Aptekarev A.I., Tulyakov D.N.

Geometry of Hermite-Padé approximants for system of functions $\{f,f^2\}$ with three branch points

Recommended form of bibliographic references: Aptekarev A.I., Tulyakov D.N. Geometry of Hermite-Padé approximants for system of functions $\{f,f^2\}$ with three branch points. Keldysh Institute preprints, 2012, No. 77, 25 p. URL: http://library.keldysh.ru/preprint.asp?id=2012-77&lg=e

РОССИЙСКАЯ АКАДЕМИЯ НАУК
ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
ИМ. М.В. КЕЛДЫША

A. I. Aptekarev, D. N. Tulyakov

Geometry of Hermite-Padé approximants for system of functions $\{f, f^2\}$ with three branch points

МОСКВА, 2012 г.

Аптекарев А. И., Туляков Д. Н.

Геометрия аппроксимаций Эрмита-Паде для системы функций $\{f, f^2\}$ с тремя точками ветвления [1]

Аннотация. В задаче об асимптотике аппроксимаций Эрмита-Паде для набора из двух аналитических функций с точками ветвления преобразование Коши предельной меры распределения полюсов аппроксимаций является алгебраической функцией третьего порядка. В общей ситуации это утверждение известно как гипотеза Наттолла. В предположении справедливости этой гипотезы мы хотим найти эти алгебраические функции. Мы ограничиваемся случаем, когда точек ветвления всего три и они у данной пары функций общие. В настоящем препринте мы обсуждаем постановку задачи, общие подходы к ее решению и исследуем возникающие алгебраические функции нулевого рода. Случаи, соответствующие алгебраическим функциям более высокого рода, будут рассмотрены в другой работе.

Ключевые слова. Алгебраические функции, римановы поверхности, аппроксимации Эрмита-Паде.

Aptekarev A. I., Tulyakov D. N.

Geometry of Hermite-Pade approximants for system of functions $\{f, f^2\}$ with three branch points

Abstract

In the problem on asymptotics of Hermite-Padé approximants for two analytic functions with branch points an algebraic function of the third order appears as the Cauchy transform of the limiting measure of poles distributions of the approximants. In general stuation this statement is known as the Nuttall's conjecture. Our goal is, assuming that this conjecture holds true to describe the algebraic functions for the case when approximated two functions have common three branch points. In this preprint we discuss statement of the problem, general approaches to its solutions, and we carry out analysis of the appearing algebraic functions of genus zero. We plan to consider the cases corresponding to the algebraic functions of higher genus in the future paper.

Key words. Algebraic functions, Riemann surfaces, Hermite-Pade approximants.

[1] Работа первого автора частично поддержана грантом научных школ НШ–8033.2010.1, программой № 1 ОМН РАН, грантами РФФИ-11-01-00245, РФФИ-11-01-12045-офи-м. Работа второго автора также частично поддержана грантами РФФИ 10-01-00682, РФФИ 12-01-00988 и РФФИ 12-01-33050.

Посвящается Е.М. Чирке, в связи с его 70-летием

1 Introduction

1.1 Class of functions and historical remarks

We consider

$$f \in \mathcal{A}(\mathbb{C}\backslash A)\,, \quad A := \{a, b, c\} \quad \text{are branch points}\,. \tag{1.1}$$

It means that the germ

$$f(z) := \sum_{\nu=0}^{\infty} \frac{s_\nu}{z^\nu} \tag{1.2}$$

is analytically continuable along any path in $\mathbb{C} \setminus A$ and this continuation is multi-valued in $\overline{\mathbb{C}} \setminus A$, i.e., f has branch-type singularities at some points in A. The most famous model representative of the class is a function

$$f(z) = \prod_{j=1}^{3} (z - a_j)^{\alpha_j}\,, \qquad \sum \alpha_j = 0\,, \quad \alpha_j \notin \mathbb{Z}, \quad j = 1, 2, 3\,. \tag{1.3}$$

Nuttall in [1], [2] has initiated study of rational Pade approximants $\pi_n := \frac{Q_n}{P_n}$:

$$R_n(z) := P_n(z)f(z) - Q_n(z) = O\left(\frac{1}{z^{n+1}}\right)\,, \quad \deg P_n, Q_n \le n\,. \tag{1.4}$$

for functions (1.3), as important illustration of behavior of rational approximants for functions with branch points. The strong asymptotics of the Nuttall-Jacobi polynomials P_n as $n \to \infty$ was studied recently by Baratchart, Yattselev in [3] and Martinez-Finkel-shtein, Rakhmanov, Suetin in [4].

S.P. Suetin (in his talk at Complex Analysis Seminar in Moscow in September 2010) has stated a problem of study the asymptotical behavior of Hermite-Pade approximants of type I (definition will be given in the next subsection) for the system of functions $\{f, f^2\}$, where f is (1.3). It is quite well understood that solution of this problem requires deep geometrical analysis of the expected asymptotics. We begin with this analysis in this preprint. Here we discuss statement of the problem, general approaches to its solutions, and we carry out analysis of the appearing algebraic functions of genus zero. We plan to consider the cases corresponding to the algebraic functions of higher genus in the nearest future.

1.2 Definition of Hermite-Padé approximants for $\{f_1, f_2\}$

Given

$$f_j(z) := \sum_{\nu=0}^{\infty} \frac{s_\nu^{(j)}}{z^\nu}, \quad j = 1, 2 . \tag{1.5}$$

For the system $\{f_1, f_2\}$ the diagonal (i.e. the multiindex is (n, n)) Hermite-Padé (H–P) approximants of the type I are defined by:

$$\begin{aligned} &R_n(z) := P_{n,1}(z) f_1(z) + P_{n,2}(z) f_2(z) + Q_n(z) = \mathcal{O}\left(\frac{1}{z^{2n+1}}\right), \\ &\deg P_{n,1}, P_{n,2} \le n\,, \ \deg P_{n,1} \ge \deg P_{n,2} . \end{aligned} \tag{1.6}$$

Correspondingly, the Hermite-Padé approximants of the type II are defined by:

$$\begin{aligned} &R_n^{(j)}(z) := P_n(z) f_j(z) - Q_n^{(j)}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad j = 1, 2, \\ &\deg\, P_n \le 2n . \end{aligned} \tag{1.7}$$

When $\deg P_n = 2n$, the index (n, n) is called *normal*, and we normalise P_n as monic. For details see survey papers [5], [6], [7], [8] and monography [11].

1.3 Model system with the same branch points — Nikishin system

A model system of functions – called Nikishin system – plays the crucial role for understanding the limiting behavior of the Hermite-Padé approximants of the system of functions with the same set of branch points. We recall this notion.

Given

$$\begin{aligned} &\Gamma := [a, b] \subset \mathbb{R}, \quad \rho > 0, \quad \text{a.e. on } \Gamma \\ &\delta := [c, d] \subset \mathbb{R}, \quad \Gamma \cap \delta = \emptyset, \quad \sigma > 0, \quad \text{a.e. on } \delta \end{aligned} . \tag{1.8}$$

In [9] Nikishin has introduced a system

$$\begin{cases} f_1(z) := \widehat{\rho}(z) := \displaystyle\int_\Gamma \frac{\rho(t)\, dt}{z - t}\,; \\ f_2(z) := \widehat{u\rho}(z) := \displaystyle\int_\Gamma \frac{u(t)\, \rho(t)\, dt}{z - t}, \quad u(t) = \widehat{\sigma}(z) = \displaystyle\int_\delta \frac{\sigma(t)\, dt}{z - t} . \end{cases} \tag{1.9}$$

Thus

$$f_1, f_2 \in H(\overline{\mathbb{C}} \setminus \Gamma), \quad u := \left.\frac{f_2^+ - f_2^-}{f_1^+ - f_1^-}\right|_\Gamma =: \widehat{\sigma} \in H(\overline{\mathbb{C}} \setminus \delta) . \tag{1.10}$$

The type II polynomials $\{P_n\}$ and the type I forms $\{L_n := P_{n,1} + \widehat{\sigma} P_{n,2}\}$ satisfy biortho-gonality relations

$$\int\limits_{\Gamma} P_n(z)\, L_m(z)\, \rho(z)\, dz = \mathrm{const}_n\, \delta_{n,m} \ .$$

In [10] Nikishin has proved for the type I (see (1.6))

1)
$$\begin{cases} L_n \text{ has } 2n \text{ zeros on } \Gamma \ ; \\ \\ P_{n,2} \text{ has } n \text{ zeros on } \delta \ . \end{cases}$$

2) For the zero counting measure ν_n (i.e. in each finite zero the $\dfrac{1}{n}$ mass point is placed)

$$\nu_n\, [L_n] \overset{*}{\longrightarrow} \lambda \ , \quad |\lambda| = 2$$

$$\nu_n\, [P_{n,2}] \overset{*}{\longrightarrow} \mu \ , \quad |\mu| = 1$$

where

$$\begin{cases} 2V^{\lambda} - V^{\mu} = \gamma_{\Gamma} \text{ on } \Gamma \\ \\ 2V^{\mu} - V^{\lambda} = \gamma_{\delta} \text{ on } \delta \end{cases} \ , \quad V^{\tau}(z) := \int \ln \left| \frac{1}{t-z} \right| d\tau(t) \ , \qquad (1.11)$$

for some equilibrium constants γ_{Γ} and γ_{δ}.

This system of the equilibrium conditions is equivalent to the following extremal problem for the energy functional of a vector measure

$$\mathcal{L} \ : \qquad \overrightarrow{\tau} := (\tau_1, \tau_2) \in \mathcal{L} \ : \quad \begin{cases} \operatorname{supp} \tau_1 \subseteq \Gamma \ , \ |\tau_1| = 2 \\ \operatorname{supp} \tau_2 \subseteq \delta \ , \ |\tau_2| = 1 \end{cases} \ , \qquad (1.12)$$

with respect to the matrix of interaction $\mathcal{N} := \{\mathcal{N}_{ij}\}_{i,j=1}^{2}$

$$\mathcal{N} := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \qquad (1.13)$$

An energy functional is defined as

$$I_{\mathcal{N}}(\overrightarrow{\tau}) := \sum_{i,j} \mathcal{N}_{ij}\, I(\tau_i, \tau_j) \ , \quad I(\tau_i, \tau_j) := \int\int \ln \left| \frac{1}{t-z} \right| d\tau_i(t)\, d\tau_j(t) \ .$$

The unique solution $\overrightarrow{\tau}^{*}$ of the extremal problem

$$\overrightarrow{\tau}^{*} \ : \quad I_{\mathcal{N}}(\overrightarrow{\tau}^{*}) \ = \ \min_{\overrightarrow{\tau} \in \mathcal{L}} I_{\mathcal{N}}(\overrightarrow{\tau}) \ , \qquad (1.14)$$

from class (1.12) and with the matrix of interaction (1.13) satisfies to the equilibrium conditions (1.11), where

$$\lambda = \overrightarrow{\tau}^*_1 , \quad \mu = \overrightarrow{\tau}^*_2 . \tag{1.15}$$

Analogous result holds true for type II polynomials (1.7). See, for example, [12]:

1)

$$\begin{cases} P_n \text{ has } 2n \text{ zeros on } \Gamma \text{ ;} \\ \\ R_n^{(j)} \text{ , } j = 1, 2 \text{ have } n \text{ zeros on } \delta \text{ .} \end{cases}$$

2) Limiting zero counting measures

$$\begin{aligned} \nu_n [P_n] &\xrightarrow{*} \lambda \, , \quad |\lambda| = 2 \, ; \\ \nu_n [R_n^{(j)}] &\xrightarrow{*} \mu \, , \quad |\mu| = 1 \, , \end{aligned} \tag{1.16}$$

are solutions (1.15) of the extremal problem (1.14), (1.13), (1.12), which satisfy the equilibrium condition (1.11).

A generalization of Nikishin system (1.8) — (1.9), which considers a case

$$\Gamma \cap \delta = [e_1, e_2] \neq \emptyset \, ,$$

was studied in [13] and [14].

1.4 Non-Hermitian Nikishin system – tails and membranes

Consider two functions (1.5), each belongs to class (1.1)

$$\{f_1, f_2\} \, , \qquad f_j \in \mathcal{A}(\overline{\mathbb{C}} \backslash A) \, , \quad A = \{a, b, c\} \, , \quad j = 1, 2 \, . \tag{1.17}$$

Take a point $\widetilde{e} \notin \{a, b, c\}$, define $\Gamma := \gamma_a^{\widetilde{e}} \cup \gamma_b^{\widetilde{e}} \cup \gamma_c^{\widetilde{e}}$, where $\gamma_{e_1}^{e_2}$ is an oriented Jordan ark joining points e_1 and e_2. Thus

$$f_1, f_2 \in H(\overline{\mathbb{C}} \backslash \Gamma) \, .$$

Using differences of the boundary values of f_1, f_2 from the both sides of Γ, the system $\{f_1, f_2\}$ can be written as

$$f_1(z) = \int\limits_\Gamma \frac{\rho(t)\, dt}{z - t} \, , \qquad f_2(z) = \int\limits_\Gamma \frac{u(t)\rho(t)\, dt}{z - t} \, , \tag{1.18}$$

where

$$u(t) := \frac{f_2^+ - f_2^-}{f_1^+ - f_1^-}(t) , \quad \rho(t) := (f_1^+ - f_1^-)(t) , \qquad t \in \Gamma . \tag{1.19}$$

Now, we consider the single valued holomorphic continuation of u from Γ to complex plane. Denote a union of the domains where u is a piecewise holomorphic function as $\Omega[u]$, and a set of boundary points of Ω as δ

$$u \in H(\Omega) , \quad \partial(\Omega) =: \delta , \quad \Omega \Supset \Gamma .$$

Then, using boundary values of u on δ, by means of Cauchy integral we can present

$$u(z) = \int_{\delta} \frac{\sigma(t)\, dt}{z - t} .$$

Thus function (1.17) can be viewed as a Nikishin system (1.8), (1.9), (1.10), but now Γ and Ω are contours in complex plane and ρ, σ are complex valued functions.

We note details regarding the case of (1.17), when $f_1 = f$, $f_2 = f^2$:

$$\{f, f^2\} .$$

For this case we have in (1.19)

$$u(t) = f^+(t) + f^-(t) \, ; t \in \Gamma . \tag{1.20}$$

Now couple of words about δ. There are two typical obstacles for the holomorphic (singlevalued) continuation of the function $u(t)$, from Γ to $\Omega[u]$, see (1.20).

The first reason is when function u preserves branching at the branch points of f. It happens when winding at this point differs from the square root winding. Then a piece of δ performs the cut starting from the corresponding point of A. We call such piece of δ as "tail".

The second reason which stops the holomorphic continuation of u is when the result at some point of the continuation of u started from one component (Jordan arc $\gamma_e^{\widetilde{e}}$) of Γ is different from the result which comes from another component of Γ. In this situation the pieces of δ make boundaries of the domains of the holomorphic continuation u from these different components. We call the corresponding pieces of δ as "membranes".

On Figure 1.1 and 1.2 there are possible pictures of the cut Γ and the set Ω with its subdivision by δ.

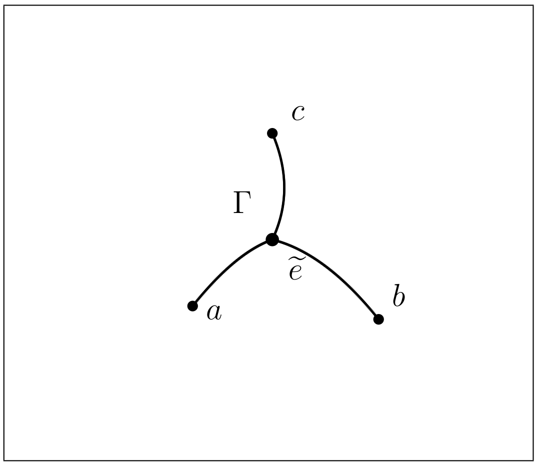

Рис. 1.1: Cut Γ.

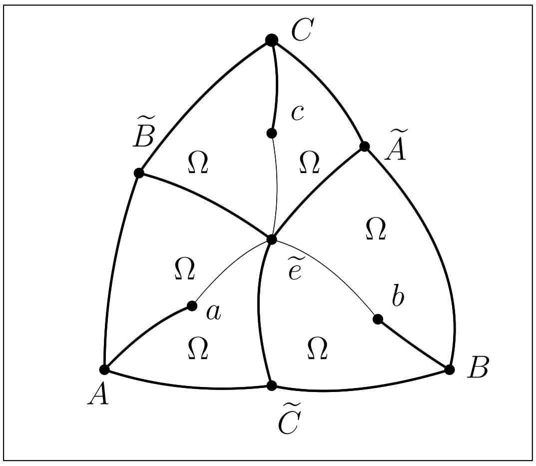

Рис. 1.2: Set Ω and its subdivision by δ.

Here we have. The cut $\Gamma := \bigcup\{\gamma_a^{\widetilde{e}}, \gamma_b^{\widetilde{e}}, \gamma_c^{\widetilde{e}}\}$ making f a holomorphic function. Tails — Jordan arcs $\gamma_A^a, \gamma_B^b, \gamma_C^c$. Membranes — $\gamma_{\widetilde{B}}^{\widetilde{e}}, \gamma_{\widetilde{A}}^{\widetilde{e}}, \gamma_{\widetilde{C}}^{\widetilde{e}}$. Boundary set $\delta := \bigcup\{\gamma_A^a, \gamma_B^b, \gamma_C^c, \gamma_{\widetilde{B}}^{\widetilde{e}}, \gamma_{\widetilde{C}}^{\widetilde{e}}, \gamma_{\widetilde{A}}^{\widetilde{e}}, \gamma_A^C, \gamma_C^B, \gamma_B^A\}$.

1.5 Geometry of the problem — general approach

Here we state an extremal problem for saddle point of energy functional (1.13) of non Hermitian Nikishin system (1.17)— (1.20). It is an example of Rakhmanov's "max–min"problem, when one minimize the energy varying measure with fixed support and maximize it varying the support (see [17] and statement for H-P approximants in [16]).

Given f — (1.1). Consider $\overrightarrow{f} = (f, f^2)$. Define a class $\mathcal{T}_f$ of vector compacts $\overrightarrow{E} := (E_1, E_2)$

$$\mathcal{T} := \{\overrightarrow{E}\}\ , \tag{1.21}$$

such that

$$E_1 = \Gamma\ , \quad f \in H(\overline{\mathbb{C}}\backslash\Gamma)$$

is a cut, making f holomorphic, and

$$E_2 = \delta, \quad u \in H(\Omega), \quad \delta := \partial(\Omega)\ ,$$

is the boundary of a set where $u := f^+ + f^-$ from the cut Γ has a piece wise holomorphic continuation.

Next, define a class of measures.

$$\mathcal{L}_{\overrightarrow{E}} := \{\overrightarrow{\tau}\} \quad : \quad \begin{cases} |\tau_1| = 2\,,\ |\tau_2| = 1\ , \\ \operatorname{supp}(\tau_k) \in E_k\,,\ k = 1,2\ . \end{cases} \tag{1.22}$$

We expect, that a saddle point

$$\exists\ !\ \ \overrightarrow{\tau}^*\ : \qquad I_{\mathcal{N}}(\overrightarrow{\tau}^*) = \max_{\overrightarrow{E}\in\mathcal{T}_f}\ \min_{\overrightarrow{\tau}\in\mathcal{L}_{\overrightarrow{E}}} I_{\mathcal{N}}(\overrightarrow{\tau})\ , \tag{1.23}$$

exists and unique, and provide the weak limits (1.10), (1.16) for the Hermite-Pade approximations of $\overrightarrow{f}$:

$$\lambda = \tau_1^* , \quad \mu = \tau_2^* .$$

The main goal of these notes is to discuss representations of the solution of problem (1.21)— (1.23) for function $\overrightarrow{f} = (f, f^2)$, where f is from the class (1.1).

1.6 Geometry of the problem — special solutions

J.Nuttall has put forward in [5] a general conjecture on asymptotics of H-P approximants. The conjecture states, that for vector-analytic function

$$\overrightarrow{f} := \{f_1, \ldots , f_d\} \in \mathcal{A}(\overline{\mathcal{C}} \backslash A) ,$$

with a finite set of branch points A, the main term of H-P asymptotics is governed by an abelian integral $G(\xi)$ on some $(d + 1)$ Riemann surface $\mathcal{R}$. If an appropriate $\mathcal{R}$ is given, then G uniquely (up to an additive constant) defined by two conditions. The only singularities of $G(\xi)$ are

$$G(\xi) = \begin{cases} -d \ \ \ln \xi \, , \ \xi \to \infty^{(0)} \, , \\ \\ \ \ \ \ \ \ \ \ln \xi \, , \ \xi \to \infty^{(j)} \, , \ j = 1, 2, \ldots , d \, , \end{cases} \tag{1.24}$$

and the periods of G are purely imaginary on $\mathcal{R}$, i.e. function $g := Re\, G$ is singlevalued on $\mathcal{R}$

$$g := Re\, G = \{g_j\}_{j=0}^d . \tag{1.25}$$

In the example of our situation, the weak form of the Nuttall conjecture states for the main term of asymptotics of type II H-P polynomial P_n

$$\frac{1}{n} \ln |P_n| \ \longrightarrow \ -g_0 . \tag{1.26}$$

However, general Nuttall conjecture has a deficiency — it does not say how to get the Riemann surface $\mathcal{R}$. At this point we can say that general approach related to Rakhmanov's "maxmin" saddle points of energy functional for vector-measures is able to fill this gap. The existence of the saddle point leads to the existence of the Riemann surface required for the Nuttall approach.

Conversely, for some specific problems, using input information on $\overrightarrow{f}$ (like character of branching, differential properties) it possible (ad hoc) to find an appropriate $\mathcal{R}$, which through (1.24), (1.25) leads to (1.26). In result we also obtain a constructive solution of the saddle point energy problem solution. In this notes we attack the problem (1.21) — (1.23) for (f, f^2), where f is (1.1), using the Nuttall approach.

2 Algebraic curves for H-P asymptotics

2.1 Nuttall approach and Nikishin system (d=2)

Our plan is the following. Using information about the function f from (1.1) we find a possible algebraic equations for

$$h(z) := \frac{dG(z)}{dz} \,. \tag{2.1}$$

The derivative of abelian integral G is meromorphic function h on a three sheeted Riemann surface $\mathcal{R} := \{\mathcal{R}^{(j)}\}_{j=0}^{2}$. For this Riemann surface we define a global sheet structure (we call it Nuttall sheet structure), taking

$$\{g_i\}_{j=0}^{2} \;:\quad g_0(z) \leqslant g_1(z) \leqslant g_2(z)\,, \quad z \in \overline{\mathbb{C}}\,. \tag{2.2}$$

Then we put

$$\begin{aligned} \Gamma &:= \{z \in \mathbb{C} \;:\; g_0(z) = g_1(z)\}\,, \\ \delta &:= \{z \in \mathbb{C} \;:\; g_1(z) = g_2(z)\}\,. \end{aligned} \tag{2.3}$$

We note, that g_0 is a superharmonic function and g_2 is a subharmonic function. Therefore (up to an additive harmonic function) g_0 is potential of a measure (denote it λ) and g_2 is minus potential of a measure (denote it μ)

$$\begin{aligned} \lambda \;&:\; |\lambda| = 2\,, \quad \operatorname{supp} \lambda \subseteq \Gamma\,, \\ \mu \;&:\; |\mu| = 1\,, \quad \operatorname{supp} \mu \subseteq \delta\,. \end{aligned} \tag{2.4}$$

Because of $g_0 + g_1 + g_2 = \text{const}$ in $\mathbb{C}$, for the measures (2.4) we have the Nikishin equilibrium (1.11).

Thus our main goals are

1) To find the algebraic equation for h — (2.1)

2) To describe in a constructive way set $\mathcal{T}$

$$\mathcal{T} := \Gamma \cup \delta = \{z \in \mathbb{C} \;:\; g_j(z) = g_k(z)\,,\; j \neq k,\; j,k = 0,1,2\}\,. \tag{2.5}$$

Definition of $\mathcal{T}$ is equivalent to

$$z \in \mathcal{T} \Leftrightarrow \;\operatorname{Re} \int [h_j(z) - h_k(z)]\, dz = 0 \Leftrightarrow \left(\int [h_j(z) - h_k(z)]\, dz\right)^2 < 0\,, j \neq k\,.$$

Therefore $\mathcal{T}$ is union of the critical orthogonal trajectories on $\mathcal{R}$ (since $(h_j - h_k)^2$ being the discriminant of h is meromorphic function in $\mathcal{R}$).

2.2 General equation for h

We start with pointing properties of the algebraic function h

$$h(z) \in \mathcal{M}(\mathcal{R}) \ : \quad G = \int h\, dz\,. \tag{2.6}$$

Behavior of G at infinity (see (1.24)) leads to

$$h(z) = \begin{cases} -\ \dfrac{2}{z} + \ldots\ , & z \to \infty^{(0)}\ , \\ \\ \dfrac{1}{z} + \ldots\ , & z \to \infty^{(j)}\ ,\ j = 1, 2\,. \end{cases}$$

Note, that function h can have its poles at the set B of the branch points of $\mathcal{R}$ only (otherwise, it would be a contradiction with (1.24) – the fact that G is regular in $\mathcal{R} \setminus \{\bigcup_{j=0}^{2} \infty^j\}\}$). Moreover, there are reasons to assume that poles h belongs to $A \in B$. Since h is rational (meromorphic) function on a three-sheeted Riemann surface, the maximal order of winding at $e \in B$ can be 3 (i.e. locally like $\sqrt[3]{z}$). Then in order to keep regularity of G in e the pole of h there should be of order less or equal 2. (If the order of the winding is equal 2, then the order of the pole is one). This observation together with (2.6) leads to the general form of equation for h

$$h^3\ -\ \frac{3P_4(z)}{\Pi_A^2(z)}\, h\ +\ \frac{2P_3(z)}{\Pi_A^2(z)}\ =\ 0, \tag{2.7}$$

where

$$\Pi_A(z) = \prod_{j=1}^{3} (z - a_j)\,,$$

and monic $P_j,\ \deg P_j = j,\ j = 3, 4$. The discriminant of equation (2.7) is equal

$$\triangle := (h_1 - h_2)^2\, (h_1 - h_3)^2\, (h_3 - h_2)^2\ =\ \frac{P_4^3 - \Pi_A^2 P_3^2}{\Pi_A^6}\,,$$

Since the maximal order of pole of h is equal 2 then the degree of the denominator has to be diminished (by cancelation) and we have

$$\triangle = \frac{P_4^3 - \Pi_A^2 P_3^2}{\Pi_A^6} = \frac{(z-d)^3 \Pi_A(z) - P_3^2(z)}{\Pi_A^4(z)} =: \frac{\widetilde{\triangle}}{\Pi_A^4}\,. \tag{2.8}$$

Therefore for the monic polynomials P_j in (2.7) have form

$$P_4(z) := (z - d)\, \Pi_A(z)\ , \quad P_3(z) := \prod_{j=1}^{3} (z - \widetilde{c}_j) =: \Pi_C(z), \tag{2.9}$$

where unknown parameters $d, \{\widetilde{c}_j\}$ should be determined.

Thus

$$h^3 \;-\; \frac{3(z-d)}{\Pi_A(z)}\,h \;+\; \frac{2\Pi_C(z)}{\Pi_A^2(z)} \;=\; 0. \tag{2.10}$$

We point out one linear equation connecting unknown parameters in (2.10). We restrict our choice of h assuming that the smaller terms in the expansion (2.6) at infinity for all branches are

$$h_j(z) \;=\; \frac{C_j}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad j = 1, 2, \tag{2.11}$$

(we think it can be rigorously deduced from the variational principle (1.22) - (1.23)). Then (2.11) implies that the degree of numerator has to be even

$$\deg \widetilde{\triangle} \;\leqslant\; 4\,. \tag{2.12}$$

It gives

$$3\,d \;=\; \widetilde{c}_1 + \widetilde{c}_2 + \widetilde{c}_3\,. \tag{2.13}$$

We also remark that if the branch point $e \in A$ of h has winding of the second order then one factor of $\Pi_C(z)$ is equal $(z-e)$ and we have cancelation in (2.7) – (2.10).

2.3 Monodromy classes of $\{f\}$ and choice of curve h

Different monodromy properties of input function f – (1.1) lead to various Riemann surfaces $\mathcal{R}$ (different genus) and to various sets $\mathcal{T} = \Gamma \cup \delta$ (different topology of pictures of $\mathcal{T}$) for H–P problem for $\{f, f^2\}$. A slight hope to realize the exhaustive search of all possible $\mathcal{R}$ and $\mathcal{T}$ follows from boundedness of the degree of the numerator $\widetilde{\triangle}$ of the discriminant h (see (2.12)). Zeros of $\mathcal{R}$ of the odd order are the extra (with respect to input set A) branch points for h. Appearance additional branch points increases genus of $\mathcal{R}$. In accordance with (2.8) and (2.12) genus of the algebraic curve h (with cubic root winding at the branch points) formally varies from 1 to 3. We shall relate the monodromy properties of f with various classes of $\mathcal{T}$ – (1.21). It gives us a procedure to find unknown parameters $d, \{\widetilde{c}_j\}$ in equation (2.10).

The main idea which we borrow from the Nuttall conjecture for our analysis is the following. If we take the first two sheets ($\mathcal{R}^{(0)}$ and $\mathcal{R}^{(1)}$) of $\mathcal{R}_h$ (three sheeted Riemann surface of function h) cut such that

$$\mathcal{R}^{(0)} = \pi_0^{-1}\left(\overline{\mathbb{C}} \setminus \Gamma\right)\,, \qquad \mathcal{R}^{(1)} = \pi_1^{-1}\left(\overline{\mathbb{C}} \setminus (\Gamma \cup \delta)\right)\,, \tag{2.14}$$

give domains of global definitions of two single valued branches $\{h_j\}_{j=0,1}$ of function h

$$h_0 \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus \Gamma\right) , \quad h_1 \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus (\Gamma \cup \delta)\right) , \quad h_{0\pm} = h_{1\mp} \text{ on } \Gamma . \qquad (2.15)$$

Then (due to the Nuttall conjecture) the same cut two sheets (2.14) are the first two sheets of the Riemann surface $\mathcal{R}_f$ of our input function f, which define globally two its single valued branches $\{f_j\}_{j=0,1}$:

$$f_0 \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus \Gamma\right) , \quad f_1 \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus (\Gamma \cup \delta)\right) , \quad f_{0\pm} = f_{1\mp} \text{ on } \Gamma . \qquad (2.16)$$

We note that when $\overline{\mathbb{C}} \setminus \Gamma$ is a domain, then the branch f_0 is a result of holomorphic continuation of the input germ (1.2).

For the better understanding of the correspondence (2.15) $\longleftrightarrow$ (2.16) we recall that the poles of the Padé approximants (1.4) of f define the cut of the first ("physical") sheet of its Riemann surface $\mathcal{R}_f^{(0)}$, and this sheet is, in some sense, an optimal (the cut of this sheet has the minimal capacity). In a similar fashion the Hermite–Padé approximats for $\{f, f^2\}$ by means of (2.16) $\Longleftrightarrow$ (2.15) $\Longleftrightarrow$ (2.14) define in an "optimal" way the first two sheets $\mathcal{R}_f^{(0)}$, $\mathcal{R}_f^{(1)}$.

3 Classification of algebraic curves h (beginig)

3.1 General rules for classification of h

We recall that our goals are:

1) To find the unknown parameters for the algebraic function h – (2.10), which necessarily has to correspond to the input function f from the class (1.1), in the way that (2.15) corresponds to (2.16).

2) To draw a portrait of the set $\mathcal{T}$ – (1.21).

Concerning to item 1) we mention that we have two sorts of conditions for determination of the unknown parameters – algebraic and transcendental conditions.

The trancendental conditions express the fact that periods of the Nuttall's Abelian integral $G = \int h\, dz$ – (1.24) are purely imaginary on $\mathcal{R}$ – (1.25). For example, if all zeros of the numerator $\widetilde{\triangle}$ of the discriminant h (see (2.12)) are simple, then $\mathcal{R}_h$ has a maximal possible genus which is equal 3, therefore $\mathcal{R}_h$ has 6 cycles and we have 6 real, transcendental conditions (Re $\int h\, dz = 0$ along 6 cycles) for determination of 3 unknown complex parameters in (2.9), (2.13).

The algebraic conditions appears when genus of $\mathcal{R}_h$ diminishes, that happens due to even multiplicity of some zeros of the numerator $\widetilde{\triangle}$ of the discriminant h. For example, if all zeros of $\widetilde{\triangle}$ (i.e. due to (2.12) 2 zeros) have order

2, then we get $\text{gen}(\mathcal{R}_h) = 1$ and for 3 unknown complex parameters we have 2 real, transcendental conditions and 2 complex algebraic conditions.

Thus reduction of the number of the transcendental conditions is compensated by the same number of algebraic conditions imposed on even multiplicity of zeros of $\widetilde{\triangle}$, and we always have in total the same number of conditions (transcendental and algebraic) as the number of the unknown parameters.

There is also a scenarios when transcendental condition disappears together with unknown parameter. It happens if the input function f has a branch point of order 2 (like a square root). Then (in accordance with (2.15) $\longleftrightarrow$ (2.16)) the algebraic function h also has the same winding at this point, that caused from one side the reduction of the $\text{gen}(\mathcal{R}_h)$ and therefore lost of the transcendental condition, but from another side we have the cancelation of one unknown parameter $\widetilde{c}$ in (2.10), (see remark at the end of subsection 2.2). Again the number of the unknown parameters is balanced with the number of the conditions required for their determination.

We also remark, that degeneration to the square root winding at the branch points of f causes for the set $\mathcal{T}$ lost of the tail at this branch point. We find it convenient to use number of tails of the set $\mathcal{T}$ as the main characteristic for our classification of all possible algebraic curves h, appearing in the problem.

- Three tails case. It is a general situation. The equation for h is (2.10) (as we already mentioned above).

- Two tails case. Denote $A := \{a_1, a_2, b\}$ and we set that point b has lost the tail. The equation for h takes a form
$$h^3 \; - \; \frac{3(z-d)}{\Pi_A(z)}\,h \; + \; \frac{2(z-c_1)(z-c_2)}{(z-a_1)^2(z-a_2)^2(z-b)} \; = \; 0, \tag{3.1}$$
and genus of $\mathcal{R}_h$ varies from 0 to 2.

- One tail case. Denote $A := \{a, b_1, b_2\}$ and we set that points b_1, b_2 has lost their tails. The equation for h takes a form
$$h^3 \; - \; \frac{3(z-d)}{\Pi_A(z)}\,h \; + \; \frac{2(z-c)}{(z-a)^2(z-b_1)(z-b_2)} \; = \; 0, \tag{3.2}$$
and genus of $\mathcal{R}_h$ may take values 0 or 1.

- Zero tails (tailless) case. For this case the algebraic curve h is defined explicitly in the unique manner. Curve h has genus 0 and its equation is
$$h^3 \; - \; \frac{3z \; - \; (a_1+a_2+a_3)}{\Pi_A(z)}\,h \; + \; \frac{2}{\Pi_A(z)} \; = \; 0, \tag{3.3}$$

Finally, we say a couple of words about item 2) from the above list of our goals.

To draw the set $\mathcal{T}$ – (1.21) for the cases when $\text{gen}(\mathcal{R}_h) = 0$ we employ machinery developed in [15], [19], [20]. We use the fact that

$$\Phi = \exp\left(\int h\, dz\right) \tag{3.4}$$

is also an algebraic function with the same Riemann surface as h. The function Φ satisfies the equation

$$\Phi^3(z) + q_1(z)\Phi^2(z) + q_2(z)\Phi(z) + q_0 = 0, \tag{3.5}$$

where q_j are polynomials of degree $\leq j$ ($j = 0, 1, 2$), which can be computed from the coefficients of the equation for h. In terms of Φ the set $\mathcal{T}$ – (1.21) can be expressed as

$$\mathcal{T} = \{z:\ |\Phi_j(z)| = |\Phi_k(z)|,\ 0 \leq j \neq k \leq 2\}. \tag{3.6}$$

The set $\mathcal{T}$ given in (3.6) for the function Φ satisfying (3.5) is also an algebraic curve, which can be expressed as

$$\mathcal{T} = \{z:\ J(\nu, z) = \nu^3 + A(z)\nu^2 + B(z)\nu + C(z) = 0, \nu \in [-2, 2]\}, \tag{3.7}$$

where

$$\begin{aligned} A(z) &= \frac{3q_0 - q_1(z)q_2(z)}{q_0} \\ B(z) &= \frac{q_0 q_1^3(z) + q_2^3(z) - 5q_0 q_1(z)q_2(z) + 3q_0^2}{q_0^2} \\ C(z) &= \frac{2q_0 q_1^3(z) - q_1^2(z)q_2^2(z) + 2q_2^3(z) - 4q_0 q_1(z)q_2(z) + q_0^2}{q_0^2} \end{aligned} \tag{3.8}$$

and q_0, q_1, q_2 are the coefficients of the equation (3.5) for Φ - (3.4).

To draw the set $\mathcal{T}$ for the cases when $\text{gen}(\mathcal{R}_h) > 0$ we use the fact (2.5) that $\mathcal{T}$ is union of the critical orthogonal trajectories on $\mathcal{R}$, i.e. $z \in \mathcal{T} \Leftrightarrow$

$$\left(\int [h_j(z) - h_k(z)]\, dz\right)^2 < 0 \quad \Leftrightarrow \quad \text{Re} \int [h_j(z) - h_k(z)]\, dz = 0, \quad j \neq k\ .$$

The last equality leads to differential equation

$$\text{Re}\ (h_j - h_k)(z)dx - \Im(h_j - h_k)(z)dy = 0, \quad z = x + iy. \tag{3.9}$$

This equation can be solved numerically (starting from the branch points of h) by various methods (from which the most appropriate one for each particular case can be chosen). Speaking about numerical approaches for obtaining the set $\mathcal{T}$ we would like to thank Bernd Beckermann for providing us with software which solves (3.9) and simultaneously doing optimization of obtaining trajectories compute the unknown parameters of h.

3.2 Tailless class $\{h\}$ (zero tails)

Example of function f from class (1.1), whose corresponding function h – (1.10) has not tails is

$$f(z) := \sqrt{(z-a_1)(z-a_2)} + \sqrt{(z-a_2)(z-a_3)} = 0. \qquad (3.10)$$

As we already mentioned above, for this case equation of function h has explicit form (3.3), h in (3.3) has four branch points at

$$a_1,\ a_2,\ a_3,\ b^* = \frac{a_1a_2a_3 - \left(\dfrac{a_1+a_2+a_3}{3}\right)^3}{a_1a_2+a_1a_3+a_2a_3 - 3\left(\dfrac{a_1+a_2+a_3}{3}\right)^3}\,. \qquad (3.11)$$

its genus equal zero

$$\mathrm{gen}(h) = 0\,.$$

The algebraic equation was studied in details in [15]. From there we present an equation for the algebraic curve $\mathcal{T}$. Without loss of generality, we put

$$a_1 := -1, \quad a_2 := 0, \quad a_3 := a. \qquad (3.12)$$

The exponential function of the abelian integral

$$\Phi = \exp\left(\int h(z)\,dz\right)$$

of the function h in (3.3) is, up to a multiplicative constant, an algebraic function satisfying the equation

$$\Phi^3 + q_1(z)\Phi^2 + q_2(z)\Phi + q_0 = 0, \qquad (3.13)$$

where $\deg q_j \le j$ for $j = 0,1,2$ and the q_j are rational functions of a_1, a_2, b_2. If we take into account (3.12) then we have

$$\begin{aligned} q_1(z) &= z(a-1)(a^2+\frac{5a}{2}+1) + \frac{a(a^2+4a+1)}{2} \\ q_2(z) &= -\tilde{\kappa}\left(\frac{27z^2}{4} - \frac{9}{2}(a-1)z - \frac{1}{4}(a^2+10a+1)\right) \\ q_0 &= \tilde{\kappa}^2 \end{aligned} \qquad (3.14)$$

where $\tilde{\kappa} = a^2(a+1)^2/4$. Now substitute the q_0, q_1, q_2 from (3.14) into the expressions (3.8) for the coefficients A, B, C (see (3.8)) of

$$\mathcal{T} = \{z\ :\ J(\nu, z) = \nu^3 + A(z)\nu^2 + B(z)\nu + C(z) = 0,\ \nu \in [-2,2]\}\,. \qquad (3.15)$$

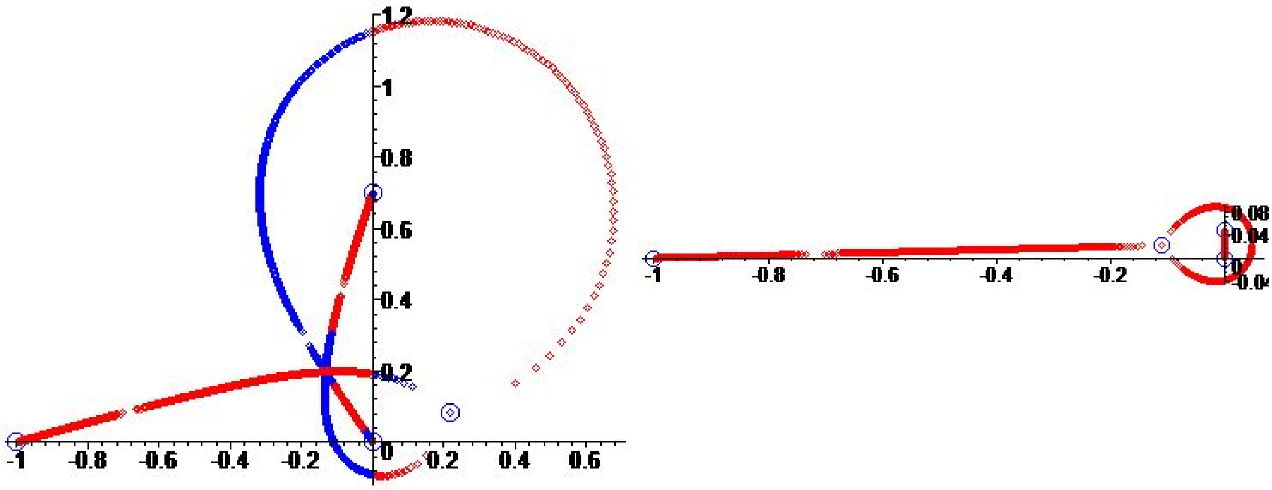

Рис. 3.1: Plot of set $\mathcal{T}$. The tailless case.

Plots of the algebraic curve $\mathcal{T}$ for purely imaginary parameters $a := 0.7 * I$ and $a := 0.02 * I$ are presented on Figure 3.1.
Various sketches of the curve $\mathcal{T}$ of different topology are presented below (see Figures 3.2 and 3.4).

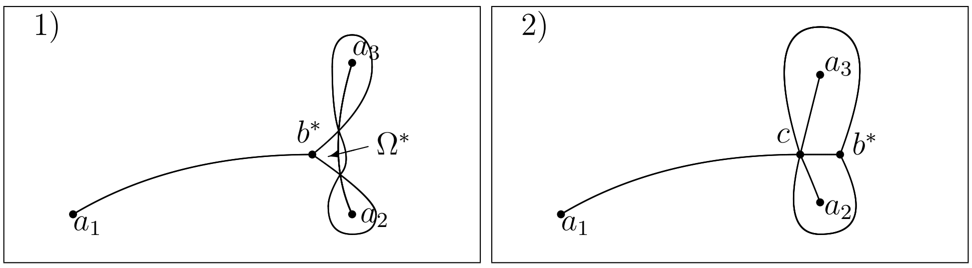

Рис. 3.2: Sketches of the curve $\mathcal{T}$. The tailles case.

We see, that for some values of the input parameters (i.e. set of branch points A) the extremal cut Γ, making the function f holomorphic is a union of analytic arcs, like in Figure 3.2–2), see on the Figure 3.3 corresponding subdivision of the extremal contour $\mathcal{T}$ into Γ and δ. For such cases the extremal set of

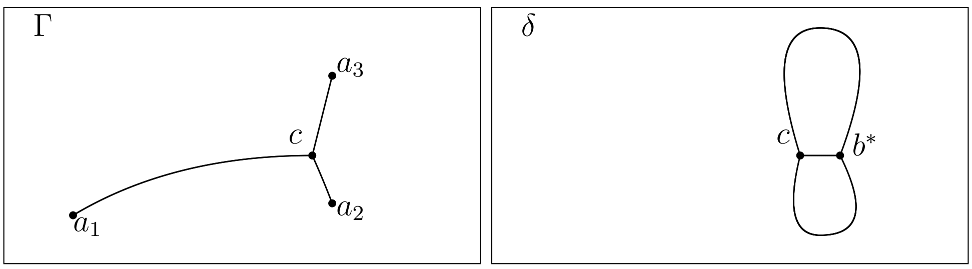

Рис. 3.3: Subdivision of $\mathcal{T}$ into Γ and δ for Figure 3.2–2).

the holomorphic continuation $f \in \mathcal{H}(\mathbb{C} \setminus \Gamma)$ is a domain (i.e. connected set).

We see also, that there are values of the input parameters A such that the extremal cut Γ contains an analytic curve (closed) or closed union of analytic arcs, like in Figure 3.4 or Figure 3.2–1), see on the Figure 3.10 corresponding subdivision of the extremal contour $\mathcal{T}$ into Γ and δ. For such cases the extremal set for the holomorphic branch $f \in \mathcal{H}(\mathbb{C} \setminus \Gamma)$ is disconnected. It is a union of domain which has a component, not containing infinity point $\infty \notin \Omega^*$. Appearence of the component Ω^* on the main (or "physical") sheet $\mathcal{R}_0$ of the

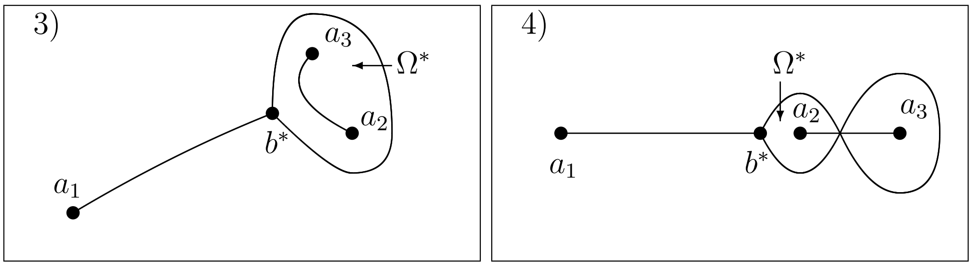

Рис. 3.4: Sketches of the curve $\mathcal{T}$. The tailles case.

Riemann surface of the functions f and h (they have the same first two sheets due to (2.16), (2.15)) causes (for some cases) down shift from $\mathcal{R}_0$ one (see Figure 3.4 – 2)) or even two (see Figure 3.4 – 1)) branch points of the input set A to another ("nonphysical") sheets. For such cases we say, that the branch point is not activated on the "physical" sheet.

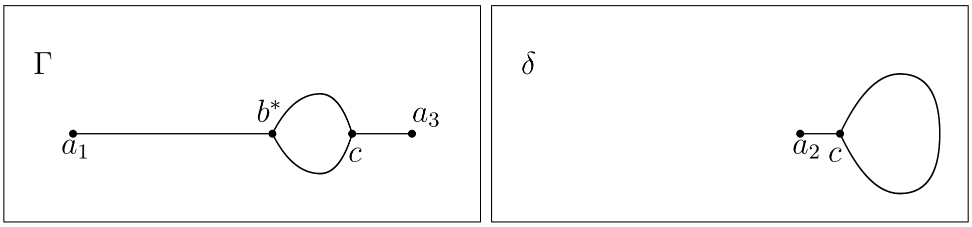

Рис. 3.5: Subdivision of $\mathcal{T}$ into Γ and δ for Figure 3.4–1).

In [15] the complete description of the bifurcations of the topology of the extremal set was obtained. We recall here that result. Assuming (3.12) we denote:

D_3 is a set of positions for the point $a := a_3$ inside of the unit circle D for which $\mathbb{C} \setminus \Gamma$ will be a domain (connected set).

D_2 is a set of position for $a := a_3$ for which $\mathbb{C} \setminus \Gamma$ is not connected, but the point $a_2 = 0$ is still active on the "physical" sheet. The rest in D is a domain D_1.

We have (see [15]).

1. The algebraic curve

$$\widehat{P}(a,\nu) := 16a^{12}+96a^{11}+(336-108\nu)a^{10}+(800-540\nu)a^9+(2169-1404\nu)a^8$$
$$+\ (4932-2376\nu)a^7+(6630-2808\nu)a^6+(4932-2376\nu)a^5$$
$$+\ (2169-1404\nu)a^4+(800-540\nu)a^3+(336-108\nu)a^2+96a+16=0,$$

which for $\nu = 2$ is factorized as

$$(a-1)^4(2a+1)^4(a+2)^4 = 0,$$

has six branches in D for $\nu \in [-2, 2]$. These six branches start (for $\nu = 2$) from the points 1 (two branches) and $-1/2$ (four branches) and give the inner boundary ∂D_3^{in} of the region D_3. The outer boundary ∂D_3 consists of $\{|a| = 1\}$.

2. Two branches of the algebraic curve

$$a^4 + (4-4\nu)a^3 + (22-8\nu)a^2 + (4-4\nu)a + 1 = 0$$

lying in D for $\nu \in [-2, 2]$ form the boundary ∂D_1 of the domain D_1.

3. The region D_2 is the open set bounded by ∂D_3^{in} and ∂D_1.

See Figure 3.6:

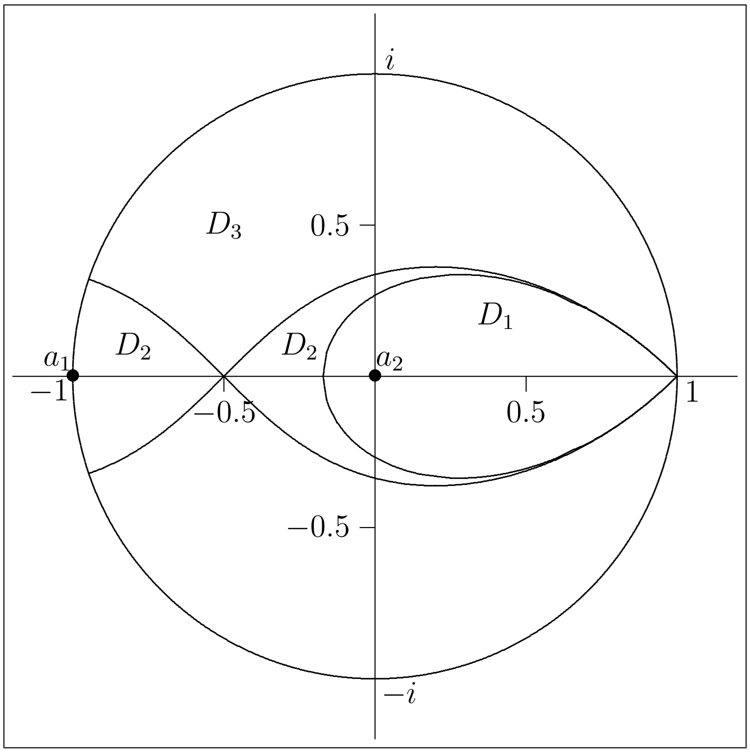

Рис. 3.6: The regions D_1, D_2, D_3 and their boundaries.

3.3 One tail class $\{h\}$

3.3.1 Genus zero subcases

We find the unknown parameters d and c for the equation (3.2):

$$h^3 - \frac{3(z-d)}{\Pi_A(z)}h + \frac{2(z-c)}{(z-a)^2(z-b_1)(z-b_2)} = 0,$$

To do it we compute the discriminant of this equation and vanishing the coefficient for the highest (third) degree of the numerator we obtain one linear equation $-2c - b_1 - b_2 + 3d + a = 0$ for c and d.

If we express c via d and substitute it back to the numerator of the discriminant, we get a polynomial of the second degree in z. Since we are in zero genus case, we take discriminant of this polynomial equal to zero, then we obtain an equation for d of degree 6, which has nice factorization $(b_1 + b_2 - a - d)^3 \left(-16d^3 + (24b_1 + 24b_2)d^2 + (-30b_1b_2 - 9b_1^2 - 9b_2^2)d + d_0\right) = 0$, where $d_0 := 7b_1^2b_2 - ab_1^2 - ab_2^2 + b_1^3 + b_2^3 + 7b_1b_2^2 + 2b_1b_2a$. This equation has six different solutions, which depend on a monodromy class of the input function f. On the Figure 3.7 is plot of the one tail case of genus zero without membrane. For this

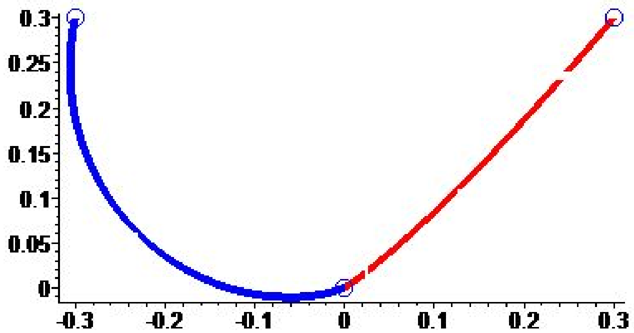

Рис. 3.7: One tail, no membranes case, $\mathrm{gen}(\mathcal{R}_h) = 0$. Set $\mathcal{T}$: Γ is blue, δ is red.

case the branch point $b_2 = 0.3 + 0.3 * I$ is not active on the "physical" sheet. We note, that although the constellation of the branch point A is symmetric with respect to imaginary axis, nevertheless the plot have lost its symmetry, due to the quite understandable reasons. Another monodromy properties of the finction f for the same input data (set and character of the branch points A) give us the case with one tail and membranes (see Figure 3.8):

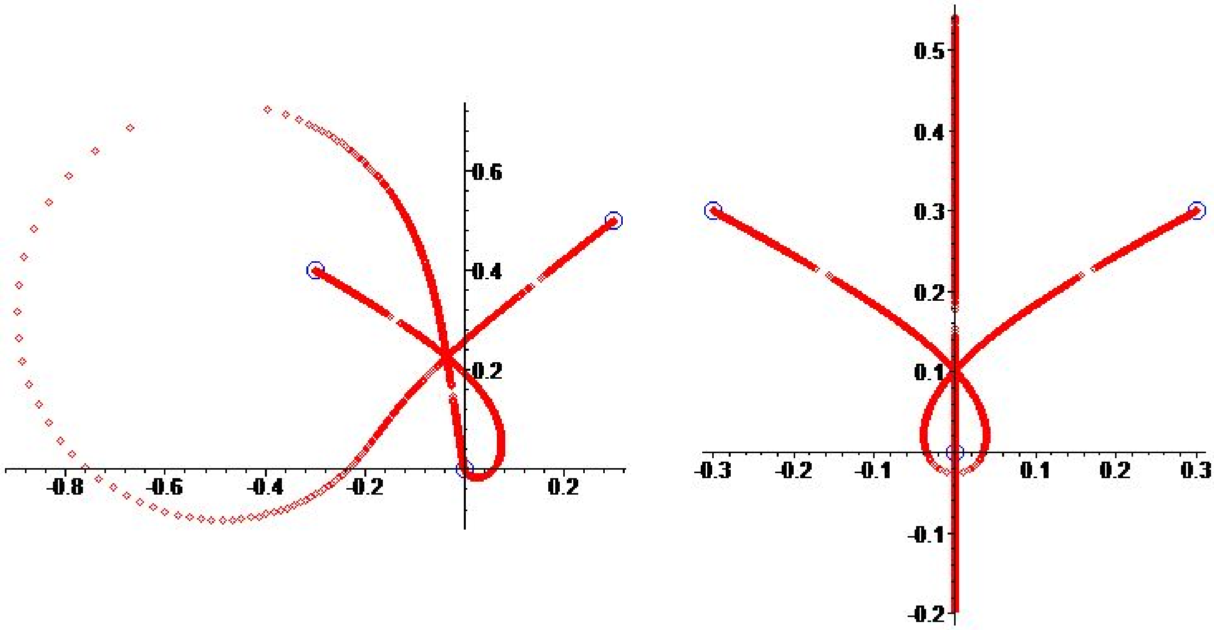

Рис. 3.8: Set $\mathcal{T}$ for one tail with membranes case ($\text{gen}(\mathcal{R}_h) = 0$).

3.3.2 Genus one subcase

Below (see Figure 3.9) we present a result of computer aided analysis of genus one subcase. Using MatLab procedure (written by Bernd Beckermann) we find numerically the unknown parameter d for the equation (3.2) and plot the set $\mathcal{T}$ for the input values $b_1 = -1$, $b_2 = 0$, $a = 1$ of the branch points A.

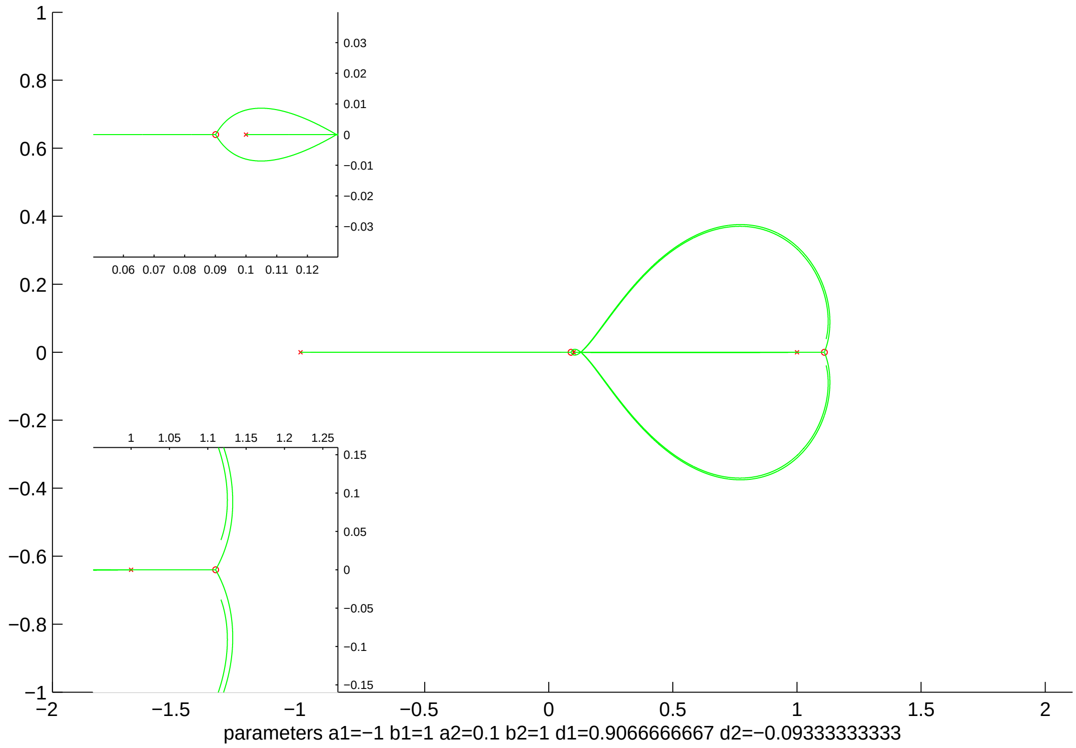

Рис. 3.9: One tail with membranes case ($\text{gen}(\mathcal{R}_h) = 1$). Set $\mathcal{T}$.

We note, that the parameters in caption of Figure 3.9 are related with the parameters in (3.2) as $z^2 - d_1 z + d_2 = (z-d)(z-b)$ and symbols a and b for Figure 3.9 and for the equation (3.2) should be swapped.

3.4 Two tails class $\{h\}$ (genus zero subcase)

To present an example of the function f with two tails and genus zero we choose a function with two branch points a_1 and a_2 of higher order (let say winding in a_1 higher than 2 and in a_2 higher than 3) and branch point a_3 is not active not only on the "physical" sheet, but not active on the next sheet neither. Other words the Riemann surface $\mathcal{R}_f$ should have at least four sheets with monodromy matrices at the branch points as follows

$$a_1 \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; a_2 \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; ; \; a_3 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Such function evidently exists. For this function f the corresponding three sheeted Riemann surface $\mathcal{R}_h$ (in accordance with) has the following monodromy matrices at the branch points

$$a_1 \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; ; \; a_2 \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; ; \; a_3 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Thus the winding at a_3 disappears from $\mathcal{R}_h$ and we have that h is an algebraic function of the third order with the only two branch points at a_1 and a_2 with the cubic root type of winding. It is not difficult to write down an equation for h. Taking into account (2.11) we get

$$h^3 - \frac{3}{(z-a_1)(z-a_2)} h + \frac{2(z-c)}{(z-a_1)^2(z-a_2)^2} = 0, \qquad c = \frac{a_1+a_2}{2}.$$

It is evident, that set $\mathcal{T}$ for this case is just a straight line (see Figure 3.10):

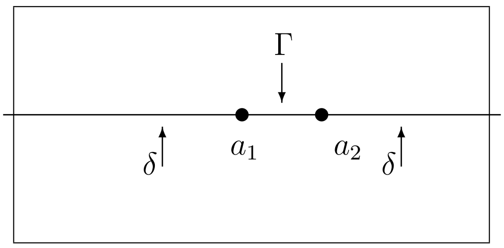

Рис. 3.10: Two tails, genus zero case. The set $\mathcal{T}$.

3.5 Nuttall-Jacobi class $\{h\}$ (announcement)

The most interesting case of the class (1.1) functions with three branch points is the Nuttall-Jacobi class (1.3). Using our classification, for this class we have two cases.

The first one is a degenerative case "two tails, genus one"with representative function

$$f(z) = \sqrt{(z-a_1)}\,\sqrt[4]{(z-a_2)(z-a_3)}.$$

From our experience we could say that it is very difficult case for analysis.

The second case has a general character. For this case any type of winding at the branch points allowed, except of the square root winding, i.e.

$$f(z) = \prod_{j=1}^{3}(z-a_j)^{\alpha_j}\,, \qquad \sum \alpha_j = 0\,, \quad 2\alpha_j \notin \mathbb{Z}, \quad j = 1, 2, 3\,. \tag{3.16}$$

A power of the Nuttall conjecture is that via (2.15) <–> (2.16) we immediately get that for the case (3.16) the Riemann surface for the function h is

$$\mathcal{R}_h \;=\; \mathcal{R}\left(\sqrt[3]{\prod_{j=1}^{3}(z-a_j)}\right)\,. \tag{3.17}$$

However, to find an equation for the function h – (1.24), (2.1), even knowing that h is a rational function on the Riemann surface (3.17), it is a rather difficult technical problem. We shall report about a solution of this problem in the nearest future. Plot here presents a result of this analysis.

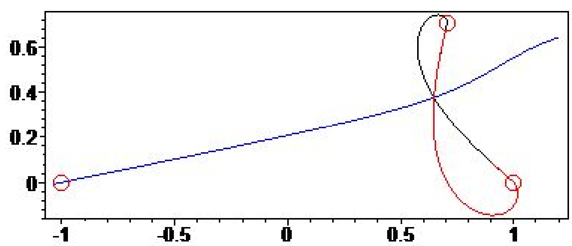

Рис. 3.11: Three tails, three membranes, genus 1. Set $\mathcal{T}$. Nuttall-Jacobi class.

Список литературы

[1] J. Nuttall, *Asymptotics of generalized Jacobi polynomials*, Constr. Approx., 2:59–77, 1986.

[2] J. Nuttall, *Note on generalized Jacobi polynomials*, The Riemann Problem, Complete Integrability and Arithmetic Applications, Lecture Notes in Mathematics, 925, Springer-Verlag, Berlin, 1982, pp. 258–270.

[3] L. Baratchart and M. Yattselev, *Asymptotics of Padé approximants to a certain class of elliptic-type functions*, Journal d'Analyse Mathematique (in press)

[4] A. Martinez-Finkelshtein, E. A. Rakhmanov, and S. P. Suetin, *Heine, Hilbert, Pade, Riemann, and Stieltjes: a John Nuttall's work 25 years later*, Contemporary Math. (in press)

[5] J. Nuttall, *Asymptotics of diagonal Hermite—Pade polynomials*, J. Approx. Theory **42** (1984), no. 4, 299–386.

[6] A. I. Aptekarev, H. Stahl, *Asymptotics of Hermite-Padé polynomials*, in 'Progress in Approximation Theory' (A. Gonchar, E. B. Saff, eds.), Springer-Verlag, Berlin, 1992, pp. 127–167.

[7] A. I. Aptekarev, *Multiple orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), no. 1–1, 423–448.

[8] W. Van Assche, *Multiple orthogonal polynomials, irrationality and transcendence,* in "Continued fractions: from analytic number theory to constructive approximation" , Contemporary Mathematics 236 (1999), 325-342.

[9] E. M. Nikishin, *Simultaneous Padé approximants*, Mat. Sb. **113** (155) (1980), 449–519 (in Russian); Math. USSR Sbornik **41** (1982), 409–425.

[10] E. M. Nikishin, *Asymptotic behavior of linear forms for simultaneous Padé approximants*, Izv. Vyssh. Uchebn. Zaved. Mat. (1986), no. 2, 33–41; Soviet Math. **30** (1986), no. 2, 43–52.

[11] E. M. Nikishin and V. N. Sorokin, *Rational Approximations and Orthogonality*, Nauka, Moscow, 1988; Translations of Mathematical Monographs **92**, Amer. Math. Soc., Providence RI, 1991.

[12] A. A. Gonchar, E. A. Rakhmanov, and V. N. Sorokin, *On Hermite-Padé approximants for systems of functions of Markov type*, Mat. Sb. **188** (1997), no. 5, 33–58 (in Russian); Sbornik Math. **188** (1997), no. 5, 671–696.

[13] A. I. Aptekarev and V. G. Lysov, *Systems of Markov functions generated by graphs and the asymptotics of their Hermite-Padé approximants*, Mat. Sb. 201(2) (2010), 29–78 (in Russian); English transl. in Sb. Math. 201 (2) (2010), 29–78.

[14] E. A. Rakhmanov, *The asymptotics of Hermite-Pade' polynomials for two Markov-type functions* , Mat. Sb. 202:1 (2011), 133–140 (in Russian); English transl. in Sb. Math. 202:1 (2011), 127–134.

[15] A. I. Aptekarev, A. B. J. Kuijlaars and W. Van Assche, *Asymptotics of Hermite-Padé rational approximants for two analytic functions with separated pairs of branch points (case of genus 0)*, International Mathematics Research Papers, **2008** (2008) , Article ID rpm007, 128 pages, (doi:10.1093/imrp/rpm007)

[16] A. I. Aptekarev, *Asymptotics of Hermite–Padé Approximants for Two Functions with Branch Points*, Doklady Akademii Nauk, 422:4, 2008, 443-445; English transl. in Dokl. Math., **78** (2), (2008), 717–719, (http://dx.doi.org/10.1134/S1064562408050207)

[17] E. A. Rakhmanov, *Orthogonal polynomials and S curves* , Contemporary Math. (in press)

[18] A. A. Gonchar and E. A. Rakhmanov, *On the convergence of simultaneous Padé approximants for systems of functions of Markov type*, Trudy Mat. Inst. Steklov. **157** (1981), 31–48; Proc. Steklov Inst. Math. **157** (1983), 31–50.

[19] A. I. Aptekarev, V. G. Lysov and D. N. Toulyakov, *Three-sheeted Riemann surfaces of genus 0 with fixed projections of the branch points*, Preprint, Inst. Appl. Math. Russia Akad. Sci. (13) (2007), (in Russian, http://www.keldysh.ru/papers/2007/source/prep2007_13.pdf)

[20] A. I. Aptekarev, V. A. Kalyagin , V. G. Lysov and D. N. Toulyakov, *Equilibrium of vector potentials and uniformization of the algebraic curves of genus 0*, Journal of Computational and Applied Mathematics, **233** (2), (2009), 602–616

Содержание