration points and the phase space around them was presented. Then the problem of finding (quasi-)periodic orbits and their invariant manifolds is considered. Here both pure numerical and semi-analytical results are mentioned. At last analytical results concerning elliptical restricted three-body problem are mentioned. In the second part of the work, station-keeping methods are given. Here a several classifications were made by an approach to developing station-keeping techniques: Target Point and Floquet Mode approaches, Linear Time-Invariant of Linear Time-Varying models to describe the dynamics around libration points, formation-keeping and etc.

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