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Асимптотика L_p-нормы многочленов Эрмита [1]

Аннотация. Асимптотика L_p-нормы многочленов Эрмита получена для $n \to \infty$ и $p > 0$. Этот результат мотивирован вычислениями Рэньи энтропии квантово-механических плотностей вероятностей высоко-энергетических (ридберговских) состояний изотропического осцилятора.

Ключевые слова. Асимптотический анализ; ортогональные многочлены; информационная энтропия.

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Abstract

The asymptotics of the weighted L_p-norms of Hermite polynomials is determined for $n \to \infty$ and $p > 0$. The result is motivated by calculations the Rényi entropy of the quantum-mechanical probability density of the highly-excited (Rydberg) states of the isotropic oscillator.

Key words. Asymptotical analysis; orthogonal polynomials; information entropy.

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1 Introduction

The study of the L_p -norms of orthogonal polynomials is of independent interest in the theory of general orthogonal and extremal polynomials. According to the methods of solution this problem is connected with the classical research of S.N. Bernstein on the asymptotics of the L_p extremal polynomials in [6], that received further development in recent papers [7], [8]. On the other hand, its statement is a generalization of a widely known problem of Steklov on the estimation of the L^∞ norms of polynomials orthonormal with respect to a positive weight (see [9]). Indeed, for $p = 1$ the norms are bounded (they just equal 1); however, for $p = \infty$ (as it has been shown by Rakhmanov [10]) they may grow to infinity. What happens with boundness of the L_p -norms of the orthonormal polynomials when $1 < p < \infty$?

The aim of this work is the asymptotic ($n \to \infty$) determination of the entropic moment of the Hermite polynomials, i.e.

$$W_p[H_n] = W_p[\rho(H_n)] = \int_{-\infty}^{+\infty} \left[e^{-x^2} H_n^2(x)\right]^p dx; \quad p \geq 0. \tag{1.1}$$

The solution of this problem is physico-mathematical relevant not only *per se* because it extends previous results obtained when $p \in \left[0, \frac{4}{3}\right]$ by means of high-brow techniques of approximation theory [3, 4, 5], but also because it paves the way for the evaluation of the pth-order Rényi entropy of the spatial probability density associated to the highly-excited (i.e., Rydberg) states of the physical systems whose radial wavefunctions are controlled by Hermite's polynomials such as, e.g. the oscillator-like systems. This calculations have been perfomed in [1].

The structure of the preprint is the following. In the next Section 2 we precise the statement of the problem, discuss the previous results and the difficulties which did not allow to find a complete solution of the problem before. Finally, in this section we present a statement of the main result obtained in our paper. In Section 3 we sketch a method from [2] for obtaining the global asymptotic portrait for solution of recurrence relations and state the corresponding result for the global asymptotics of Hermite polynomials. In Section 4, the asymptotics of L_p-norms of the Hermite polynomials is found for $p > 0$.

2 Asymptotics ($n \to \infty$) of L_p-norms of Hermite polynomials. Statement of the problem and the result.

Let us begin by keeping in mind that the standard orthogonal Hermite polynomials are defined by the following hypergeometric function (see [11])

$$H_n(x) = (2x)^n \, {}_2F_0 \left(\begin{array}{c} -\frac{n}{2}, \frac{1-n}{2} \\ - \end{array} ; -\frac{1}{x^2} \right),$$

and satisfy (see [11])

$$H_n(x) = (2x)^n \left(1 - \frac{n(n-1)}{4x^2} + O\left(\frac{n^4}{x^4} \right) \right). \tag{2.1}$$

Moreover, they fulfil the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H_0(x) = 1, \quad H_{-1}(x) = 0, \tag{2.2}$$

and have the following norm [11]

$$h_n = \int_{-\infty}^{+\infty} \left[H_n^2(x) \, e^{-x^2} \right] dx \; = \; \sqrt{\pi} \, n! \; 2^n \, . \tag{2.3}$$

The first attack on the problem of finding asymptotics (1.1) of the L_p-norms was attempted almost two decades ago. In [3] it was obtained for the orthonormal Hermite polynomials $\tilde{H}_{n-1}(x)$ that

$$\int_{-\infty}^{+\infty} \left[\tilde{H}_{n-1}^2(x) e^{-x^2} \right]^p dx = \; c_p (2n)^{\frac{1-p}{2}} \left(1 + o(1) \right) \, , \quad \text{for } p \le \frac{4}{3} \, , \tag{2.4}$$

where

$$c_p \; = \; \left(\frac{2}{\pi} \right)^p \, \frac{\Gamma(p + \frac{1}{2})}{\Gamma(p+1)} \, \frac{\Gamma(1 - \frac{p}{2})}{\Gamma(\frac{3}{2} - \frac{p}{2})} \, . \tag{2.5}$$

Keeping in mind (2.3) and Stirling's formula, this expression together with (1.1) produce the following asymptotics for the entropic moment of the orthogonal Hermite polynomial $H_{n-1}(x)$:

$$\begin{aligned} W_p[H_{n-1}] &= c_p \; h_{n-1}^p \; (2n)^{\frac{1-p}{2}} \; (1 + o(1)) \\ &= c_p \pi^p (2n)^{p(n-1)+1/2} \, e^{-pn} \; (1 + o(1)) \, . \end{aligned} \tag{2.6}$$

Despite the limitation in p (we note that constant c_p in (2.5) remains bounded until $p < 2$), the result of [3] had a big importance, because it allowed to differentiate (2.4) by variable p at point $p = 1$ and it lead to obtain for the first time in 1995 the asymptotics of quantum-information entropy for the classical harmonic oscillator!

The derivation of (2.4) in [3] was based on asymptotics for $H_n(x)$ on $\mathbb{R}$, in particularly on Plancherel–Rotach formulae (and their generalization for Freud weights from [14]). The first strong asymptotics formulae for Hermite polynomials are due to Plancherel and Rotach [12] (see also [13]). They describe polynomials H_n when $n \to \infty$ in the following subdomain of $\mathbb{R}$:

$$\begin{array}{lll} \text{a)} & x = \sqrt{2n+1}\cosh y, & \epsilon \le y < \infty; \\ \text{b)} & x = \sqrt{2n+1} + n^{-\frac{1}{6}} t, & t \in K \subset \mathbb{C}; \\ \text{c)} & x = \sqrt{2n+1}\cos y, & \epsilon \le y \le \pi - \epsilon, \end{array} \tag{2.7}$$

where $0 < \epsilon < \pi$ is a small positive real number, and K is an arbitrary compact subset of the complex plane.

The restriction on p in (2.4) appeared because of the Plancherel–Rotach formulae in (2.7) do not match each other; i.e., subdomains in (2.7) do not intersect. Particularly, there is a gap between zone a) and zone b) in (2.7), which plays an important role for the limit (1.1) when $n \to \infty$. In the asymptotics of (2.4) the main contribution in the left hand side integral gives the part of the integral described in a) of (2.7). The gap between zone a) and zone b) gives the main contribution in the integral for bigger p.

The asymptotical description of Hermite polynomials in the subdomain covering all $\mathbb{R}$ was obtained not so long ago. In 1999 Deift et al [16] (see also [17]) have obtained the global asymptotical portrait of polynomials orthogonal with respect to exponential weights by means of the powerful matrix Riemann-Hilbert method. As a corollary for the scaled Hermite polynomials $H_n(\sqrt{2n}z)$, they obtained asymptotics as $n \to \infty$ and z belongs to

$$\begin{array}{ll} \text{a)} & |z| \ge 1 + \delta; \\ \text{b)} & 1 - \delta \le |z| \le 1 + \delta; \\ \text{c)} & |z| \le 1 - \delta. \end{array} \tag{2.8}$$

for small $\delta > 0$. Evidently, there are no gaps between zones and Deift et al's asymptotics can be used for obtaining asymptotics of (1.1) for bigger p. However, during more than a decade since this global asymptotics was available, there were no successful attempts to increase the bound for p from $p = \frac{4}{3}$.

Recently a new approach for obtaining the global asymptotical portrait of orthogonal polynomials has appeared [2]. Contrary to the matrix Riemann-Hilbert method which starts from the weight of orthogonality, the starting point in [2] is the recurrence relation which characterizes the orthogonal polynomials. In the next section we present a general description of this method and its application to derivation of the global Plancherel–Rotach type asymptotics for Hermite polynomials. Using this asymptotics we can obtain the main result of this work:

Theorem 2.1. *Let $H_n(x)$ be the Hermite polynomials with the standard normalization (2.1). Then, the frequency or entropic moments $W_p[H_{n-1}]$ have for $n \to \infty$ the following asymptotical values*

$$W_p[H_{n-1}] = \begin{cases} c_p \pi^p \, (2n)^{p(n-1)+1/2} \, e^{-pn} \, (1+o(1)) \,, & p < 2, \\ 2(2n)^{2n-\frac{3}{2}} e^{-2n} \, (\ln(n) + O(1)) \,, & p = 2, \\ 2C_p \, 2^{-p} \, (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} \, (1+o(1)) \,, & p > 2. \end{cases} \tag{2.9}$$

where the constant c_p is defined in (2.5) and the constant C_p is equal to

$$C_p = \int_{-\infty}^{+\infty} \left[\frac{2\pi}{\sqrt[3]{2}} \, \mathrm{Ai}^2 \left(-\frac{z\sqrt[3]{2}}{2} \right) \right]^p dz \,.$$

We note that the first asymptotical formula in the right hand side of (2.9) coincides with (2.6), but now it holds true in the maximal range of p (when $p = 2$, then $c_p = \infty$). We also note that the leading term of all three formulae in the right hand side of (2.9) match each other when $p \to 2$.

3 A global asymptotics of solutions of recurrence relations and Hermite polynomials

3.1 General description of the method

Consider the linear recurrence relations

$$Q_{n+1}(x) = \sum_{j=0}^{p-1} a_{n,j}(x) Q_{n-j}(x), \qquad n \in \mathbb{N}, \tag{3.1}$$

with coefficients $a_{n,j}(x)$ that are polynomials in the spectral parameter x. We investigate the asymptotic behaviour of the solutions $Q_n(x)$ as the independent variable (the index n) and the spectral parameter x of the relation approach infinity, under various relations between their growth. Asymptotic formulae of this kind are called *Plancherel-Rotach type asymptotics* (see [17]).

In this work we use a method suggested in [2] for constructing asymptotic expressions for bases of solutions of difference equations (3.1) in overlapping domains of (n, x)-space extending to infinity. Matching solutions in the intersections of these domains lets us obtain a global asymptotic picture of the behaviour of solutions of (3.1) in the complex domain of the spectral parameter x, for a suitable scaling depending on n. Thus our approach is focused on finding global asymptotic representations for solutions of (3.1) starting from initial data in the form of the coefficients of the recurrence relations: this is similar to using the method of steepest descent for the matrix Riemann-Hilbert problem (proposed in [15] and developed further in [16] and [17]) to solve the same problem for orthogonal polynomials starting from the orthogonality weights.

We use a convenient re-formulation of the recurrence relation (3.1) in the form of the vector relation

$$F_{n+1} = A_n(x)F_n, \qquad n \in \mathbb{N}, \tag{3.2}$$

where F_n is a column vector and $A_n(x)$ is a $p \times p$-matrix whose entries are polynomials in x with coefficients which are rational in n. Because they are rational in n, we can assume that the $A_n(x)$ are well-defined for all $n \in \mathbb{C}$, so that expressions of the form A_{n+t} with $t \in \mathbb{C}$ are legitimate. The concrete example in this work concern the case when the entries of A_n are linear in x, but the determinant of the matrix is independent of x.

The central problem is to find (in the form of asymptotic series) bases for the solutions of equation (3.2) in some domains of the (n, x)-plane and to match these expansions in the transition zones. Then, to find asymptotic formulae for concrete polynomial solutions (in x) of (3.2) (for instance, for orthogonal polynomials) we consider bases of solutions for values of n small in comparison with large x and by calculating the leading coefficients of these concrete polynomial solutions from the initial data, to find their expressions in terms of the basis solutions.

To obtain bases for the solutions of equation (3.2) we design a diagonalizing transformation V_n. Passing to another basis, which depends on n, we obtain a problem similar to the original one:

$$F_n = V_n U_n \quad \Longrightarrow \quad U_{n+1} = (V_{n+1}^{-1} A_n V_n) U_n. \tag{3.3}$$

Our aim is to select appropriate matrices V_n (depending on x) so that the new transition matrix $D_n := V_{n+1}^{-1} A_n V_n$ is close to a diagonal one. The procedures for obtaining V_n depend significantly on the dynamics of how the eigenvalues λ_i of the matrix A_n change when regarded as functions of n and x. We split the plane (n, x) in two regions.

By *nice diagonalization regions* we shall mean connected components of the open subset of the (n,x)-plane in which the condition

$$\Xi := \left\{ (n,x) : \max_{i\neq j,\, 1\le i,j\le k} \frac{|\lambda_i| + |\lambda_j|}{|\lambda_i - \lambda_j|} < C \right\} \tag{3.4}$$

is fulfilled for some fixed $C > 1$. We call *transition zones* Ξ^c to the sets in which (3.4) fails.

In a nice diagonalization region Ξ the determination of the diagonalizer V_n has a universal character; in that case fairly general results are stated and proven in [2]. On the other hand, the construction of a diagonalizing transformation in the transition zones Ξ^c is specific. In these zones the difference problem of finding a diagonalizing map reduces to a differential problem. Further analysis depends on the type of this differential problem. For the case of Hermite polynomials the differential problem reduces to Airy's equation (see, for example, [13], § 1.81 for the definition).

In a nice diagonalization region Ξ shall seek an asymptotic expansion for V_n by introducing an additional parameter t and finding a formal power series in t which solves the following problem:

$$V_n = V_n^{(0)} + t\varphi_1(x,n) + t^2\varphi_2(x,n) + \cdots, \qquad V_{n+t}^{-1} A_n V_n \approx \operatorname{diag}(\,\cdot\,). \tag{3.5}$$

As an initial approximation, and at the same time the first term of the expansion $V_n^{(0)}$, we take the matrix formed by the column of the eigenvectors $(\vec{b}_i)_1^p$ of the matrix $A_n(x)$ we have obtained:

$$V_n^{(0)} = (\vec{b}_1, \ldots, \vec{b}_p), \qquad D := (V_n^{(0)})^{-1} A_n V_n^{(0)} = \operatorname{diag}(\lambda_1(x,n), \ldots, \lambda_p(x,n)). \tag{3.6}$$

Then, for $V_n = V_n^{(0)}$ off-diagonal entries of the matrix $V_{n+t}^{-1} A_n V_n$ have order $O(t)$.

Consider the sequence of formal series $V_n^{(l)}$ generated by the following process (when we shift the variable n, the series must be re-expanded):

$$S^{(l)} := (V_{n+t}^{(l)})^{-1} A_n V_n^{(l)}, \qquad X_n^{(l)} := (X_{ij}^{(l)}), \quad \text{where} \begin{cases} X_{ii}^{(l)} = 0; \\ X_{ij}^{(l)} = \dfrac{S_{ij}^{(l)}}{\lambda_i - \lambda_j}, \quad i \neq j, \end{cases}$$
$$V_n^{(l+1)} := V_n^{(l)} (E + X_n^{(l)})^{-1}. \tag{3.7}$$

Then, the series $V_n^{(l)}$ and $V_n^{(l+1)}$ coincide up to $o(t^l)$, and the series $S^{(l)}$ and $S^{(l+1)}$ coincide up to $o(t^{l+1})$. This iterative procedure allows for the basis solutions of the recurrence relations (3.1) and (3.2) to construct expansions in formal series, which can be bounded by asymptotic series provided with an estimation of the reminder term:

$$\Pi_j(x,n)=\pi_j(x,n)\Big(1+\sum_{m=1}^{s}\psi_m^{(j)}(x,n)+o\big(\tilde\varphi(x,n)\varphi^s(x,n)\big)\Big),\;\; j=1,\,\dots,p. \quad (3.8)$$

where the estimation of the reminder has a form $\psi_m^{(j)}=O\big(\tilde\varphi\varphi^m\big)$; $\tilde\varphi$ depending on the choice of the domain in the plane (n,x).

In the transition zone Ξ^c for the case when the difference equation approximates differential equation Airy we have two basis solution of the form

$$\exp\Big(E(z,x)\Big)\,\mathrm{Ai}\big(h(z,x)\big) \qquad \text{and} \qquad \exp\Big(E(z,x)\Big)\,\mathrm{Bi}\big(h(z,x)\big)\,, \quad (3.9)$$

where Ai and Bi are linearly independent solution of the Airy equation (Airy functions). In [2] a procedure for determination of the explicit form of the terms of the series

$$E(z,x)\;=\;E_0(z,x)\Big(1+\sum_{m=1}^{\infty}\gamma_m(z,x)\Big)\,,\qquad h(z,x)\;=\;\sum_{m=0}^{\infty}\eta_m(z,x)\,, \quad (3.10)$$

is described. These terms are bounded (in the same sense as in (3.8)) by asymptotic series. However, in this case φ and $\tilde\varphi$ from the reminder term in (3.8) have a simpler form : $|z|^{r_1}|x|^{r_2}$ (r_1,r_2 for φ and for $\tilde\varphi$ depend on the problem under consideration only).

Finally, to solve the main problem we must match the bases constructed in different subdomains of the (n,x)-space. We can do this because the formal series representing the basis of the solutions in Ξ are also meaningful in a wider domain, when the constant C in (3.4) is replaced by $|x|^\alpha$ for some $\alpha>0$. Thus, in subdomains of transition zones Ξ^c adjacent to Ξ both representations for bases of solutions are valid, and therefore, since (3.2) is homogeneous in F, there exists a function independent of n but depending on x:

$$K(x)=\sum_{m=0}^{\infty}\kappa_m(x)\,, \quad (3.11)$$

so that, multiplying one of the solutions by this function, we do not only level off the growth of both solutions, but also ensure that the other terms of their asymptotic series coincide. Finding such a function gives one an additional test for the correctness of the expansions of solutions to (3.2) in various domains of the (n,x)-plane.

3.2 The global asymptotic behavior of Hermite polynomials

For the Hermite polynomials (2.2) the recurrence relations have the following matrix form (3.2) (here we set $Q_n(x) := H_n(x)$):

$$\begin{pmatrix} Q_{n+1} \\ Q_n \end{pmatrix} = \begin{pmatrix} 2x & -2n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Q_n \\ Q_{n-1} \end{pmatrix}. \tag{3.12}$$

Since we can find the two leading coefficients from the initial data, this specifies our choice of the solution:

$$Q_n(x) = (2x)^n \left(1 - \frac{n(n-1)}{4x^2} + O\left(\frac{n^4}{x^4}\right)\right). \tag{3.13}$$

As the first step we find the roots

$$\lambda_{1,2} = x \pm \sqrt{x^2 - 2n}$$

of the characteristic polynomial of the transition matrix

$$A_n = \begin{pmatrix} 2x & -2n \\ 1 & 0 \end{pmatrix}$$

and also the regions of nice separation of the eigenvalues

$$\Xi = \left\{n < \frac{(1-\varepsilon)|x|^2}{2}\right\} \cup \left\{n > \frac{(1+\varepsilon)|x|^2}{2}\right\}.$$

In these regions we find an approximation for V_n. The initial approximation is

$$V_n^{(0)} = \begin{pmatrix} x + \sqrt{x^2 - 2n} & x - \sqrt{x^2 - 2n} \\ 1 & 1 \end{pmatrix}.$$

Refining V_n by the iterations (3.7), we obtain an expansion for it given by the following power series:

$$\begin{pmatrix} x + \sqrt{x^2-2n} - \dfrac{nt}{2(\sqrt{x^2-2n})^3} + \dfrac{n(5x - \sqrt{x^2-2n})t^2}{8(\sqrt{x^2-2n})^6} + \cdots & x - \sqrt{x^2-2n} + \dfrac{nt}{2(\sqrt{x^2-2n})^3} + \dfrac{n(5x + \sqrt{x^2-2n})t^2}{8(\sqrt{x^2-2n})^6} + \cdots \\ 1 - \dfrac{x + \sqrt{x^2-2n}}{4(\sqrt{x^2-2n})^3} t + \dfrac{3x^2 - n + 3x\sqrt{x^2-2n}}{8(\sqrt{x^2-2n})^6} t^2 + \cdots & 1 + \dfrac{x - \sqrt{x^2-2n}}{4(\sqrt{x^2-2n})^3} t + \dfrac{3x^2 - n - 3x\sqrt{x^2-2n}}{8(\sqrt{x^2-2n})^6} t^2 + \cdots \end{pmatrix}.$$

Simultaneously we obtain the diagonal entries of the matrix S in (3.7), which becomes D_n (3.3) for $t = 1$:

$$d_1(n) = x + \sqrt{x^2 - 2n} + \frac{x + \sqrt{x^2 - 2n}}{2(\sqrt{x^2 - 2n})^2} t + \frac{3x^2 - 5n + 3x\sqrt{x^2 - 2n}}{4(\sqrt{x^2 - 2n})^5} t^2 + \cdots ,$$
$$d_2(n) = x - \sqrt{x^2 - 2n} + \frac{x - \sqrt{x^2 - 2n}}{2(\sqrt{x^2 - 2n})^2} t - \frac{3x^2 - 5n - 3x\sqrt{x^2 - 2n}}{4(\sqrt{x^2 - 2n})^5} t^2 + \cdots .$$

We renormalize the columns of V_n so that the entries in the second row are equal to 1. This changes the $d_1(n)$ and $d_2(n)$, and V_n will be conveniently expressed in terms of these quantities:

$$V_n = \begin{pmatrix} d_1(n) & d_2(n) \\ 1 & 1 \end{pmatrix}, \tag{3.14}$$

where now

$$d_1(n) = x + \sqrt{x^2 - 2n} + \frac{x + \sqrt{x^2 - 2n}}{2(\sqrt{x^2 - 2n})^2} t - \frac{2x^2 + n + 2x\sqrt{x^2 - 2n}}{4(\sqrt{x^2 - 2n})^5} t^2 + \cdots ,$$
$$d_2(n) = x - \sqrt{x^2 - 2n} + \frac{x - \sqrt{x^2 - 2n}}{2(\sqrt{x^2 - 2n})^2} t + \frac{2x^2 + n - 2x\sqrt{x^2 - 2n}}{4(\sqrt{x^2 - 2n})^5} t^2 + \cdots .$$

We obtain asymptotic expressions for the solutions by finding the asymptotic expansion in powers of t for the product

$$V_n \begin{pmatrix} \Pi_1(n) & 0 \\ 0 & \Pi_2(n) \end{pmatrix}, \qquad \Pi_m := \prod_k^{n/t-1} d_m(tk), \quad m = 1, 2,$$

which is a matrix of two column solutions. (Here number products are analogues of antiderivatives). We also define $\vec{\Pi}_m$, $m = 1, 2$, by the formulae

$$\vec{\Pi}_m = V_n \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix} \vec{e}_m = \Pi_m V_n \vec{e}_m. \tag{3.15}$$

As a result, we obtain a formal expansion:

$$\Pi_1 = \frac{(x + \sqrt{x^2 - 2n})^{n/t - 1/2}}{\sqrt[4]{x^2 - 2n}} \exp\Bigg(-\frac{n + x\sqrt{x^2 - 2n}}{2t} - \frac{x(x^2 + 3n) + (8n - x^2)\sqrt{x^2 - 2n}}{24n(\sqrt{x^2 - 2n})^3} t + \frac{5x(x - \sqrt{x^2 - 2n})}{16(\sqrt{x^2 - 2n})^6} t^2 + \cdots \Bigg),$$
$$\Pi_2 = \frac{(x - \sqrt{x^2 - 2n})^{n/t - 1/2}}{\sqrt[4]{x^2 - 2n}} \exp\Bigg(-\frac{n - x\sqrt{x^2 - 2n}}{2t} + \frac{x(x^2 + 3n) + (x^2 - 8n)\sqrt{x^2 - 2n}}{24n(\sqrt{x^2 - 2n})^3} t + \frac{5x(x + \sqrt{x^2 - 2n})}{16(\sqrt{x^2 - 2n})^6} t^2 + \cdots \Bigg),$$

and setting $t = 1$ in it we arrive at (3.21).

In the transition zone we can construct a standard continuous system and then make a comparison with it (see details in [2]). We take another basis, and in place of n we introduce a continuous variable z by the following formulae :

$$\widetilde{V}_n = \begin{pmatrix} x & \sqrt[3]{x} \\ 1 & 0 \end{pmatrix}, \qquad 2n = x^2 + zx^{2/3}. \tag{3.16}$$

Then substituting in z we obtain

$$U_{n+1} = x\left[I + x^{-2/3}\begin{pmatrix} 0 & 1 \\ -z & 0 \end{pmatrix}\right]U_n,$$

which gives us the difference equation:

$$U_n = x^n u(z) \quad \Longrightarrow \quad u(z + 2x^{-2/3}) = \left[I + x^{-2/3}\begin{pmatrix} 0 & 1 \\ -z & 0 \end{pmatrix}\right]u(z).$$

Transforming the right-hand side to adapt it to a shift of n, we obtain a continuous system:

$$u'(z) = \begin{pmatrix} \dfrac{z}{4x^{2/3}} - \dfrac{1}{3x^{4/3}} - \dfrac{z^2}{8x^2} + \cdots & \dfrac{1}{2} - \dfrac{z}{6x^{4/3}} + \dfrac{1}{6x^2} + \cdots \\ -\dfrac{z}{2} + \dfrac{1}{2x^{2/3}} + \dfrac{z^2}{6x^{4/3}} - \dfrac{z}{3x^2} + \cdots & \dfrac{z}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{z^2}{8x^2} + \cdots \end{pmatrix} u(z).$$

The solution of this system (the first component of the vector) has the following formal representation:

$$u_1^{(A)}(z) = \exp\left(\frac{z^2}{8x^{2/3}} - \frac{5z}{12x^{4/3}} - \frac{z^3}{24x^2} + \cdots\right)\left(1 + \frac{2z}{15x^{4/3}} - \frac{2}{15x^2} + \cdots\right)\mathrm{Ai}(h(z)),$$

where

$$h(z) := -\sqrt[3]{2}\left(\frac{z}{2} - \frac{1}{2x^{2/3}} - \frac{z^2}{15x^{4/3}} + \frac{2z}{15x^2} + \cdots\right);$$

we obtain a second linearly independent solution $u_1^{(B)}$ by replacing Ai by Bi here.

We know we can use such expansions when $|z| = O(|x|^{2/3-\varepsilon})$ for some $\varepsilon > 0$. But in fact the expansions are valid in a wider domain

$$|z| = O(|x|^{4/3-\varepsilon}).$$

To see this we can take the conjugation by a diagonal matrix with entries $|x|^{2/3-\varepsilon}$ and 1, which is independent of n (or, equivalently, of z).

We see that the continuous system has linearly independent solutions of the form

$$u^{(A)} = e^{E(z)} \begin{pmatrix} \mathrm{Ai}(h(z)) \\ f_1\,\mathrm{Ai}(h(z)) + f_2\,\mathrm{Ai}'(h(z)) \end{pmatrix},$$
$$u^{(B)} = e^{E(z)} \begin{pmatrix} \mathrm{Bi}(h(z)) \\ f_1\,\mathrm{Bi}(h(z)) + f_2\,\mathrm{Bi}'(h(z)) \end{pmatrix},$$

where f_1 and f_2 have the expansions (we use notation: $|z|_+ := \max(1, |z|)$)

$$f_1 = \frac{1}{10x^{4/3}} + \cdots + O\left(\frac{|z|_+^{(k-2)/2}}{|x|^{2k/3}}\right),$$
$$f_2 = -\sqrt[3]{2}\left(1 + \frac{z}{15x^{4/3}} - \frac{1}{15x^2} + \cdots\right) + O\left(\frac{|z|_+^{k/2}}{|x|^{2k/3}}\right);$$

and the expansion for $E(z)$ has the form

$$E(z) = \frac{z^2}{8x^{2/3}} - \frac{17z}{60x^{4/3}} - \frac{5z^3+16}{120x^2} + \cdots + O\left(\frac{|z|_+^{k+3/2}}{|x|^{2k/3}}\right).$$

We must bear in mind that the solutions constructed display exponential growth. Factoring out the exponential we obtain columns proportional to the solutions, which display a powerlike behaviour:

$$\exp\left(-E(z) + \frac{2}{3}h^{3/2}(z)\right)u^{(A)} = \exp\left(\frac{2}{3}h^{3/2}(z)\right) \begin{pmatrix} \mathrm{Ai}(h(z)) \\ f_1\,\mathrm{Ai}(h(z)) + f_2\,\mathrm{Ai}'(h(z)) \end{pmatrix},$$
$$\exp\left(-E(z) - \frac{2}{3}h^{3/2}(z)\right)u^{(B)} = \exp\left(-\frac{2}{3}h^{3/2}(z)\right) \begin{pmatrix} \mathrm{Bi}(h(z)) \\ f_1\,\mathrm{Bi}(h(z)) + f_2\,\mathrm{Bi}'(h(z)) \end{pmatrix}.$$

The second column displays powerlike behaviour only in the sector $\arg(h) \in (-\frac{\pi}{3}, \frac{\pi}{3})$.

To obtain a wider domain with polynomial behaviour, in place of Bi we can take two special linear combinations of the Airy functions.

Finally we obtain formulae for V_n, ensuring diagonalization to within the required accuracy, in a transition zone in the form of the sector $\arg(z) \in (-\pi, \frac{\pi}{3})$, where $2n = x^2 + zx^{2/3}$:

$$V_n = \widetilde{V}_n \begin{pmatrix} 1 & 0 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} \mathrm{Ai}(h(z)) & \mathrm{Ai}(h(z)) + i\,\mathrm{Bi}(h(z)) \\ \mathrm{Ai}'(h(z)) & \mathrm{Ai}'(h(z)) + i\,\mathrm{Bi}'(h(z)) \end{pmatrix}$$
$$\times \begin{pmatrix} \exp\left(\frac{2}{3}h^{3/2}(z)\right) & 0 \\ 0 & \exp\left(-\frac{2}{3}h^{3/2}(z)\right) \end{pmatrix}.$$

The expansions we found earlier are valid in the domains $|z| > |x|^{\varepsilon}$. We now see that there are scales in which both results are valid, so that comparing them we can find the correspondence between the bases. The coefficients of this correspondence do not depend on n, but they can depend on x, and indeed they do.

Assume that $\mathrm{Im}(x) \geq 0$. Then for z in (3.16) and arbitrary $n \in \mathbb{R}_+$ $\arg(z)$, ranges between $\arg(x^{-2/3}) = -\frac{2}{3}\arg(x)$ and $\arg(-x^{4/3}) = -\pi + \frac{4}{3}\arg(x)$. These values differ by $\pi - 2\arg(x)$, which is less than π in absolute value for $\arg(x) \in (0, \pi)$. This means that the notion of 'between' from the viewpoint of the values of the arguments and the viewpoint of the directions in the complex plane agree. Hence all the possible values of $\arg(z)$ lie in the interval $(-\pi, \frac{\pi}{3})$. The smallest value of $|z|$ that is actually taken is $|x|^{4/3}|\sin(2\arg(x))|$ for $\cos(2\arg(x)) > 0$ and $|x|^{4/3}$ otherwise. The formulae for the transition zone can be used for $|z| = O(|x|^{4/3-\varepsilon})$; that is, $\arg(x) < |x|^{-\varepsilon}$ or $\arg(x) > \pi - |x|^{-\varepsilon}$. In these cases $\min(|z|) \sim \mathrm{Im}(x)\sqrt[3]{|x|}$. The formulae for nice diagonalization (3.14) and (3.21) can be used for $|z| \geq |x|^{\varepsilon}$.

Thus, for $\mathrm{Im}(x) = O(|x|^{1-\varepsilon})$ there exists an interval of values of n in which we can use the formulae for the transition zone; on the other hand, for $\mathrm{Im}(x) = O(|x|^{\varepsilon-1/3})$ we cannot do without these formulae. Now we find the correspondence between the solutions

$$x^n \exp\biggl(E(z) - \frac{2}{3}h^{3/2}(z)\biggr)V_n\vec{e}_1, \qquad x^n \exp\biggl(E(z) + \frac{2}{3}h^{3/2}(z)\biggr)V_n\vec{e}_2 \tag{3.17}$$

in the transition zone and the basis solutions (3.15), (3.21). We limit ourselves to a neighbourhood of the positive half-axis; that is, to $\arg(x)$ close to zero. The second case, when $\arg(x)$ is close to π, is similar.

In our case, for small n (before we move into the transition zone and upon entering it) $\arg(z) \approx -\pi$ while for large n (as we leave the transition zone and beyond it) $\arg(z) \approx 0$. Within the transition zone

$$h(z) = -\frac{\sqrt[3]{2}}{2z} + o(|z| + 1);$$

that is, $\arg(h(z)) \in [0, \pi]$. This interval lies in the interval $(-\frac{\pi}{3}, \frac{5\pi}{3})$, where we have an asymptotic representation for the second solution. As a result, with coefficient

$$\frac{-i\sqrt{\pi}}{\sqrt[6]{2}}x^{2/3}\exp\biggl(-\frac{x^2}{4} + \frac{1}{4x^2} + \cdots\biggr),$$

the second solution in (3.17) corresponds to the second solution in (3.21) both at the entrance to and at the exit from the transition zone.

The situation for the first solution in (3.17) is different. At the entrance $\arg(h) \approx 0$ and the first solution of (3.17) corresponds to the first solution in (3.21) with coefficient

$$\frac{2\sqrt{\pi}}{\sqrt[6]{2}} x^{2/3} \exp\biggl(-\frac{x^2}{4} + \frac{1}{4x^2} + \cdots\biggr).$$

At the exit $\arg(h) \approx \pi$, and we must represent the solution column as the following linear combination:

$$\begin{pmatrix} \mathrm{Ai}(h(z)) \\ \mathrm{Ai}'(h(z)) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{Ai}(h(z)) + i\,\mathrm{Bi}(h(z)) \\ \mathrm{Ai}'(h(z)) + i\,\mathrm{Bi}'(h(z)) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathrm{Ai}(h(z)) - i\,\mathrm{Bi}(h(z)) \\ \mathrm{Ai}'(h(z)) - i\,\mathrm{Bi}'(h(z)) \end{pmatrix},$$

and write out the asymptotic formulae setting $\arg(h) \approx \pi$ in the first term and $\arg(h) \approx -\pi$ in the second. Thus, in the sense of the correspondence of formal series one has

$$\frac{2\sqrt{\pi}}{\sqrt[6]{2}} x^{2/3} \exp\biggl(-\frac{x^2}{4} + \frac{1}{4x^2} + \cdots\biggr) \begin{pmatrix} \mathrm{Ai}(h(z)) \\ \mathrm{Ai}'(h(z)) \end{pmatrix} = \vec{\Pi}_1 + i\vec{\Pi}_2.$$

For the Hermite polynomials we determine the behaviour of some leading coefficients and use them as initial data for a decomposition with respect to the basis obtained, assuming that x is sufficiently large ($\hat{n} \ll |x|^{\varepsilon}$):

$$Q_{\hat{n}}(x) = (2x)^{\hat{n}} \biggl(1 - \frac{\hat{n}(\hat{n}-1)}{4x^2} + O\Bigl(\frac{\hat{n}^4}{x^4}\Bigr)\biggr),$$

which yields

$$\begin{pmatrix} Q_n(x) \\ Q_{n-1}(x) \end{pmatrix} = \frac{\exp(x^2/2 + o(x^{-2}))}{\sqrt{2}} \vec{\Pi}_1 + o\biggl(x^{2\hat{n}} \exp\Bigl(-\frac{x^2}{2}\Bigr)\biggr) \vec{\Pi}_2.$$

Eventually, this gives us the asymptotic formulae (3.18)–(3.20) from Theorem 3.1.

Applying the procedure described above the following result is obtained in [2].

Theorem 3.1. *The Hermite polynomials (defined by* (3.12)–(3.13)*) have the following asymptotic expansions for fixed* $\varepsilon > 0$ *and* $\mathrm{Im}(x) \geq 0$*:*

a) *for* $2n < |x|^2 - |x|^{\varepsilon + 2/3} + \mathrm{Im}(x)$

$$Q_{n-1}(x) = \frac{\exp(x^2/2 + o(x^{-2}))}{\sqrt{2}} \Pi_1; \tag{3.18}$$

b) *for* $2n = x^2 + zx^{2/3}$, $|z| = O(|x|^{4/3-\varepsilon})$

$$Q_{n-1}(x) = \frac{\sqrt{2\pi}}{\sqrt[6]{2}} x^{n+2/3} \exp(\widetilde{E}(z)) \operatorname{Ai}(h(z)); \tag{3.19}$$

c) *for* $2n > |x|^2 + |x|^{\varepsilon+2/3} - \operatorname{Im}(x)$

$$Q_{n-1}(x) = \frac{\exp(x^2/2 + o(x^{-2}))}{\sqrt{2}} (\Pi_1 + i\Pi_2). \tag{3.20}$$

Here

$$\Pi_1 := \frac{(x + \sqrt{x^2 - 2n})^{n-1/2}}{\sqrt[4]{x^2 - 2n}} \exp\Big(-\frac{n + x\sqrt{x^2 - 2n}}{2}\Big)\Big(1 + \sum_{j=1}^{\infty} \Phi_j(x, n)\Big), \tag{3.21}$$

where the first term of the series in (3.21) *is equal to*

$$\Phi_1(x, n) := \frac{-x(x^2 + 3n) + (x^2 - 8n)\sqrt{x^2 - 2n}}{24n(\sqrt[3]{x^2 - 2n})},$$

and the general term of the series has the following estimate for $|x^2-2n| < \frac{1}{2}|x|^2$*:*

$$\Phi_k(x, n) = \mathcal{O}\Big(\Big(\frac{|x|}{|x^2 - 2n|^{3/2}}\Big)^k\Big).$$

The expression for Π_2 *is obtained from* (3.21) *by changing the sign of* $\sqrt{x^2 - 2n}$*; and, in the expressions for* Π_1 *and* Π_2*,* $\arg(\sqrt[4]{x^2 - 2n}) \in [0, \pi/4]$ *for* $\operatorname{Re}(x) \geq 0$.

Furthermore,

$$\widetilde{E}(z) = \frac{x^2}{4} + \frac{z^2}{8x^{2/3}} - \frac{17z}{60x^{4/3}} - \frac{5z^3 + 46}{120x^2} + \cdots + O\Big(\frac{|z|_+^{(k+3)/2}}{|x|^{2k/3}}\Big), \tag{3.22}$$

$$h(z) := -\sqrt[3]{2}\Big(\frac{z}{2} - \frac{1}{2x^{2/3}} - \frac{z^2}{15x^{4/3}} + \frac{2z}{15x^2} + \cdots\Big) + O\Big(\frac{|z|_+^{(k+3)/2}}{|x|^{2k/3}}\Big). \tag{3.23}$$

The formal series in (3.21)–(3.23) *are dominated by asymptotic series as* $(n, x) \to \infty$*, so that finite sums can be used with remainders in the form of 'big-O' of the first term dropped.*

Now we compare this result with the known asymptotic formulae for Hermite polynomials. Note that the domains (2.7), in which the classical Plancherel-Rotach formulae can be used (see [12] and [13]), are nonoverlapping; so, these formulae do not produce a global asymptotic description of the Hermite polynomials. On the other hand, the regions in which we can use the asymptotic formulae provided by Theorem 3.1 can be 'glued together' because the domains where formulae a), b) and c) can be applied are wider (by comparison with (2.7)). Another example we know of, giving a global asymptotic description of the Hermite polynomials are the asymptotic formulae in [16] (see [17]), which were derived using the method of the matrix Riemann-Hilbert problem. Note that in the asymptotic analysis of [16], domains a) and c) are the same as in the classical Plancherel-Rotach description (2.7): the asymptotic formulae in [16] are "'stitched together" thanks to extending domain b), by comparison with (2.7); in it an asymptotic expression in terms of Airy functions is used. We see that both Theorem 3.1 and the corresponding result in [16] provide global asymptotic descriptions of the Hermite polynomials, but they accomplish this at different scales.

4 Asymptotics ($n \to \infty$) of L_p-norms of Hermite polynomials. Prove Theorem 2.1.

Thus, our approach applied to the Hermite polynomials brings an asymptotical description in the following subdomains of $\mathbb{R}$:

$$\begin{array}{ll} \text{a)} & x^2 \in [2n + n^{\frac{1}{3}+\theta}; \infty); \\ \text{b)} & x^2 \in [2n - n^{\frac{1}{3}+\theta}; 2n + n^{\frac{1}{3}+\theta}]; \\ \text{c)} & x^2 \in [0; 2n - n^{\frac{1}{3}+\theta}]. \end{array} \tag{4.1}$$

for $\theta \in (0, \frac{2}{3})$. Theorem 3.1 gives us

$$\begin{aligned} &\text{in a) } H_{n-1}(x) = \frac{1}{\sqrt{2}} \frac{\left(x + \sqrt{x^2 - 2n}\right)^{n-\frac{1}{2}}}{\sqrt[4]{x^2 - 2n}} \exp\left(\frac{x^2 - n - x\sqrt{x^2 - 2n}}{2}\right)(1 + o(1)), \\ &\text{in c) } H_{n-1}(x) = \sqrt{2} \frac{\left(\sqrt{2n}\right)^{n-\frac{1}{2}}}{\sqrt[4]{2n - x^2}} \exp\left(\frac{x^2 - n}{2}\right) \\ &\times \cos\left(\left(n - \frac{1}{2}\right) \arcsin\left(\sqrt{1 - \frac{x^2}{2n}}\right) - \frac{x\sqrt{2n - x^2}}{2} - \frac{\pi}{4} + o(1)\right)(1 + o(1)), \\ &\text{in b) } H_{n-1}(x) = \frac{\sqrt{2\pi}}{\sqrt[6]{2}} x^{n-\frac{2}{3}} \exp\left(\frac{x^2}{4} + o(1)\right) \mathrm{Ai}\left(-\frac{\sqrt[3]{2}}{2} z + o\left(n^{-\frac{1}{3}}\right)\right)(1 + o(1)), \end{aligned} \tag{4.2}$$

where $z := \frac{2n}{x^{\frac{2}{3}}} - x^{\frac{4}{3}}$.

Now we prove Theorem 2.1.

Proof. Doing identical transformations and some elementary asymptotical estimates, we have from (4.2) that:

$$\text{in a) } H_{n-1}^2(x)e^{-x^2} = \frac{(2n)^{n-1}}{2} e^{-n} \exp\left[(2n-1)\operatorname{arccosh}\frac{x}{\sqrt{2n}} - x\sqrt{x^2-2n}\right]$$
$$\times \left(\tfrac{x^2}{2n} - 1\right)^{-\frac{1}{2}} (1+o(1))\,;$$
$$\text{in c) } H_{n-1}^2(x)e^{-x^2} = (2n)^{n-1}e^{-n}$$
$$\times \left\{1 - \sin\left[(2n-1)\arcsin\sqrt{1-\tfrac{x^2}{2n}} - x\sqrt{2n-x^2} + o(1)\right]\right\}$$
$$\times \left(1 - \tfrac{x^2}{2n}\right)^{-\frac{1}{2}} (1+o(1))\,;$$
$$\text{in b) } H_{n-1}^2(x)e^{-x^2} = (2n)^{n-\frac{2}{3}}e^{-n}\frac{2\pi}{\sqrt[3]{2}}\left(\frac{x}{\sqrt{2n}}\right)^{2\left(n-\frac{2}{3}\right)} \exp\left(\frac{2n-x^2}{2}\right)$$
$$\times \mathrm{Ai}^2\left(-\tfrac{\sqrt[3]{2}}{2}z + o\left(n^{-\frac{1}{3}}\right)\right)(1+o(1))\,. \tag{4.3}$$

Now we start to estimate the integral in (1.1). We consider the interval of integration $[0,\infty)$ (since the integrand is an even function) and split it in the subintervals a), b), c) as in (4.1). And we split the interval b) in (4.1) into three subintervals:

$$\begin{aligned} &\mathrm{b}_1) \quad x^2 \in \left[2n - n^{\frac{1}{3}+\theta}; 2n - Mn^{\frac{1}{3}}\right]; \\ &\mathrm{b}_2) \quad x^2 \in \left[2n - Mn^{\frac{1}{3}}; 2n + Mn^{\frac{1}{3}}\right]; \\ &\mathrm{b}_3) \quad x^2 \in \left[2n + Mn^{\frac{1}{3}}; 2n + n^{\frac{1}{3}+\theta}\right]. \end{aligned} \tag{4.4}$$

Thus, we have splitted $x \in [0,\infty)$ into five zones (see Figure 4.1).

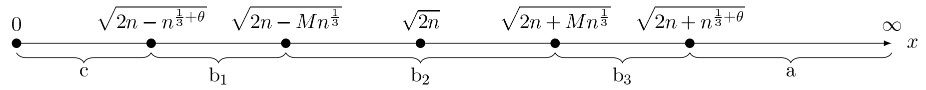

Рис. 4.1: Zones of $\mathbb{R}_+$, which gives different contribution to the integral, depending on p.

We recall, that θ is a fixed small number, such that $0 < \theta < \frac{1}{6}$, and M is an arbitrary positive constant that will be chosen depending on p.

Making the change of variables $\frac{x}{\sqrt{2n}} = t$ in (4.3), we obtain for the integrals along the interval a) in (4.1):

$$I_a = \int\limits_{\sqrt{2n+n^{\frac{1}{3}+\theta}}}^{\infty} \left(H_{n-1}^2(x)e^{-x^2}\right)^p dx = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn} 2^{-p}$$

$$\times \int\limits_{1+\frac{1}{4}n^{\theta-\frac{2}{3}}+\epsilon_n}^{\infty} \exp\left[p(2n-1)\operatorname{arccosh} t - 2ntp\sqrt{t^2-1} + o(1)\right] \frac{dt}{(t^2-1)^{\frac{p}{2}}}, \quad (4.5)$$

and for the interval c) in (4.1):

$$I_c = \int_0^{\sqrt{2n-n^{\frac{1}{3}+\theta}}} \left(H_{n-1}^2(x)e^{-x^2}\right)^p dx = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn}$$

$$\times \int\limits_0^{1-\frac{1}{4}n^{\theta-\frac{2}{3}}+\epsilon_n} \left[1-\sin\left((2n-1)\arcsin\sqrt{1-t^2} - 2nt\sqrt{1-t^2}\right) + o(1)\right]^p \frac{dt}{(1-t^2)^{\frac{p}{2}}}$$

$$= (2n)^{p(n-1)+\frac{1}{2}} e^{-pn}$$

$$\times \int\limits_0^{1-\frac{1}{4}n^{\theta-\frac{2}{3}}+\epsilon_n} \sin^{2p}\left(\frac{(2n-1)\arcsin\sqrt{1-t^2} - 2nt\sqrt{1-t^2}}{2} - \frac{\pi}{4} + o(1)\right) \frac{2^p dt}{(1-t^2)^{\frac{p}{2}}},$$

where $\epsilon_n = o\left(n^{\theta-\frac{2}{3}}\right)$.

Then we pass to the integrals along b)-(4.1). The idea to split interval b) into the three subintervals (4.4) was because we are trying to use in the subintervals b_1) and b_3) in (4.4) the asymptotics of the Airy function from b)-(4.3); in b_2)-(4.4) we shall use the explicit expression for the Airy function. Notice, that for $n \to \infty$, we have from definition of z in (4.2)

$$z = \frac{2n}{x^{\frac{2}{3}}} - x^{\frac{4}{3}} \Rightarrow x \simeq \sqrt{2n} - \frac{z}{2\sqrt[6]{2n}} \Rightarrow dx = -\frac{dz}{2\sqrt[6]{2n}}$$

Thus,

$$\begin{aligned} I_{\mathrm{b}_1} &= \int\limits_{\sqrt{2n-n^{\frac{1}{3}+\theta}}}^{\sqrt{2n-Mn^{\frac{1}{3}}}} \left(H_{n-1}^2(x)e^{-x^2}\right)^p dx \\ &\simeq (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} \frac{1}{2} \int\limits_M^{n^\theta} \left[1+\sin\left(\frac{2}{3}z^{\frac{3}{2}}\right)\right]^p z^{-\frac{p}{2}} dz, \end{aligned}$$

$$
\begin{aligned}
I_{\mathrm{b}_3} &= \int\limits_{\sqrt{2n+Mn^{\frac{1}{3}}}}^{\sqrt{2n+n^{\frac{1}{3}+\theta}}} \left(H_{n-1}^2(x)e^{-x^2}\right)^p dx \\
&\simeq (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} 2^{-p-1} \int\limits_M^{n^\theta} \exp\left(-\frac{2}{3}pz^{\frac{3}{2}}\right) z^{-\frac{p}{2}} dx; \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
I_{\mathrm{b}_2} &= \int\limits_{\sqrt{2n-Mn^{\frac{1}{3}}}}^{\sqrt{2n+Mn^{\frac{1}{3}}}} \left(H_{n-1}^2(x)e^{-x^2}\right)^p dx \\
&\simeq (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} 2^{-p-1} \int\limits_{-M}^{M} \left[\frac{2\pi}{\sqrt[3]{2}} \mathrm{Ai}^2\left(-\frac{z\sqrt[3]{2}}{2}\right)\right]^p dz. \qquad (4.7)
\end{aligned}
$$

Now we can analyse the contributions of the various p-depending parts of the integral of the left hand side of (2.9), when $n \to \infty$. We note that all $o(1)$ terms in our asymptotical analysis are differentiable; therefore, they will not make contributions in our further estimates of the integrals.

First, we notice that the integral part of I_{a} in the right hand side of (4.5) is exponentially small, and there exist constants $\alpha, c > 0$, such that this integral is estimated as $O\left(n^\alpha \exp\left[-cn^{\frac{3}{2}\theta}\right]\right)$, and for I_{a} we have

$$
I_{\mathrm{a}} = \frac{(2n)^{p(n-1)+\frac{1}{2}}}{2^p e^{np}} O\left(n^\alpha \exp\left[-cn^{\frac{3}{2}\theta}\right]\right).
$$

Therefore this part is negligible for (2.9).

Second, we notice that the integral parts of I_{b_3} and I_{b_2} in the right hand side of (4.6) and (4.7), respectively, are $O(1)$ and we have

$$
I_{\mathrm{b}_2}, I_{\mathrm{b}_3} = (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} 2^{-p-1} O(1).
$$

When $p < 2$, the integral parts of I_{c} and I_{b_1} behave as $O(1)$ and $O\left(n^{\theta\frac{2-p}{2}}\right)$ and we have

$$
I_{\mathrm{c}} = (2n)^{p(n-1)+\frac{1}{2}} e^{-pn} O(1),
$$

$$
I_{\mathrm{b}_1} = (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} \frac{e^{-pn}}{2} O\left(n^{\theta\frac{2-p}{2}}\right),
$$

for $0 < p < 2$. Therefore, when $p < 2$, only I_{c} gives contribution in (2.9). Thus, we have proved that (2.4) is valid for $0 < p < 2$.

When $p = 2$, then both integral parts $I_{\rm c}$ and $I_{{\rm b}_1}$ have the same logarithmic rate of growth $O(\ln n)$. Computing the constant in O we obtain that

$$\int_0^\infty \left(H_{n-1}^2(x)e^{-x^2}\right)^2 dx = (2n)^{2n-\frac{3}{2}} e^{-2n} \left(\ln(n) + O(1)\right), \text{ for } p = 2.$$

Finally, for $p > 2$ as we see from (4.5)-(4.7), the integral over b)-(4.1) dominates in (2.9). Thus taking $M \to \infty$, we obtain

$$\int_{-\infty}^\infty \left[H_{n-1}^2(x)e^{-x^2}\right]^p dx$$
$$= (2n)^{p\left(n-\frac{2}{3}\right)-\frac{1}{6}} e^{-pn} 2^{-p} \left(\int_{-\infty}^\infty \left[\frac{2\pi}{\sqrt[3]{2}} \mathrm{Ai}^2\left(-\frac{z\sqrt[3]{2}}{2}\right)\right]^p dz\right)(1 + o(1))\,;\ p > 2.$$

Theorem is proved.

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