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FAMILIES OF SYMMETRIC PERIODIC SOLUTIONS OF THE GENERALIZED HILL'S PROBLEM

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Рассматривается некоторое обобщение известной небесно–механической задачи — задачи Хилла. Это обобщение позволяет исследовать семейства периодических решений задачи Хилла как продолжение порождающих решений двух интегрируемых задач: синодической задачи Кеплера и задачи Энона. Показано, что для полного исследования семейств периодических решений задачи Хилла необходимо рассматривать ее вариант, для которого потенциал притяжения центрального тела заменен потенциалом отталкивания.
Ключевые слова: периодические решения, система Гамильтона, симметричное решение, задача Хилла.

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Families of symmetric periodic solutions of the Generalized Hill's Problem

Certain generalization of the well–known celestial mechanics problem, namely the Hill's problem, is considered. This generalization makes possible to study families of periodic solutions of the Hill's problem as continuation of generating solutions of two integrable problems: the Sinodical Kepler problem and the Hénon's problem. To provide complete study of families of periodic solutions it is necessary to consider a special case of the Hill's problem, called the anti–Hill's problem, where the Newtonian potential of attraction of the central body is replaced by the Newtonian potential of repulsion.
Key words: periodic solutions, Hamiltonian system, symmetric solution, Hill's problem.

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Contents

In memory of Professor Michel Hénon

1. Introduction

A common schema of studying of any complex problem consists of steps which consequently simplify the original problem up to the case that can be investigated by known methods or was considered earlier. Sometimes we have to make a backward step by embedding the obtained simplified problem into more general one. Usually such embedding looks quite naturally, because during the simplification steps we could loose some essential properties of the original complex problem.

We consider in this work a certain generalization of a celestial mechanics problem called the *Hill's problem.* The original planar Hill's problem is a limiting case of the well-known restricted three body problem (RTBP) and was proposed by G. Hill for the Moon motion theory [1, 2]. For detail information about the Hill's problem see, for example, [1, 3, 4]. This generalization links together three different problems and makes it possible to investigate periodic solutions of these problems simultaneously.

The first problem is the famous Kepler problem which is considered in the uniformly rotating frame and is called here the *Sinodical Kepler problem* (or in brief the SKP). The SKP is integrable, its families of symmetric periodic solutions are completely investigated (see [5, Ch. III]) and, more over, the SKP is often used as an unperturbed part for many non-integrable problems, e. g. RTBP.

The second problem is so called *Hénon problem* (see [6]), which gives us the description of motion of massless particle that guides the smallest massive body in RTBP and moves outside the Hill's region of this primary. The Hénon problem provides the first approximation of so called *quasi-satellite motion.*

Finally, **the third problem** is the mentioned above the *Generalized Hill's problem* (or in brief the GHP) in which the Newtonian potential of attraction of the massive body can be changed by the Newtonian potential of repulsion or even switched off. We note that the Hénon's problem takes an intermediate place between the SKP and the Hill's problem and it is also integrable.

Hamiltonian $H(\mathbf{z})$ of the GHP is

$$H(\mathbf{z},\varepsilon) = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 + \frac{\sigma}{|\mathbf{x}|} + \varepsilon\left(-x_1^2 + \frac{1}{2}x_2^2\right), \tag{1}$$

or in matrix form

$$H(\mathbf{z},\varepsilon) = \langle G\mathbf{z},\mathbf{z}\rangle + \frac{\sigma}{|\mathbf{x}|},\ \text{with } G = \begin{pmatrix} -\varepsilon & 0 & 0 & -1/2 \\ 0 & \varepsilon/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix}, \tag{2}$$

where $\mathbf{x}$ is vector of coordinates, $\mathbf{y}$ – vector of momenta, $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, $\sigma \in \{-1,0,1\}$ and $\varepsilon \in [0; 1]$.

The canonical equations of Hamiltonian (1) has first integral called Jacobi integral

$$J = (1 + 2\varepsilon)x_1^2 + (1 - \varepsilon)x_2^2 - \frac{2\sigma}{|\mathbf{x}|} - \dot{x}_1^2 - \dot{x}_2^2 = C, \quad C = -2H(\mathbf{z}, \varepsilon) = -2h. \quad (3)$$

The essential property of the GHP is the presence of discrete group of symmetries of extended phase space given by linear transformations

$$\begin{aligned} \Sigma_1 &: (t, x_1, x_2, y_1, y_2) \to (-t, x_1, -x_2, -y_1, y_2) \\ \Sigma_2 &: (t, x_1, x_2, y_1, y_2) \to (-t, -x_1, x_2, y_1, -y_2) \\ \Sigma_{12} \equiv \Sigma_1 \circ \Sigma_2 &: (t, x_1, x_2, x_1, x_2) \to (t, -x_1, -x_2, -y_1, -y_2) \end{aligned}$$

which involves that all the periodic solutions of the GHP belong to one of the following group:

1) Asymmetric solutions, which change their form under any transformation $\Sigma_{1,2}$.
2) Single symmetric solutions, which are invariant under only the one transformation Σ_1 or Σ_2.
3) Double symmetric solutions, which are invariant under any transformation Σ_1, Σ_2 or Σ_{12}.

The presence of symmetry of equations considerably simplify the usage of generating solutions defined by corresponding limiting problems.

The following Table 1 summaries the special cases of the GHP.

Table 1. Special cases of the GHP

Abbr.	**Problem name**	σ	ε	**Hamilt.**
GHP	Generalized Hill's problem	$\{\pm1,0\}$	$[0;1]$	(1)
SKP	Sinodic Kepler problem	-1	0	(6)
–	Hénon problem	0	1	(9)
–	Parametric Hénon problem	0	$[0;1]$	(9)
HP	Hill's problem	-1	1	(1)
AHP	anti–Hill's problem	$+1$	1	(1)

For small values of the parameter ε the GHP can be considered as the SKP with regular perturbation $\varepsilon R = \varepsilon\left(-x_1^2 + x_2^2/2\right)$. We show below that for large values h of Hamiltonian $H(\mathbf{z}, \varepsilon)$ the GHP can be considered as singular perturbation of the so called Parametric Hénon problem (see it definition on page 7) by function $\sigma/|\mathbf{x}|$ as well. Thus there are two main approaches for investigation of periodic solutions of the GHP:

1) using generating solutions of the Parametric Hénon problem;
2) using generating solutions of the SKP.

The first approach was partially implemented in [4, 7] by the author for the case of $\sigma = -1$, $\varepsilon = 1$ and for symmetrical orbits only. The second approach was proposed by Professor A.D. Bruno in [8] but still was not implemented. This paper is an attempt to realize the second approach for doubly symmetric periodic solutions.

2. Limiting cases of the GHP

The following steps provide obtaining all the limiting cases of the GHP and corresponding Hamiltonians as well. For details see [9, Ch. 4]. All the following computations are carried out in the space of vector power exponents, where each term $Ax_1^{r_1}x_2^{r_2}y_1^{r_3}y_2^{r_4}\varepsilon^{r_5}$ of the function $H(\mathbf{x}, \mathbf{y}, \varepsilon)$ derives the point with coordinates $(r_1, r_2, r_3, r_4, r_5)$.

1) Define the cone $\mathbf{K}$ of the problem.

2) Compute the support $\mathbf{S}$ of the Hamiltonian $H(\mathbf{z})$ – the set of vector power exponents of all the terms of function $H(\mathbf{z}, \varepsilon)$. Then the projection $\mathbf{S}'' = \pi\mathbf{S}$ is computed, where $\pi : (r_1, r_2, r_3, r_4, r_5) \mapsto (r_1 + r_2, r_3 + r_4, r_5)$.

3) Compute the convex hull $\mathbf{\Gamma}$ of the support $\mathbf{S}''$, which is a polyhedron in $\mathbb{R}^3$. The boundary of $\mathbf{\Gamma}$ is the union of generalized faces $\mathbf{\Gamma}_j^{(d)}$ (2D faces, edges and vertices). For each face $\mathbf{\Gamma}_j^{(d)}$ the corresponding external cone $\mathbf{U}_j^{(d)}$ is computed.

4) Only those faces $\mathbf{\Gamma}_j^{(d)}$ are selected which cone $\mathbf{U}_j^{(d)}$ has non empty intersection with the cone $\mathbf{K}$ of the problem.

5) The truncated Hamiltonian $\widehat{H}_j^{(d)}$ is computed for each selected face $\mathbf{\Gamma}_j^{(d)}$. This Hamiltonian gives the first approximation of the original function $H(\mathbf{z}, \varepsilon)$ in the certain domain of the space $(x_1, x_2, y_1, y_2, \varepsilon)$.

6) Often it is useful to make an additional power transformation of the phase space that is able to simplify the truncated polynomial $\hat{H}_j^{(d)}$ and represents the original Hamiltonian $H(\mathbf{z}, \varepsilon)$ in the form

$$H = \widehat{H}_j^{(d)} + \mu_j R_j, \tag{4}$$

where μ_j is a small parameter and R_j is a perturbation of the first approximation Hamiltonian $\hat{H}_j^{(d)}$.

The implementation of above schema of computation to Hamiltonian (1) is following.

1) In the case of the GHP the cone of the problem is

$$\mathbf{K} = \{\mathbf{W} \in \mathbb{R}^3_* : w_3 \leqslant 0\},$$

because we consider the parameter $\varepsilon \in [0;1]$.

2) Support $\mathbf{S}$ of the Hamiltonian (1) contains points with coordinates

$$(0,0,2,0,0), (0,0,0,2,0), (0,1,1,0,0), (1,0,0,1,0), (2,0,0,0,1), (0,2,0,0,1)$$

and the support of the monomial σ/r which is segment $\mathbf{J}$ connecting the points $(-1,0,0,0,0)$ and $(0,-1,0,0,0)$. Projection π yields the projection $\mathbf{S}''$ which contains the points

$$\Gamma_1^{(0)} = (-1,0,0), \Gamma_2^{(0)} = (0,2,0), \Gamma_3^{(0)} = (1,1,0), \Gamma_4^{(0)} = (2,0,1). \tag{5}$$

3) The convex hull $\boldsymbol{\Gamma}$ is a tetrahedron with 4 vertices with coordinates given by (5), 6 edges and 4 faces and is shown in Figure 1. Only three faces have external

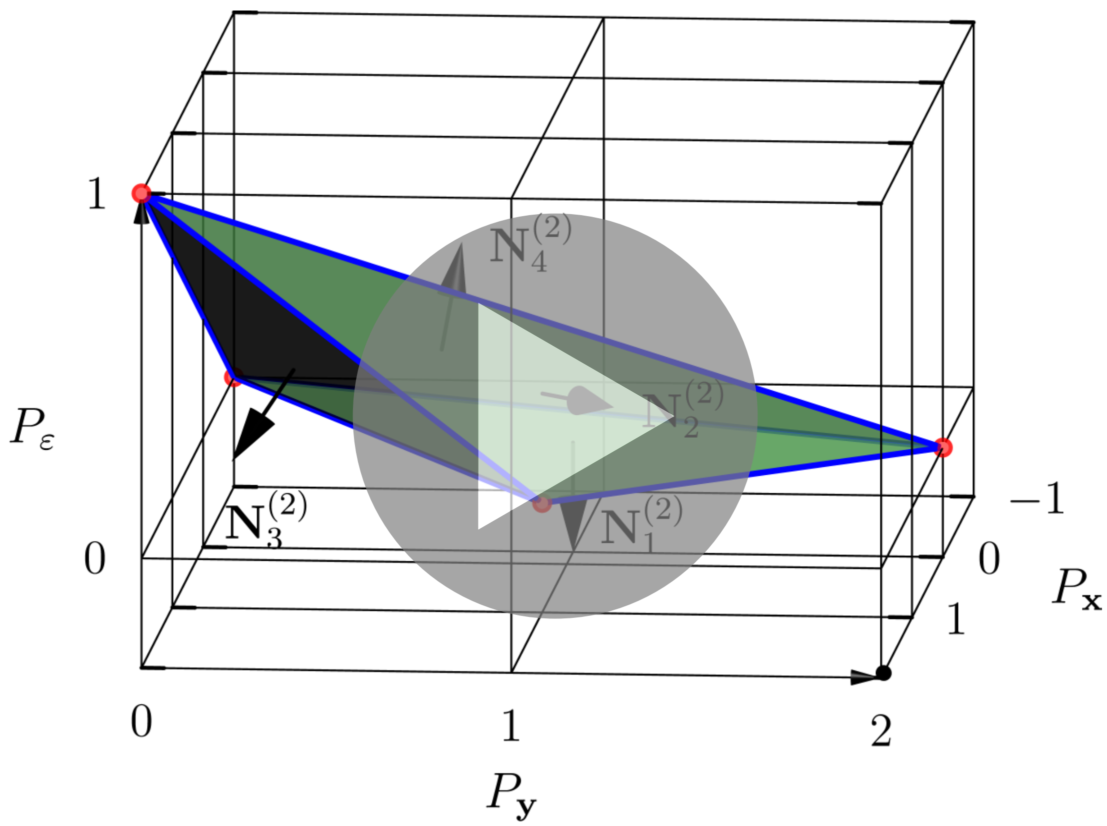


Figure 1. Convex hull $\boldsymbol{\Gamma}$ of the GHP Hamiltonian (1). Normal vectors $\mathbf{N}_i^{(2)}$, $i = 1,\ldots,4$ are shown.

normal with non–positive third coordinate, i. e. these faces and its common edges could give suitable truncated polynomial.

A) The face $\Gamma_1^{(2)}$ contains the vertices $\Gamma_1^{(0)}$, $\Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$. Its normal vector $\mathbf{N}_1^{(2)} = (0{,}0,-1)$ and truncated Hamiltonian is

$$\hat{H}_1^{(2)}(\mathbf{z},\varepsilon) = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 + \frac{\sigma}{r}, \tag{6}$$

which is Hamiltonian of the SKP, if $\sigma = -1$. The case $\sigma = +1$ is not interesting because it has not any periodic solution. Thus, the SKP is a limiting case of the GHP and the next step of investigation is isolation such periodic solutions of the SKP that can be continued into periodic solutions of the GHP. This truncation is suitable for $\varepsilon \to 0$.

B) The edge $\Gamma_1^{(1)}$ contains the vertices $\Gamma_1^{(0)}$, $\Gamma_2^{(0)}$. Its normal cone $\mathbf{U}_1^{(1)}$ is spanned on vectors $\mathbf{N}_1^{(2)}$ and $\mathbf{N}_4^{(2)} = (-2{,}1{,}6)$:

$$\mathbf{U}_1^{(1)} = \left\{\mathbf{W} \in \mathbb{R}^3 : \mathbf{W} = (1-\lambda)\mathbf{N}_1^{(2)} + \lambda\mathbf{N}_4^{(2)} = (-2\lambda, \lambda{,}7\lambda - 1),\ \lambda \in [0;1]\right\}.$$

For $0 \leqslant \lambda \leqslant 1/7$ the truncated Hamiltonian

$$\hat{H}_1^{(1)} = \frac{1}{2}\left(y_1^2 + y_2^2\right) + \frac{\sigma}{|\mathbf{x}|} \tag{7}$$

provides approximation of (1) for finite values of the parameter ε and for motion near the origin, i. e. for $|\mathbf{x}| \to 0$. Function (7) is a Hamiltonian of the Kepler problem in inertial frame and is a particular case of the Hamiltonian (6). The vector $\mathbf{N} = \left(-1, \frac{1}{2}, \frac{7\lambda-1}{2\lambda}\right)$ from the cone $\mathbf{U}_1^{(1)}$ defines the following canonical transformation

$$\mathbf{x} = |h|^{-1}\mathcal{X}, \mathbf{y} = |h|^{1/2}\mathcal{Y}, \varepsilon = |h|^{\frac{7\lambda-1}{2\lambda}}\mathcal{E}, t = |h|^{3/2}\tau, 0 \leqslant \lambda \leqslant \frac{1}{7}, \tag{8}$$

which blows up the vicinity of the origin in the coordinate space and shrinks the vicinity of infinity in momentum space. Applying transformation (8) to the whole Hamiltonian (1) one can rewrite the last in the form

$$\widetilde{H}_1 = \frac{1}{2}\left(\mathcal{Y}_1^2 + \mathcal{Y}_2^2\right) + \frac{\sigma}{|\mathcal{X}|} + \frac{\mathcal{X}_2\mathcal{Y}_1 - \mathcal{X}_1\mathcal{Y}_2}{|h|^{3/2}} + |h|^{\frac{3\lambda-1}{2\lambda}}\mathcal{E}\left(-\mathcal{X}_1^2 + \frac{1}{2}\mathcal{X}_2^2\right).$$

So, the coefficient $|h|^{-3/2}$ plays the role of small parameter μ_j for $h \to \infty$.

C) The face $\Gamma_2^{(2)}$ contains the vertices $\Gamma_2^{(0)}$, $\Gamma_3^{(0)}$ and $\Gamma_4^{(0)}$. Its normal vector

$\mathbf{N}_2^{(2)} = (1{,}1{,}0)$ and truncated Hamiltonian is

$$\hat{H}_2^{(2)}(\mathbf{z}, \varepsilon) = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 + \varepsilon\left(-x_1^2 + \frac{1}{2}x_2^2\right), \tag{9}$$

which is Hamiltonian of the *Parametric Hénon problem.* This truncated Hamiltonian provides the approximation for large values of phase coordinates, i. e. for $|\mathbf{x}| \to \infty$, $|\mathbf{y}| \to \infty$, and for finite value of the parameter ε. The canonical power transformation $\mathbf{z} = \sqrt{|h|}\mathbf{Z}$ shrinks the vicinity of the infinity and applying it to the whole Hamiltonian (1) one can rewrite the last in the form

$$\widetilde{H}_2 = \langle G\mathbf{Z}, \mathbf{Z}\rangle + |h|^{-3/2}\frac{\sigma}{|\mathbf{X}|},$$

where G is the matrix from (2) and h is the value of the Hamiltonian $H(\mathbf{z}, \varepsilon)$. So, the coefficient $|h|^{-3/2}$ plays the role of small parameter μ_j for $h \to \infty$. Thus, this limiting problem can be used to obtain periodic solutions of the GHP. It was done in [4, 7] for $\varepsilon = 1$ and symmetric periodic solutions. Note that (9) is limiting case for both Hill's and anti–Hill's problem. This fact is used below to link with each other families of periodic solutions of last two problems.

D) The face $\Gamma_3^{(2)}$ contains the vertices $\Gamma_1^{(0)}$, $\Gamma_3^{(0)}$ and $\Gamma_4^{(0)}$. Its normal vector $\mathbf{N}_3^{(2)} = (1, -2, -3)$ and truncated Hamiltonian is

$$\widehat{H}_3^{(2)}(\mathbf{z}, \varepsilon) = x_2 y_1 - x_1 y_2 + \frac{\sigma}{r} + \varepsilon^{1/3}\left(-x_1^2 + \frac{1}{2}x_2^2\right). \tag{10}$$

This Hamiltonian can be simplified by canonical transformation

$$\mathbf{x} = \varepsilon^{-1/3}\mathbf{X}, \mathbf{y} = \varepsilon^{2/3}\mathbf{Y}, t = \varepsilon\tau,$$

and in new variables it takes form (10) with $\varepsilon = 1$. The system of canonical equations, defined by $\hat{H}_3^{(2)}$ is integrable but does not contain periodic solutions. So this case is out of our interest.

3. Generating solutions and their properties

The idea of generating solution, introduced by Poincare [10], is used in this study. It was very fruitful in studying periodic solutions of the RTBP (see [5, 11]).

Definition 1. Let at small parameter value $\mu > 0$ there exists a periodic solution $\mathbf{z}(t, \mu)$ to canonical system defined by Hamitonian (4), and it could be smoothly continued over μ. Then its limit at $\mu \to 0$ (if it does exist) is called *generating solution.*

Regular generating solution could be found by the methods of normal forms.

In case of Hamiltonian system with two degrees of freedom this technique is described in [5].

Singular generating solutions are computed in more complex way and strict proof of their existence could be done usually in certain cases only.

Let the system has a periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ with period T: $\mathbf{z}(T, \mathbf{z}_0) = \mathbf{z}_0$ for definite value of Hamiltonian $H(\mathbf{z}_0) = h_0$. Let ρ'_h and ρ''_h are the minimal and the maximal distance from the origin to orbit $\mathbf{x}(t, \mathbf{z}_0)$. There are three possible ways of continuation of periodic solution $\mathbf{z}(t, \mathbf{z}_0)$ while $h \to \infty$:

1) If $\lim\limits_{h\to\infty} \rho'_h > 0$ then we get generating solution of the first species; such solution is regular and can be find by the normal form method [5, Ch. II, VII].

2) If $\lim\limits_{h\to\infty} \rho'_h = 0$ and $\lim\limits_{h\to\infty} \rho''_h > 0$ then we get generating solution of the second species.

3) If $\lim\limits_{h\to\infty} \rho'_h = \lim\limits_{h\to\infty} \rho''_h = 0$ then we get generating solution of the third species.

If generating solution is not regular then it may consist of solutions of special form called *arc–solution*. The arc–solutions start and finish at the singular points of the Hamiltonian (1). In our case generating solutions should be composed from arcs which start and finish at the origin.

4. Linking families of the Hill's and anti–Hill's Problems

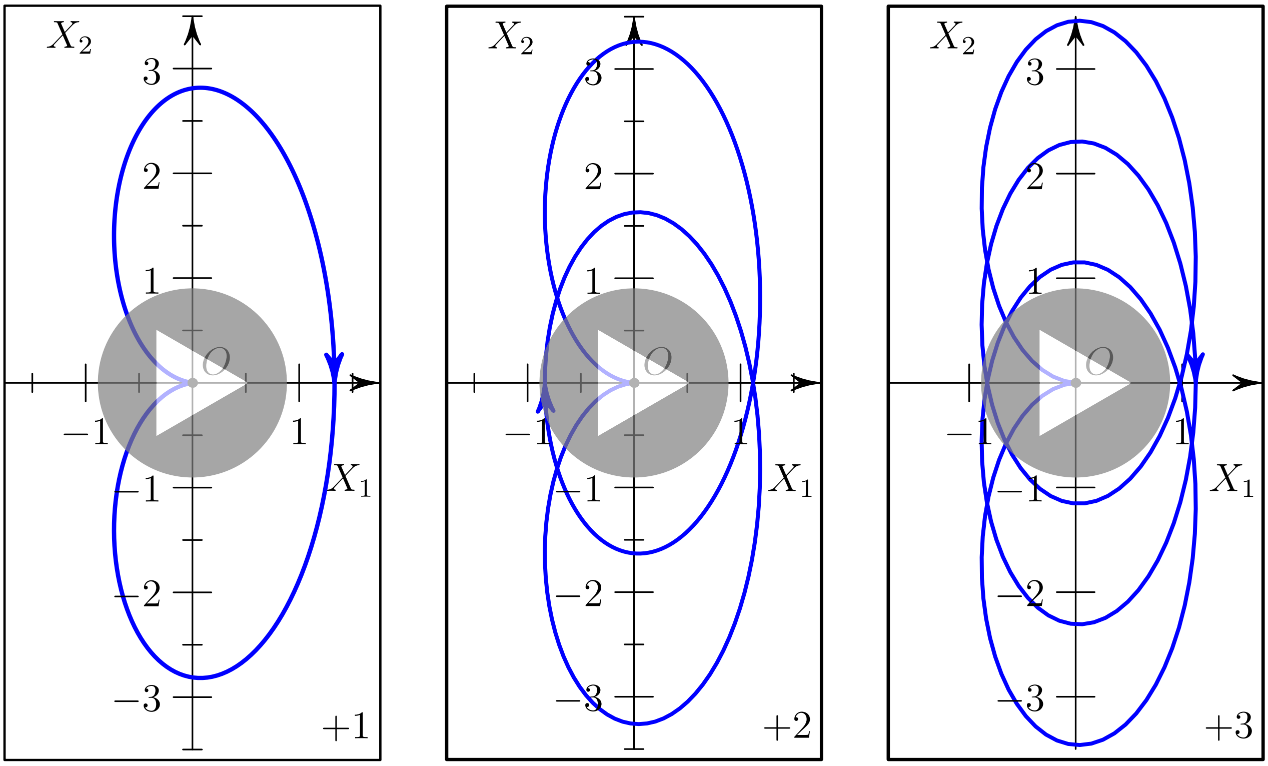
To obtain the periodic solution of the singularly perturbed system from the generating solution one has to provide a matching procedure of arc-solutions near the origin. The matching procedure of the first order matches the velocities of solution of the first limiting problem and velocities of solutions of the second limiting problem at the origin. If matching procedure is successful then we get a periodic solution of a certain family and the whole family can be computed numerically by one of the continuation algorithm (see, for example, [12]).

We apply the described above approach to the Hill's and anti–Hill's problems. Let consider the GHP for $\varepsilon = 1$ when the value $h \to -\infty$. There are two suitable limiting problems obtained in Section 2: the first is the Hénon problem with Hamiltonian (9) and the second is Kepler problem with Hamiltonian (7). The Hénon problem has two sets of suitable solutions:

1) one-parametric family of regular periodic solutions, which contains the only one generating solution of the first species (see [4]);

2) the countable set of arc-solutions of two types, which are solutions of the second species.

The arc-solutions of the first type were denoted by M. Hénon [13] by symbols $\pm j$, $j \in \mathbb{N}$. The orbits of the first type arc-solutions are epicycloid. In Figure 2 three arcs for positive values of index j are shown. The arcs with negative value of index j are symmetric with respect to the axis OY. The arc–solutions of the second type are denoted by symbols i and e and its orbits are ellipses passing through the origin (see Figure 3).



(a) Arc +1 **(b)** Arc +2 **(c)** Arc +3

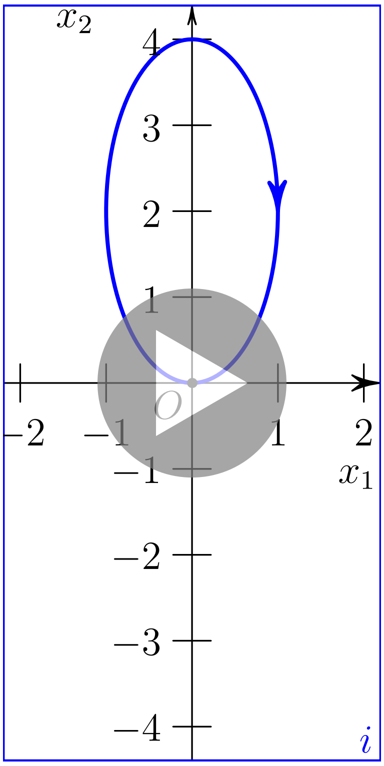
Figure 2. Arc-solutions of the first type +1, +2 and +3.

The second limiting problem is Kepler problem, which gives solutions in the form of hyperbola with semi-major axis equals to 1.

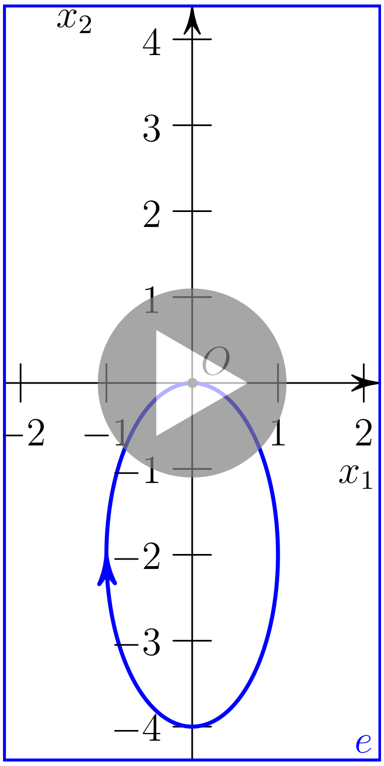
All arc-solutions pass through the origin and it is possible to compose the infinite number of sequences from arc-solutions $\pm j$, $j \in \mathbb{N}$, i, e by matching these arcs with hyperbolas of two types (see [4, 14]). Hyperbolas of the first type have its pericenter near OX axis and eccentricity $e \approx 1$, hyperbolas of the second type have pericenter near OY axis and eccentricity $e \gg 1$.

The analysis of the structure of the phase space of the Hill's problem depending on the value of Jacobi integral C was given in [4, 14], and it was shown that suitable solutions of the Hénon problem and Kepler problem can exist simultaneously only for $C < 3^{4/3} \approx 4.328$. Thus, generating solutions can give only those families of periodic orbits which are continuable up to $C \to -\infty$ (or $h \to +\infty$).

M. Hénon stated in [13] that for Newtonian potential of attraction (i. e. for $\sigma = -1$) there are two pairs of arc-solutions, namely, ii and ee, which can not be matched to each other by hyperbolas described above.



(a) Arc i



(b) Arc e

Figure 3. Arc-solutions of the second type i and e.

Statement 1 (M. Hénon [11])**.** A sequence of arc-solutions which does not contain two identical arcs of the second type in succession is a generating solution and it is called *generating sequence* for the Hill's problem.

Numerical analysis of all known families of periodic solutions of the Hill's problem allows to state the following

Statement 2. Each family of periodic solutions of Hill's problem, which is continuable to solution of the second species, is defined at the limit by generating sequence of Statement 1.

Moreover, numerical explorations of periodic solutions of the anti–Hill's problem show that there are almost no limitations in the structure of generating sequences.

Statement 3. The sequence composed from the arc-solutions j, $j \in \mathbb{N}$, i, e in arbitrary order, except two sequences consisting of arcs $\{i\}$ and $\{e\}$ only, is a *generating sequence* for anti–Hill's problem's family of periodic solutions.

The generating sequence allows to determine the following properties of corresponding family:

- the type of symmetry of periodic orbits of the family;

• global multiplicity of periodic orbits of the family;

• asymptotics of initial conditions, period and stability index of periodic orbits of the family when $C \to -\infty$.

An algorithm for studying symmetric periodic solutions defined by its corresponding generating sequences was proposed by the author in [4, 15].

1) A generating sequence is composed in according with Statement 1 or Statement 3 depending on Hill's or anti–Hill's problem is considered.
2) The type of symmetry, approximate initial conditions and period of solutions are computed.
3) An orbit of the corresponding family is computed iteratively.
4) The whole family is computed by one of the continuation method.
5) During the computation of the whole family stability of periodic solutions and its bifurcations are detected as well.

More then 20 new families of periodic solutions were found out by this algorithm. Many of them have periodic solutions useful for space flight design [7,15].

The main families of symmetric periodic solutions of the Hill's problem are given in the Table 2 (see [6]). The column marked as M gives the multiplicity of generating sequence. Characteristics of these families in coordinates (x_1, y_2) are represented in Figure 4

Table 2. Summary table of the main families of the Hill's problem

Name	**Generating sequence**	**Symmetry**	M	$C_{\max}$
f	—	$\Sigma_1,\ \Sigma_2$	1	$+\infty$
a	$\{+1\}$	Σ_1	0	$3^{4/3}$
c	$\{-1\}$			
g	$\{i, e\}$	$\Sigma_1,\ \Sigma_2$	-1	$+\infty$
g'	$\{+2\}$ $\{-2\}$	Σ_1	1	4.49998
f_3	$\{-1, +1\}$ $\{+1, i, -1, e\}$	$\Sigma_1,\ \Sigma_2$	3 -1	3.80620

Some peculiarities of anti–Hill's problem make it easier for prediction the global properties of families defined by generating sequences. Namely,

• the region of possible motion called the Hill's region is isolated from the origin, therefore, no one family has periodic orbit with collision and global multiplicity of periodic solution is invariant along the family;

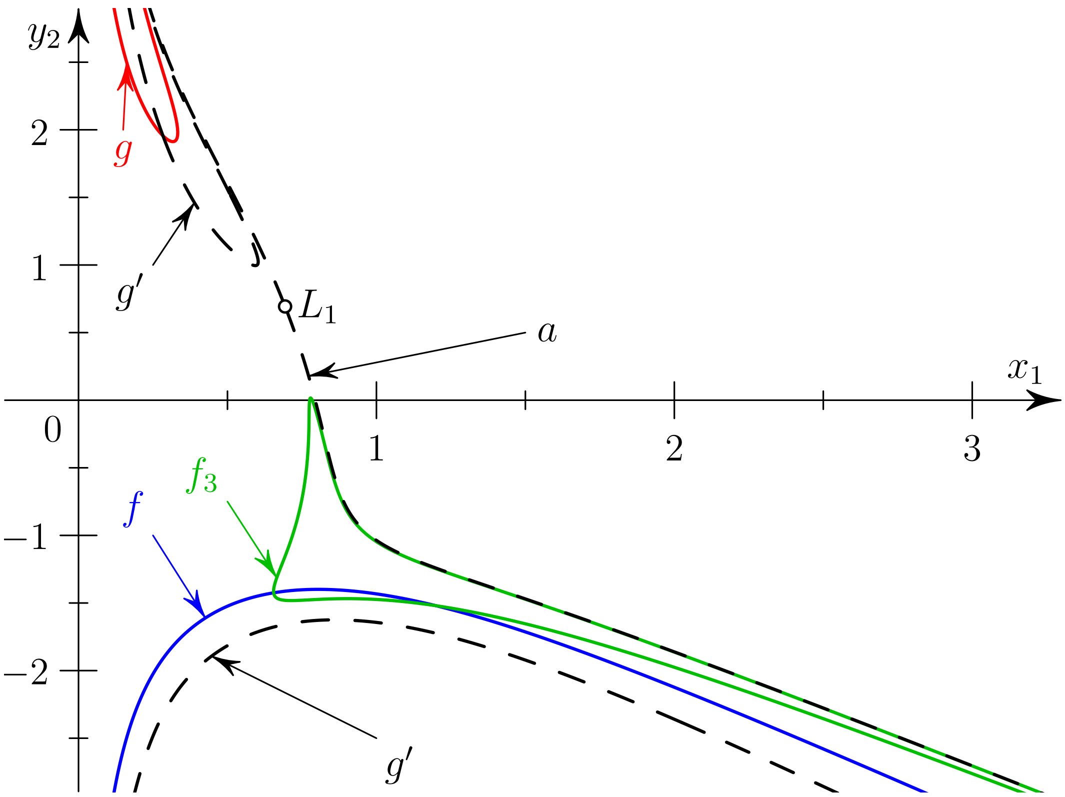


Figure 4. Characteristic of main symmetric periodic solutions of the Hill's problem (see Table 2).

• periodic solutions are possible for values $C < 0$ only, as long as configuration space is divided by OY axis for $C \geqslant 0$.

The following results were announced in [16].

It was shown above that the Hénon problem is the limiting problem both for the Hill's problem and the anti–Hill's problem. Therefore, two families of periodic solutions – one for Hill's problem and the another for the anti–Hill's – are called *linked* if they both have the same generating sequence at the limit $C \to -\infty$. Comparing Statement 1 with Statement 3 one can conclude that not every family of periodic solutions of the anti–Hill's problem can be continued into a family of Hill's problem. But, on the contrary, all known families defined by generating sequences satisfying the condition of Statement 1 are continuable to families of the anti–Hill's problem.

Figure 5 gives the result of computations in the form of schematic drawing of families of periodic solutions of the GHP, which form the common network (web) of periodic solutions in the sense that starting from an arbitrary orbit of any family one can continue to any orbit of other family. All families of the Hill's problem denoted on Figure 5 (left part of the figure) are described in [1].

Let give a short description of Figure 5. The center column is generating sequences of families. $L_{1,2}$ are libration points, O is the origin. The names of the

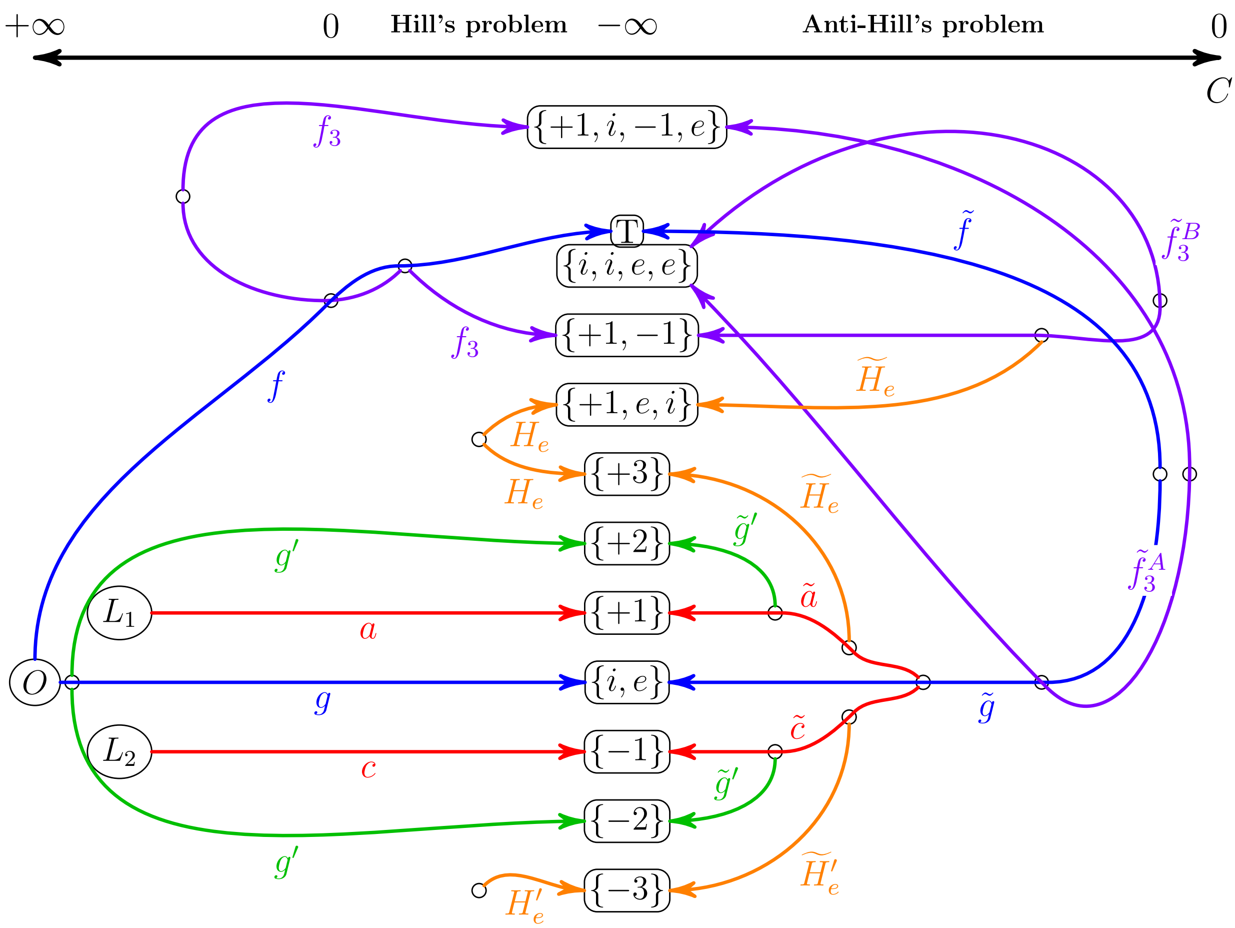


Figure 5. Diagram of connection between families of the Hill's (left part) and the Ant-Hill's problems (right part). Central columns gives generating sequences of the families.

families is used the same as in M. Hénon's papers [6, 13]. The linked families of the anti–Hill's problem have the same name with ˜ (tilde) sign above and are shown in the same color. The common orbits of two families is denoted by small circle. The same circle is used to denote the orbit at which the family riches the extremum on Jacobi integral $\mathcal{J}$ (3). To simplify the drawing we show only small part of all known at that moment families.

The family f_3 demonstrates that its continuation from the Hill's problem to the anti–Hill's and vice versa could be rather complex. Let start from the one of the common orbit of families f (blue line) and f_3 (violet line). Continuation of the family f_3 brings us to the generating sequence $\{+1, -1\}$. This sequence links together the family f_3 of the Hill's problem and family of the anti–Hill's problem denoted by $\widetilde{f}_3^B$. Family $\widetilde{f}_3^B$ riches the extreme value on C and then continues to another generating sequence $\{i, i, e, e\}$. This sequence due to Statement 1 can not be a generating sequence for any family of the Hill's problem. So, the family $\widetilde{f}_3^B$ passes through the generating sequence $\{i, i, e, e\}$ and links with the anti–Hill's problem's family denoted by $\widetilde{f}_3^A$. This family shares the common orbit with the family $\tilde{g}$, riches the extreme value on C and then continues to the generating sequence $\{+1, i, -1, e\}$. This generating sequence is also the limit of the another branch of the family f_3 of the Hill's problem.

5. Continuation of doubly symmetric periodic solutions from the SKP to the GHP

Now we give the general description of generating solutions of the SKP, which can be continued into the families of periodic solutions of the Hill's problem. We restrict our interest to doubly symmetric generating solutions, because the realization of the global picture of connection between the periodic solutions of the SKP and the Hill's problem is not yet finished and it requires considerably more computations and overcoming many computational difficulties. The following results are only the first step in understanding the connections between the two mentioned above problems.

5.1. Continuation and change of stability of libration points. It is well-known fact that the Kepler problem with Hamiltonian (7) does not have any liberation (stationary) point, but when one considers the motion of the massless particle in the gravitational field with Newtonian potential in uniformly rotating frame i. e. the SKP with Hamiltonian (6), where parameter $\sigma = -1$, then one gets the unit circle of libration points $|\mathbf{x}| = 1$. The perturbation function $\varepsilon R(\mathbf{x}) = \varepsilon\left(-x_1^2 + x_2^2/2\right)$ of the GHP destroys this unit circle. As function $\varepsilon R(\mathbf{x})$ is an even function in both variables so two pairs of librations points still remain for small values of parameter ε. The first pair is laying in the axis OX and has coordinates $\left(\pm(1+2\varepsilon)^{-1/3}, 0\right)$. These points are usually denotes by symbol L_1 for point with positive abscissa and by symbol L_2 for point with negative abscissa. The second pair of libration points is situated in the axis OY and has coordinates $\left(0, \pm(1-\varepsilon)^{-1/3}\right)$. We denote this points by symbols L_1^y and L_2^y correspondingly.

While the parameter ε changes from the value 0 to the value 1 the points $L_{1,2}$ move inside the unit circle and take the position with coordinates $\left(\pm 3^{-1/3}, 0\right)$. The characteristic polynomial $f(L_1)$ of Jacobi matrix $M(L_{1,2}) = J\,\mathrm{Hess}\,H(\mathbf{z},\varepsilon)\big|_{L_{1,2}}$, computed in libration point $L_{1,2}$ is equal

$$f(L_{1,2}) = \lambda^4 + (1-3\varepsilon)\lambda^2 - 9\varepsilon(1+2\varepsilon).$$

For any $\varepsilon \in (0; 1]$ it has one pair of real zeroes and one pair of pure imaginary zeroes, so, for all these values of parameter ε there is a family of Lyapunov periodic solutions, which is denoted by symbol a. The symmetric family around the libration point L_2 is denoted by symbol c. Families a and c exist only for those values of Jacobi integral (3) that $C \leqslant 3(1+2\varepsilon)^{1/3}$, i. e. for those values of C when points $L_{1,2}$ get into the Hill's region. So, for fixed values of parameter $\varepsilon \in [0; 1]$ family a

(c) is terminated naturally in corresponding libration point.

While the parameter ε changes from the value 0 to the value 1 the points $L^y_{1,2}$ move outside the unit circle and tend to infinity, so the Hill's problem does not have libration points $L^y_{1,2}$. The characteristic polynomial of the Jacobi matrix $M(L^y_{1,2}) = J\,\mathrm{Hess}\,H(\mathbf{z},\varepsilon)\Big|_{L^y_{1,2}}$, computed in libration point $L^y_{1,2}$ is equal

$$f(L^y_{1,2}) = \lambda^4 + \lambda^2 + 9\varepsilon(1-\varepsilon), \tag{11}$$

which has on unit interval two critical values $\varepsilon^*_{1,2} = 1/2 \mp \sqrt{2}/3$. These values divide the unit interval into three subintervals: $(0;\varepsilon^*_1)$, $(\varepsilon^*_1;\varepsilon^*_2)$, $(\varepsilon^*_2;1)$.

On interval $(0;\varepsilon^*_1)$ polynomial (11) has two pairs of pure imaginary mutually conjugated zeroes, so, there exists family of quasi-periodic orbits near the libration point $L^y_{1,2}$. For countable set of values of parameter ε, for which the ratio of zeroes is commensurable, these quasi-periodic orbits became periodic. At the value $\varepsilon = \varepsilon^*_1$ two zeroes in each pair coincide and polynomial (11) has one pair of multiple zeroes. On interval $(\varepsilon^*_1;\varepsilon^*_2)$ polynomial (11) has two pairs of complex zeroes. One pair of mutually conjugated zeroes has negative real part and on invariant manifold, which is spanned on its eigenvectors, one has a trajectory asymptotically approximating to the point L^y_1. Small non-linear perturbations shift this trajectory from the stable manifold and trajectory begins to move off the libration point. On interval $(\varepsilon^*_2;1)$ the polynomial (11) has the same zeroes as on interval $(0;\varepsilon^*_1)$. The motion near libration points $L^y_{1,2}$ is possible only for values $C \leqslant 3(1-\varepsilon)^{1/3}$.

5.2. Bifurcation of doubly symmetric periodic solutions of the SKP. Let recall the structure of families of symmetric periodic solutions of the SKP. The following description is a summary of [5, Ch. III].

All finite orbits of Kepler problem are always periodic but in the SKP the property of periodicity preserves orbits of two following types:

I) circular orbits, the shape of which does not change;
II) elliptic orbits, which period is commensurable with 2π.

Circular orbits form two families: the family of direct circular orbits, denoted by Id, and the family of retrograde circular orbits, denoted by Ir.

Let T_s is a period of elliptic orbit and $N = 2\pi T_s^{-1}$. To be periodic in the sinodic frame the orbit should have value $N = (p+q)/p$, where $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. To be symmetric the orbit should have the pericenter lying in the axes of symmetry, i.e. the argument of pericenter ϖ should be equal to values 0 or π for Σ_1-symmetric orbits and to be equal to values $\pm\pi/2$ for Σ_2-periodic solutions. To be doubly symmetric the periodic solution should have both numbers p and q to be odd and argument of pericenter $\varpi \in \{0, \pi/2, \pi, -\pi/2\}$. The families of symmetric elliptic periodic solutions are denoted by E_N.

To make the structure of symmetric orbits more obvious let consider the Poincaré section of the phase flow, defined by the system of canonic equations, by the plane of symmetry $\Pi : x_2 = y_1 = 0$. Σ_1-symmetric orbits cross this plane orthogonal and points of intersection of orbit with the plane Π have coordinates $(x_1, 0, 0, y_2)$. Along the family of periodic solutions these non-zero coordinates change smoothly and form the curve on the plane of symmetry. The following coordinates was proposed by A. D. Bruno in [5, Ch. 3] to make the form of characteristic of periodic solutions the most simple and clear:

$$\widetilde{a} = \frac{x_1}{2 - |\widetilde{e}|}, \quad \widetilde{e} = y_2 |y_2 x_1|. \tag{12}$$

The semi-major axis $a = |\widetilde{a}|$ and eccentricity $e = |1 - |\widetilde{e}||$. The part of plane of symmetry Π filled with periodic solutions is a cylinder in coordinates (12). The $\widetilde{a}$ is a coordinate along the generatrix of the cylinder and the $\widetilde{e}$ is a coordinate along the directing circle. So the dashed lines with coordinates $\widetilde{a} = \pm 2$ in Figures 6 to 8 coincide.

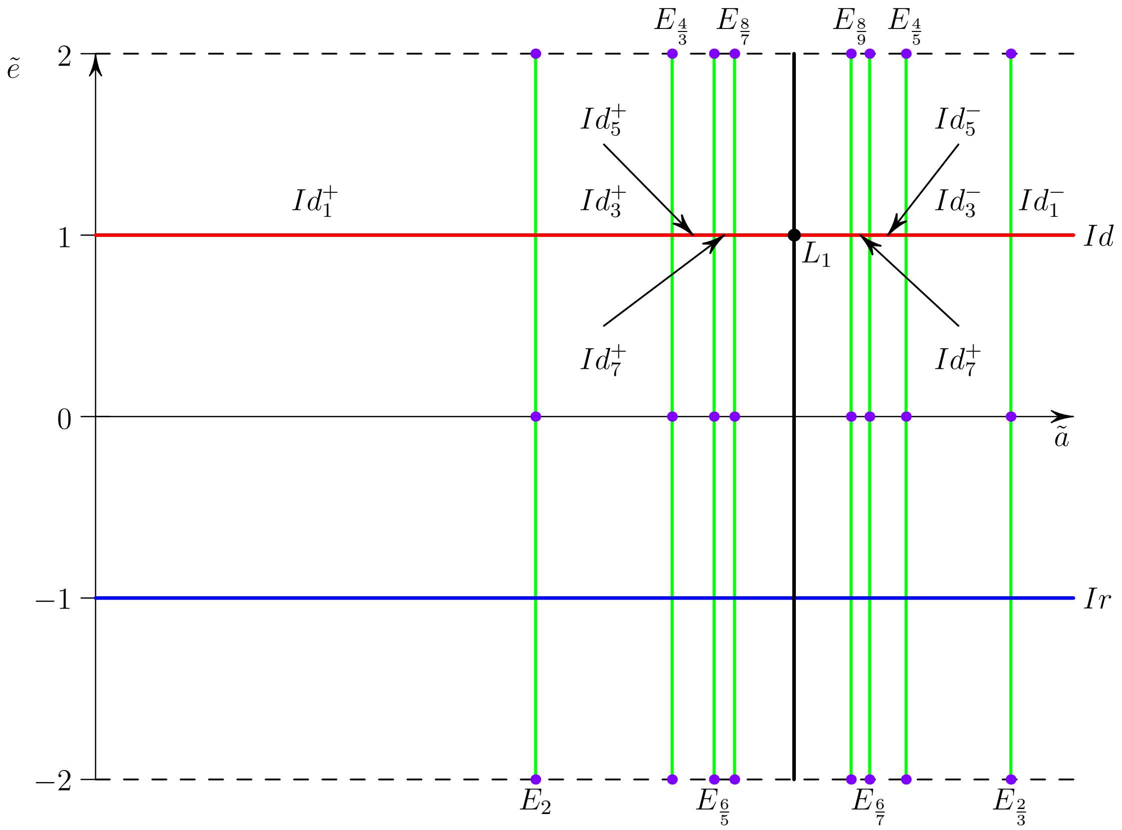


Figure 6. Main families of doubly symmetric periodic solutions of the SKP. Collision orbits are marked with purple.

The families of circular orbits are represented in coordinates (12) by straight lines parallel to axis OX with coordinate $\widetilde{e} = +1$ for family Id and with coordinate

$\widetilde{e} = -1$ for family Ir. The families E_N are represented by vertical segments with coordinate $\widetilde{a} = \pm N^{-2/3}$. Each family has two collision orbits with coordinates $(N^{-2/3}, \pm 2)$ and $(N^{-2/3}, 0)$ in plane $(\widetilde{a}, \widetilde{e})$. Each family E_N of elliptic solutions shares one common orbit with family Id at point $(N^{-2/3}, 1)$ and with family Ir at point $(N^{-2/3}, -1)$. The segments of elliptic solutions condense tending to the segment with $\widetilde{a} = 1$. The circle of stationary points is represented by black point L_1 with coordinates $(\pm 1, 1)$. The main families of doubly symmetric periodic solutions are shown in Figure 6. The part with $\widetilde{a} \geqslant 0$ is shown.

The main result of perturbation of doubly symmetric periodic solutions under the function $\varepsilon R(\mathbf{x})$ is that the points of intersection of the family Id and families E_N disappear. This bifurcation takes place for those families, which $N = (p+1)/p$ or $N = (p+1)/(p+2)$, where p is odd positive number, i. e. $p = 2k - 1$, $k \in \mathbb{N}$.

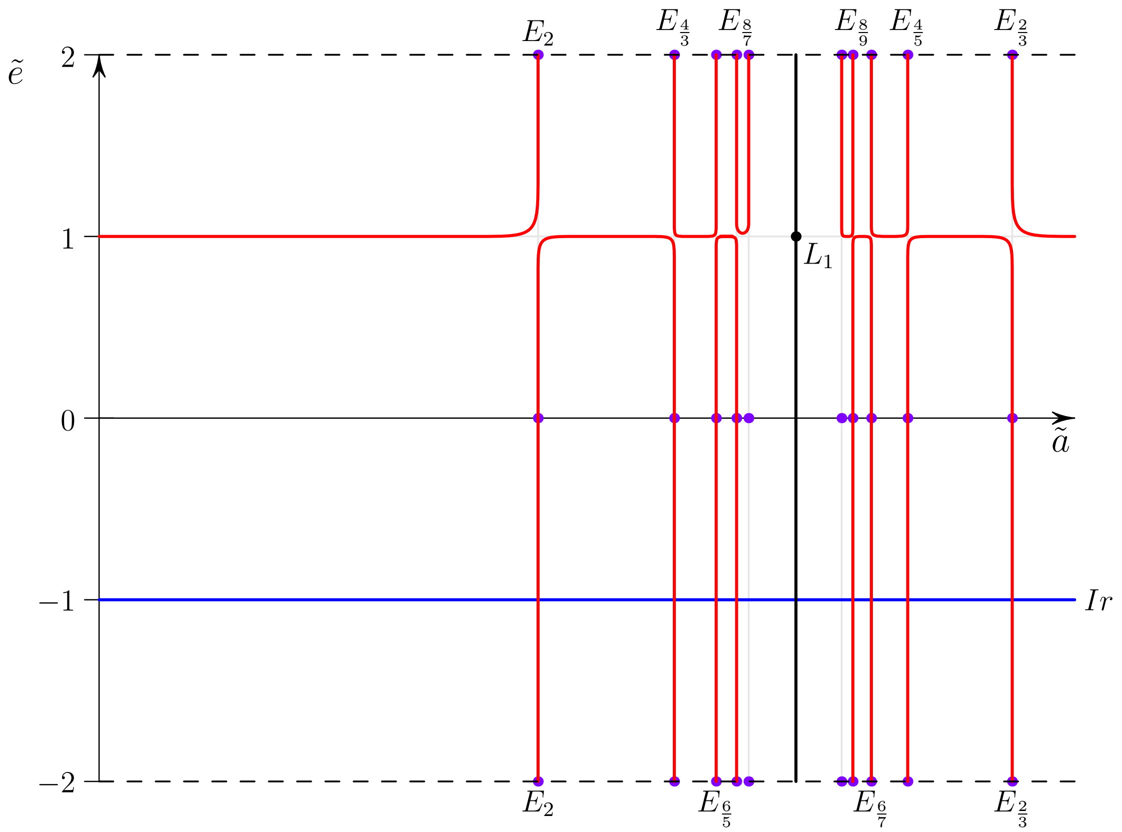


Figure 7. Bifurcations of families of doubly symmetric periodic solutions of the SKP for small value of parameter ε.

Under the perturbation εR the family Id^+ is divided into countable number of pieces by families of doubly symmetric elliptic orbits $E_{(p+1)/p}$, $p = 2k - 1$, and $E_{(p-1)/p}$, $p = 2k + 1$, $k \in \mathbb{N}$. We denote the parts of the perturbed family Id by Id_p^+ and Id_p^- (see Figure 6). Moving along the perturbed family from the circular direct orbits with small radius a one will sequentially pass through the part Id_1^+, then goes through the part of family E_2. At point with coordinates $(2^{-2/3}, \pm 2)$

family passes through the collision orbit, moving along another part of family E_2, crossing the family Ir, and goes through the part Id_3^+. This part is connected with the part of the family $E_{4/3}$, which continues through the collision orbit at the point $((4/3)^{-2/3}, \pm 2)$ and goes to another part of the family $E_{4/3}$ and so on. So, the characteristic of the perturbed families Id and $E_{(p+1)/p}$ is a labyrinth curve on the surface of cylinder, which makes infinitely many folds while tending to the point L_1. Another part of perturbed characteristic tends to point L_1 from outside, making infinitely many folds. Let denote the families of united segments Id_p^+ and $E_{(p+1)/p}$ by IdE_p^+ and united segments Id_p^- and $E_{(p-1)/p}$ by IdE_p^-. Moving through the collision points the orbits sequentially change its direction of movement and global multiplicity.

The same type of bifurcation of family Id takes place in the case of small perturbation of the RTBP, but such bifurcation happens for each integer value of number p. For details see [5, Ch. VIII].

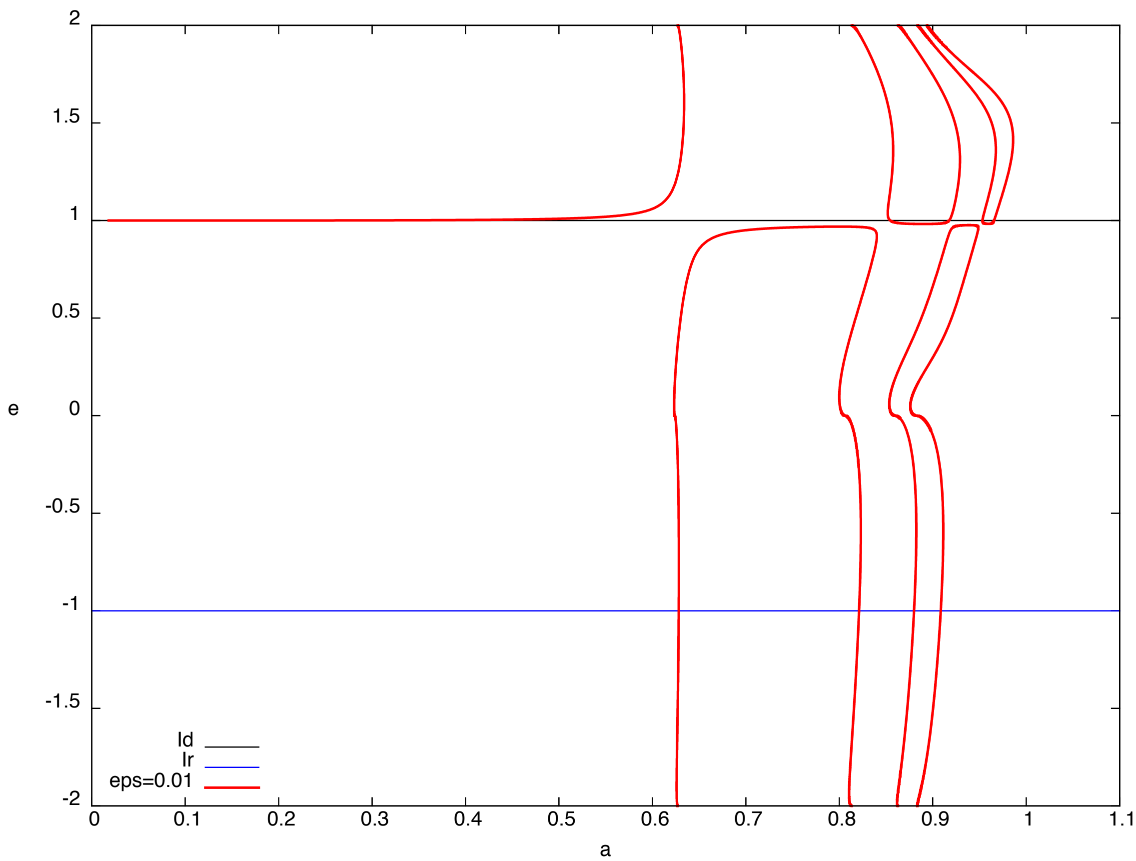


Figure 8. Deformation of characteristic of the family IdE^+ for $\varepsilon = 0.01$.

5.3. Evolution of families of doubly symmetric solutions for small value of ε. Here we give description of evolution of family IdE^+ while perturbation parameter ε changes from 0 to 1. We limit our description to periodic solutions with major semiaxis $a < 1$, i. e. which characteristic is situated at the left side of Figure 7. These results were obtained after intensive numerical computation.

Small changing of the parameter ε leads to quick deformation of the characteristic. As closer the fold of characteristic is situated to the singular point L_1 the more it is distorted by the perturbation. In the Figure 8 is shown computed characteristic of perturbed family IdE^+ for $\varepsilon = 0.01$.

At the certain values of parameter ε continuous characteristic begins to "tear" into separate pieces and these pieces should be investigated separately.

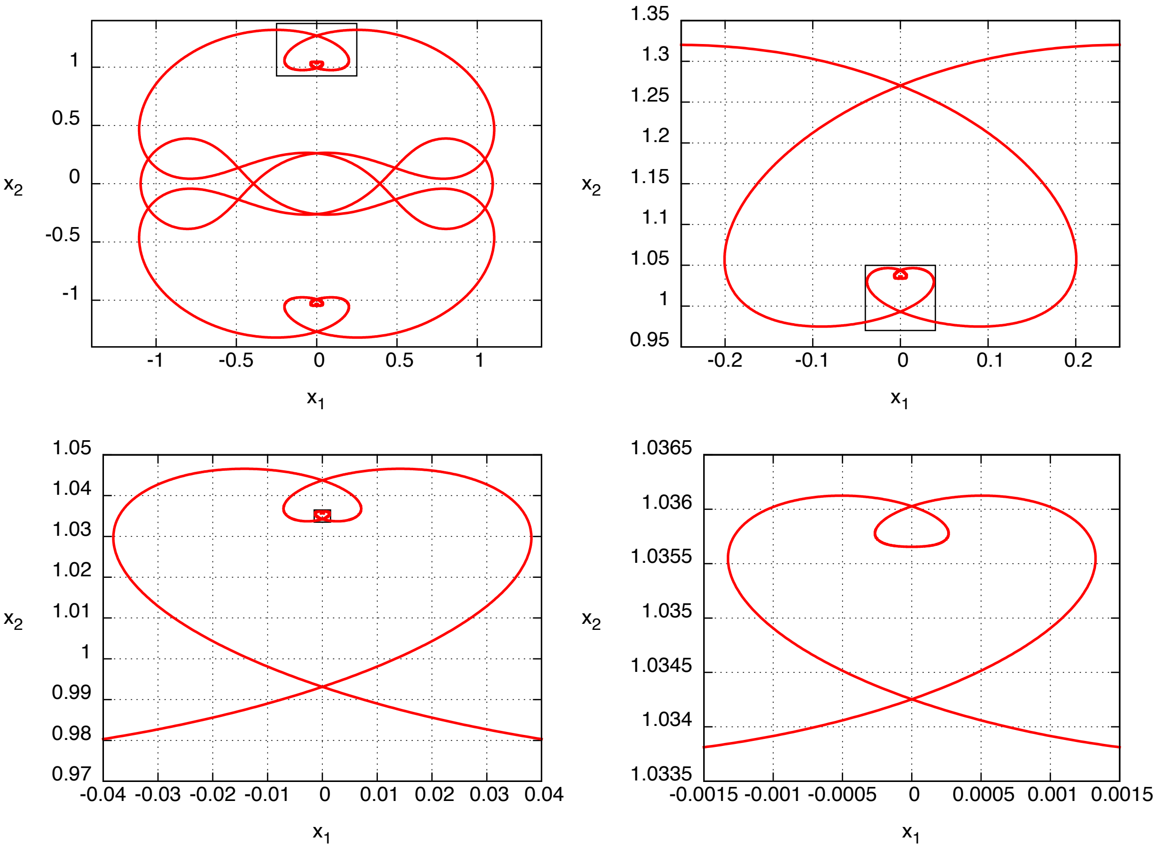


Figure 9. An example of orbit with five loops around the libration points $L^y_{1,2}$.

An important role in breaking of the whole family IdE^+ into separate pieces plays the changing of the type of stability of libration points $L^y_{1,2}$. For $\varepsilon > \varepsilon_1^* \approx 0.0286$ new type of periodic orbits appear. These orbits approach to the libration point L^y_1 for that values of Jacobi integral (3), for which this point is isolated from the region of possible motion by the zero velocity curve, i. e. for $C > 3(1-\varepsilon)^{1/3}$. The orbits have loop-shape parts of its trajectory, which are passed with very slow velocity. As Jacobi integral C tends to the critical value from upper values then the boundary from the zero velocity curve around the libration point shrinks. So, the number of decreasing loop-shape parts along the orbit increases and period of the orbit increases as well. After making a finite number of decreasing loops around the libration point L^y_1 the trajectory moves off along the increasing loops.

An example of such orbit is shown in Figure 9. The rectangles area is enlarged from left upper corner to right lower corner of the Figure 9. This orbit was computed during continuation of the family IdE_3^+ for the value of parameter $\varepsilon = 0.1$.

The breaking of the family happens in two steps. On the first step a fold appears on the characteristic of the family. On the second step the characteristic of the family is breaking at the place of the fold. Two parts of breaking characteristic during the continuation make spiral turns and tend to orbits with infinitely large period.

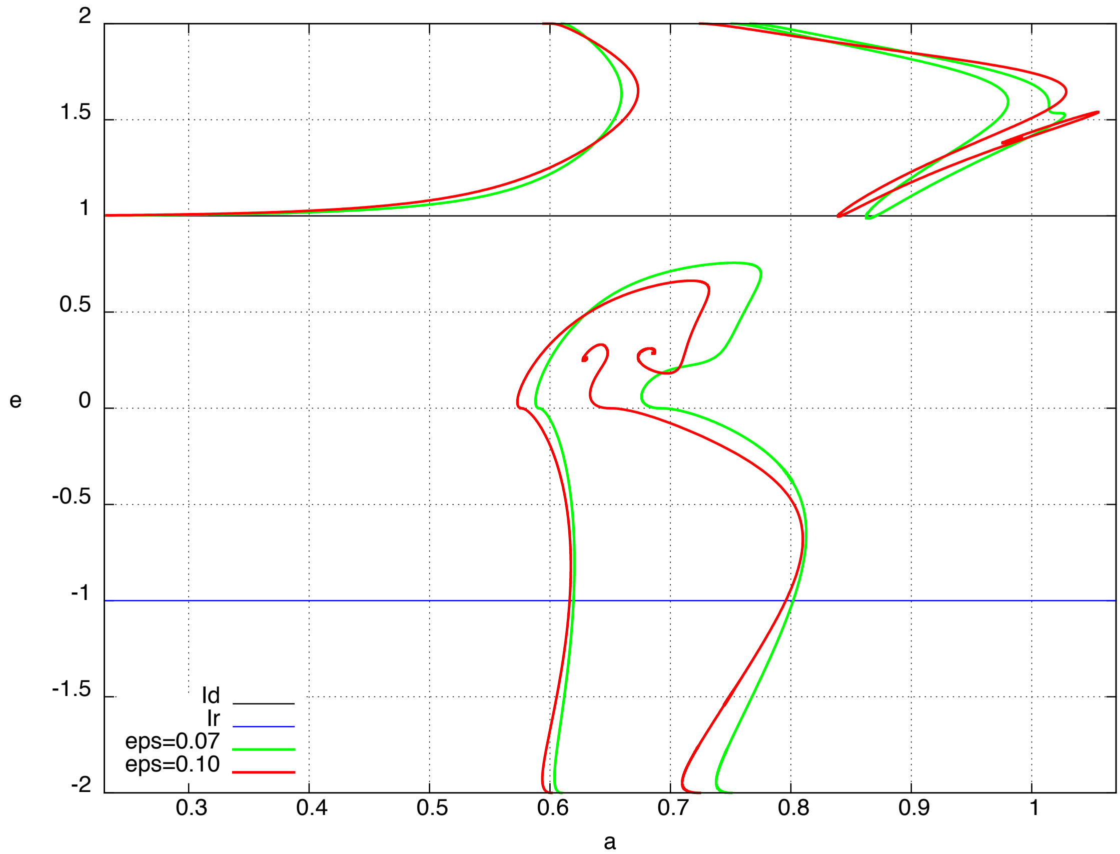


Figure 10. Characteristic of the family IdE^+ before (green line, $\varepsilon = 0.07$) and after (red line $\varepsilon = 0.1$) breaking into parts.

Consider the changes which happen, when parameter ε increases from small values to the value 0.1. One can see that for $\varepsilon = 0.07$ a fold on the right side of characteristic enlarges (green line on Figure 10) and at $\varepsilon \approx 0.0949$ this fold is breaking into parts (red line on 10). One part of the IdE^+ family consists of united families Id_1^+, E_2 and Id_3^+. This part of the family finished at solution with infinite number of loops around the libration points $L_{1,2}^y$. The next part of the family consists of the united families $E_{4/3}$ and Id_5^+. The both ends of this part are orbits with infinite number of loops around the libration points. Other parts of the family IdE^+ are arranged in the same way but are not shown in Figure 10.

The last reorganization of the family IdE^+ takes place at $\varepsilon \approx 0.436$. After this breaking the first part of the family consists of the family Id_1^+ united with part of the family E_2 for $\widetilde{e} > 1$. The limit of this part is an orbit with infinite loops around the libration points $L_{1,2}^y$.

So, the parts of the whole family can be identified by the segments Id_p^+, $p = 1{,}3{,}5, \ldots$.

Statement 4. For large values of perturbation parameter $\varepsilon > 0.435$ each part of the family IdE^+, except the first part, consists of the united segments Id_p^+ and $E_{(p-1)/(p-2)}$ for $p = 3{,}5, \ldots$. Each such part begins with and ends with the orbit, which makes infinite number of loops around the libration point $L_{1,2}^y$. The first part begins with infinitesimal small circular orbit around the origin and ends with an orbit with infinite numbers of loops. Let denote this parts by IdE_p^+, $p = 1{,}3{,}5, \ldots$.

The same type of families of periodic solutions with characteristic containing spirals was found by M. Hénon in the RTBP for the case of equals masses [17], by K. Papadakis and C. Goudas in the RTBP for mass parameter $\mu = 0.4$ [18], by A. Bruno and V. Varin for Beletsky equation [19].

5.4. Continuation parts IdE_p^+ into the Hill's periodic solutions. The author could numerically continued the family Ir and the first four parts IdE_p^+, $p = 1{,}3{,}5{,}7$ into families of doubly periodic solutions of the Hill's problem.

- The family Ir continues into the family f of retrograde satellite orbits [1, 6].
- The part IdE_1^+ continues into the family g of direct satellite orbits [1, 6].
- The part IdE_3^+ continues into the family f_3 [1, 13, 20].
- The part IdE_5^+ continues into the new family, which was described in [7]. This family has generating sequence $\{+1, +1, -1, -1\}$.
- The part IdE_7^+ continues into the new family with generating sequence $\{+1, +1, +1, -1, -1, -1\}$.

The family a (c) of Lyapunov libration orbits was also continued from the Hill's problem to the circle of stationary points of the SKP.

Families Ir, parts IdE_1^+ and IdE_3^+ can be continued into the corresponding families of the Hill's problem directly. The essential role of the anti–Hill's problem becomes clear just during continuation the parts IdE_5^+ and IdE_7^+ into the Hill's problem's families, because certain pieces of the parts IdE_p^+, $p = 5{,}7$ is able to continue through the families of the anti–Hill's problem only. For example, the certain piece of the part IdE_5^+ is continued up to the value of parameter $\varepsilon \approx 0.95$. The further continuation of this piece within the Hill's problem is impossible, but it is possible to get the limit generating sequence of this piece, which is $\{+2, -2\}$. This generating sequence within the anti–Hill's problem gives another sequence $\{+1, +1, -1, -1\}$.

6. Summary

The generalization of the Hill's problem looks quite natural and makes possible to match doubly symmetric periodic solutions of the SKP and the Hill's problem.

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