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The SPIDER Code - Solution of Direct and Inverse Problems for Free Boundary Tokamak Plasma Equilibrium

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THE **SPIDER** CODE – SOLUTION OF DIRECT AND INVERSE PROBLEMS FOR FREE BOUNDARY TOKAMAK PLASMA EQUILIBRIUM

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ВЫЧИСЛИТЕЛЬНЫЙ КОД SPIDER – РЕШЕНИЕ ПРЯМОЙ И ОБРАТНОЙ ЗАДАЧ РАВНОВЕСИЯ ПЛАЗМЫ СО СВОБОДНОЙ ГРАНИЦЕЙ В ТОКАМАКЕ

Аннотация

Код SPIDER является многомодульным вычислительным кодом для расчёта целого ряда задач, связанных с аксиально-симметричным МГД равновесием плазмы в токамаке. Данный препринт посвящен описанию модулей для решения прямой и обратной задачи равновесия плазмы со свободной границей.

В первой части препринта дается постановка прямой задачи о нахождении равновесия плазмы со свободной границей в заданных внешних удерживающих токах, которая описывается двумерным уравнением равновесия Грэда-Шафранова в неограниченной области относительно неизвестной функции полоидального магнитного потока Ψ в неизвестной занимаемой плазмой области Ω_p. Приводятся два варианта алгоритма решения задачи на фиксированной прямоугольной и априори неизвестной криволинейной адаптированной к магнитным поверхностям (линиям уровня решения Ψ) сетках.

Вторая часть препринта посвящена формулировке и методу решения обратной задачи равновесия плазмы со свободной границей в токамаке, как задачи восстановления удерживающих токов в катушках полоидального поля по заданному в определенном формате равновесию. Описывается соответствующий вычислительный алгоритм. Приводятся примеры расчетов.

В приложении приводится общая схема вывода уравнения равновесия Грэда-Шафранова.

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THE SPIDER CODE – SOLUTION OF DIRECT AND INVERSE PROBLEMS FOR FREE BOUNDARY TOKAMAK PLASMA EQUILIBRIUM

Abstract

The SPIDER code is an axisymmetric multipurpose plasma equilibrium solver for different formulations of the tokamak plasma equilibrium problems. This paper deals with solution of direct and inverse problems for free boundary plasma equilibrium.

The first part of the paper concerns the direct axisymmetric free boundary plasma equilibrium problem, which is formulated for the poloidal magnetic flux function Ψ and the unknown plasma domain Ω_p in terms of the Grad-Shafranov equation over an infinite domain. Computational algorithms for the cases of fixed rectangular grid and unknown magnetic surface adaptive grid are described.

The second part of the paper concerns the inverse free boundary plasma equilibrium problem of reconstruction of PFC currents as a module in the frame of the SPIDER code. Corresponding computational algorithm is described too. The results of simulations are presented.

Derivation of the Grad-Shafranov equilibrium equation is given in the appendix.

Contents:

1. Introduction

By now the most promising results for solving a problem of controlled thermonuclear fusion were obtained by use of tokamak type devices for magnetic confinement of plasma. Much work is in progress on the project of International Thermonuclear Experimental tokamak-Reactor (ITER) for demonstration of scientific and technological possibility of D-T fusion reaction (France, Cadarache).

Magnetically confined toroidal high-temperature plasma is the main subject of the controlled thermonuclear fusion research, which includes a broad spectrum of interdependent plasma-physical and engineering problems.

One of the necessary conditions to reach the goals of controlled magnetic fusion projects is the development of mathematical models and computational codes to simulate tokamak plasma behavior. Modern mathematical models of plasma and extensive experimental database give a possibility to successfully design tokamaks and predict plasma performance.

Formation and confinement of a toroidal plasma with an optimal shape and its position in a tokamak vessel, and obtaining of desired plasma pressure and current density profiles is one of the basic problems of the tokamak plasma modeling. Because of this, an accurate computation of toroidal plasma equilibrium with arbitrary plasma shape and with radial distribution of plasma pressure and current density varying in a wide range of realistic plasma parameters is an important problem, which provides data for optimization of the magnetic system parameters in tokamaks.

The set of toroidal plasma equilibrium problem formulations may be subdivided into two main groups – equilibrium problem over finite domain with prescribed fixed boundary of plasma and equilibrium problem over an infinite domain with unknown free boundary of plasma and prescribed external toroidal currents in the poloidal field coils (PFC). Essentially more complicated and realistic problems of plasma equilibrium with free boundary are of primary practical interest. A distinction is made between direct (plasma equilibrium computation with prescribed PFC currents) and inverse (PFC currents computation with prescribed plasma equilibrium) problems of axisymmetric free boundary plasma equilibrium in external PFC currents.

By now there are several well known computational codes over the world, which can solve direct free boundary plasma equilibrium problem on fixed rectangular grid with prescribed plasma pressure and poloidal current profiles. As an example we can mention here DINA [1], TSC [2], CORSICA [3], JETTO [4], SCED [5] and MAXFEA [6] codes. The SPIDER code [7], presented in this paper, is designed for free boundary equilibrium problem computation for both: on conventional fixed rectangular grid and on a priori unknown magnetic surface adaptive grid – so-called flux grid. In some instances the use of the adaptive grid technique provides a possibility to solve equilibrium problems, which cannot be practically solved on fixed rectangular grids.

As examples of inverse free boundary tokamak plasma equilibrium problem solvers we can mention here such well-known computational codes as EFIT [8], LIUQE [9], CLISTE [10].

The SPIDER code is an axisymmetric multipurpose plasma equilibrium solver for different formulations of the tokamak plasma equilibrium problems. This paper deals with solution of direct and inverse problems for free boundary plasma equilibrium.

The first part of the paper concerns the direct axisymmetric free boundary plasma equilibrium problem, which is formulated for the poloidal magnetic flux function Ψ and the unknown plasma domain Ω_p in terms of the Grad-Shafranov equation over an infinite domain. Computational algorithms for the cases of fixed rectangular grid and unknown magnetic surface adaptive grid are described.

The second part of the paper concerns the inverse free boundary plasma equilibrium problem of reconstruction of PFC currents as a module in the frame of the SPIDER code. Corresponding computational algorithm is described too. The results of simulations are presented.

Derivation of the Grad-Shafranov equilibrium equation is given in the Appendix.

Originally developed techniques for solution of direct and inverse free boundary equilibrium problems provide a possibility to obtain extensive information which is necessary for optimization of tokamak magnetic system parameters. They are also an integral part of the complex of computational codes used for design of thermonuclear reactors on the basis of tokamak, in particular ITER.

2. Free boundary plasma equilibrium problem

2.1. Formulation of the problem

Axisymmetric free boundary plasma equilibrium in cylindrical coordinate system (R,φ,Z) is described by the Grad-Shafranov equation [11]:

$$-R\nabla\cdot\left(\frac{1}{R^2}\nabla\Psi\right)\equiv-\frac{1}{R}\Delta^*\Psi=\mu_0 j_\varphi(a,R)+\mu_0\sum_k J_k\delta(R-R_k,Z-Z_k) \qquad (2.1)$$

over an infinite domain. The plasma domain Ω_p is taken to be unknown. Below we use the following notations: Ψ - the poloidal magnetic flux, Φ - the toroidal magnetic flux, J_{pol} - the poloidal plasma current, J_{tor} - the toroidal plasma current, μ_0 - magnetic permeability of free space. The normalized poloidal flux $a(R,Z)=\overline{\Psi}(R,Z)=\dfrac{\Psi-\Psi_{axis}}{\Psi_{boun}-\Psi_{axis}}\in[0,1]$ may be chosen as a flux coordinate - label of magnetic surfaces, where Ψ_{axis} is the Ψ value at the magnetic axis, Ψ_{boun} is the Ψ value at the plasma boundary. Magnetic axis coordinates are determined from the condition $|\nabla\Psi|=0$. The boundary of the plasma domain is denoted by $\partial\Omega_p$.

In right hand side of (2.1)

$$j_\varphi(a,R)=(\vec{j},\nabla\varphi)R=\begin{cases} R\dfrac{dP}{d\Psi}+\dfrac{1}{\mu_0 R}F\dfrac{dF}{d\Psi}, & (R,Z)\in\Omega_p \\ 0, & (R,Z)\notin\Omega_p \end{cases} \qquad (2.2)$$

– toroidal component of the plasma current density; $P(a)$ - plasma pressure; $\frac{dP}{d\Psi}(a)$ profile is taken to be prescribed as a function of flux coordinate a; $F(a) = \mu_0 \frac{J_{pol}}{2\pi}$ – normalized poloidal current; $F\frac{dF}{d\Psi}(a)$ profile is also taken to be prescribed. The cross-sections of poloidal field coils and passive conductive structures, surrounding toroidal plasma, are approximated by means of a finite set of filaments – point-wise circular conductors. R_k, Z_k are the k-th filament coordinates. J_k is prescribed value of the k-th filament current.

The value of full toroidal plasma current

$$I_{pl} = \int\limits_{\Omega_p} j_\varphi(a,R)dS \tag{2.3a}$$

is also input parameter of problem.

Equilibrium equation (2.1) is complemented by the next boundary conditions:

$$\Psi\Big|_{R\to 0} = 0\,, \qquad \Psi\Big|_{R^2+Z^2\to\infty} = 0\,, \qquad \Psi\Big|_{\partial\Omega_p} = \Psi_{boun} = const\,. \tag{2.3б}$$

Here Ψ_{boun} (Ψ value at $\partial\Omega_p$) is determined from relation

$$\alpha = \frac{\Psi_{boun} - \Psi_{axis}}{\Psi_{separ} - \Psi_{axis}} = const, \quad \alpha \in (0,1]\,, \tag{2.4}$$

where Ψ_{axis} and Ψ_{separ} – poloidal flux values at magnetic axis and unknown plasma separatrix; α – an input parameter of the problem, defining proximity of the plasma boundary to the separatrix (the case of $\alpha = 1$ corresponds to coincidence of the boundary and the separatrix). Ψ_{boun} can be also determined as Ψ value in some prescribed point of the plasma domain – so called limiter point.

It is assumed that plasma equilibrium configuration has the nested magnetic surfaces with a single magnetic axis.

2.2. Reduction to finite domain (Lackner's method)

The non-linear free boundary elliptic equilibrium problem, described above, poses additional difficulties from a numerical point of view because of the infinite original domain. To solve this problem we use Lackner's method [12] to reduce it to a finite domain. The technique using combination of a special auxiliary elliptic problem and the adaptive grid technique was described and applied to free boundary elliptic problems in [13-15]. We describe here the general scheme of Lackner's method:

1) The fundamental solution (Green's function) $G(\vec{r},\vec{r}_0)$ of the Δ^* operator is the solution of the following equation:

$$\Delta^* G = -R\mu_0\delta(\vec{r}-\vec{r}_0) \tag{2.5}$$

over an infinite domain, where $\delta(R,Z)$ is the Dirac's delta-function.

2) Free boundary plasma equilibrium equation (2.1) can be split into two following equations:

$$\Delta^*\psi_{pl} = -R\mu_0 j_\varphi \ , \tag{2.6a}$$

$$\Delta^*\psi_{ext} = -R\mu_0\sum_k J_k\delta(\vec{r}-\vec{r}_k) \tag{2.6b}$$

In doing so, we can write

$$\psi_{pl}(\vec{r}_0) = \int_{\Omega_p} j_\varphi(\vec{r})G(\vec{r},\vec{r}_0)dS \ , \tag{2.7a}$$

$$\psi_{ext}(\vec{r}_0) = \sum_k G(\vec{r}_k,\vec{r}_0)J_k \tag{2.7b}$$

on the base of definition and properties of the Green's function.

3) For two arbitrary functions u, v, which have smooth second partial derivatives in a bounded region S, the following integral Green's theorem is valid:

$$\int_S \frac{1}{R}\left(u\Delta^* v - v\Delta^* u\right)dS = \int_{\partial S}\left(\frac{v}{R}\frac{\partial u}{\partial n}dl - \frac{u}{R}\frac{\partial v}{\partial n}dl\right) \quad , \tag{2.8}$$

where $\partial \cdot/\partial n$ is the derivative with respect to an external normal to ∂S.

4) Let S to be a bounded domain with sufficiently smooth boundary ∂S which contains plasma domain Ω_p. Consider the following boundary-value problem

$$\Delta^* g = -R\mu_0 j_\varphi \ , \quad g\big|_{\partial S} = 0 \tag{2.9}$$

in S. Let us assume that $v \equiv G(\vec{r}, \vec{r}_0)$, $u \equiv g$ in (2.8). Then

$$\int_S \frac{1}{R}\left(g\Delta^* G - G\Delta^* g\right)dS = \int_{\partial S}\left(\frac{G}{R}\frac{\partial g}{\partial n}dl - \frac{g}{R}\frac{\partial G}{\partial n}dl\right)$$

and

$$\lambda g(\vec{r}_0) - \int_S j_\varphi G(\vec{r}, \vec{r}_0)dS = \int_{\partial S}\frac{G}{R}\frac{\partial g}{\partial n}dl \ , \quad \text{где } \lambda = \begin{cases} 1, & \vec{r}_0 \in S \\ 1/2, & \vec{r}_0 \in \partial S \\ 0, & \vec{r}_0 \notin S \end{cases} \quad . \tag{2.10}$$

Let us assume that $\vec{r}_0 \in \partial S$ in (2.10). Then $-\int_S j_\varphi G(\vec{r}, \vec{r}_0)dS = \int_{\partial S}\frac{G}{R}\frac{\partial g}{\partial n}dl$, where according to (2.7a) $\int_S j_\varphi G(\vec{r}, \vec{r}_0)dS = \psi_{pl}(\vec{r}_0)$, and we derive boundary-value condition for $\psi = \psi_{pl} + \psi_{ext}$ at the boundary ∂S:

$$\psi_{pl}(\vec{r}_0) = -\int_{\partial S}\frac{G}{R}\frac{\partial g}{\partial n}dl \ , \quad \psi_{ext}(\vec{r}_0) = \sum_k G(\vec{r}_k, \vec{r}_0)J_k \quad . \tag{2.11}$$

Here g is the solution of the boundary-value problem (2.9).

Applying this technique the solution of original problem over an infinite domain is reduced to the solutions of two boundary-value problems (2.6), (2.9) over the finite domains.

2.3. Computational algorithm

To solve the nonlinear problem (2.1)-(2.4) the following iteration procedure is proposed. The key point of this procedure is the use of so-called “limiter points” or “limiters”, which provide convergence of the iteration loop. The use of the limiters leads to two nested iteration loops. First the problem is solved with fixed limiters (inner iterations), and then the locations of the limiters are adjusted to satisfy the original problem (outer iterations), and the next step of the iteration procedure is performed.

Inner iterations

Picard iterations are performed for equation

$$\Delta^* \psi_{pl}^{s+1} = -R\mu_0 j_\varphi(\psi^s, R), \quad s \text{ – iteration number,}$$

$$\psi_{ext}(\vec{r}) = \sum_k G(\vec{r}_k, \vec{r}) J_k \quad ,$$

with a fixed location of the limiter point (R_m, Z_m) – the magnetic axis of the equilibrium configuration in our case. To fix the magnetic axis in the prescribed point (R_m, Z_m) we add an artificial poloidal flux as follows

$$\psi_{art}^{s+1}(\vec{r}) = C_r^{s+1} R^2 + C_z^{s+1} Z \qquad (2.12)$$

so that

$$\psi^{s+1} = \psi_{pl}^{s+1} + \psi_{ext} + \psi_{art}^{s+1} \quad . \qquad (2.13)$$

Constants C_r^{s+1} and C_z^{s+1} in (2.12) are fitted from the condition that a magnetic axis is coincides with the prescribed limiter point (R_m, Z_m). Here s is the iteration number. New plasma boundary approximation is determined according to the prescribed input parameter α (2.4) and the obtained approximate solution ψ^{s+1}.

Outer iterations

Let us consider coefficients C_r, C_z as functions of unknown magnetic axis coordinates (R_m, Z_m). The next step of the iteration procedure consist of finding such values of (R_m, Z_m), which provide the solution of the original problem (2.1)-(2.4). Required values are obtained from solution of the following nonlinear system of the equations:

$$\begin{aligned} C_r(R_m, Z_m) &= 0, \\ C_z(R_m, Z_m) &= 0. \end{aligned} \tag{2.14}$$

This two-level iteration procedure is repeated until we obtain sufficiently small values of C_r, C_z corresponding to vanishing artificial poloidal flux (2.12).

2.4. Finite-difference scheme

The computational domain is covered by a quadrangular computational grid. The difference analog of the Grad-Shafranov operator $\Delta^*(\cdot)$ (2.1) is constructed on the basis of the conservative finite-difference approximation of the operator $\nabla \times (\nabla \times (\cdot))$ by means of the operational finite-difference method [16]. The procedure of the construction of the difference scheme for the equilibrium equation (2.1) is described in [17].

2.5. Equilibrium on rectangular grid

In case of the free boundary plasma equilibrium computation on the rectangular grid the difference problem is solved in the rectangular domain, which covers the plasma domain.

In Fig.1 we demonstrate an example of a computation of the tokamak plasma equilibrium configuration with the following input parameters:

- the current density profile is prescribed by the functions $dP/d\Psi$ and $F\,dF/d\Psi$ shown in Fig.2 versus normalized poloidal flux;
- full toroidal plasma current value $I_{pl} = 15MA$;
- proximity of the plasma boundary and the separatrix is set by the parameter $\alpha = 0.995$;
- locations of the filaments, which approximate PFC cross-sections, are shown in Fig.1; the prescribed currents for each PFC are distributed over these filaments.

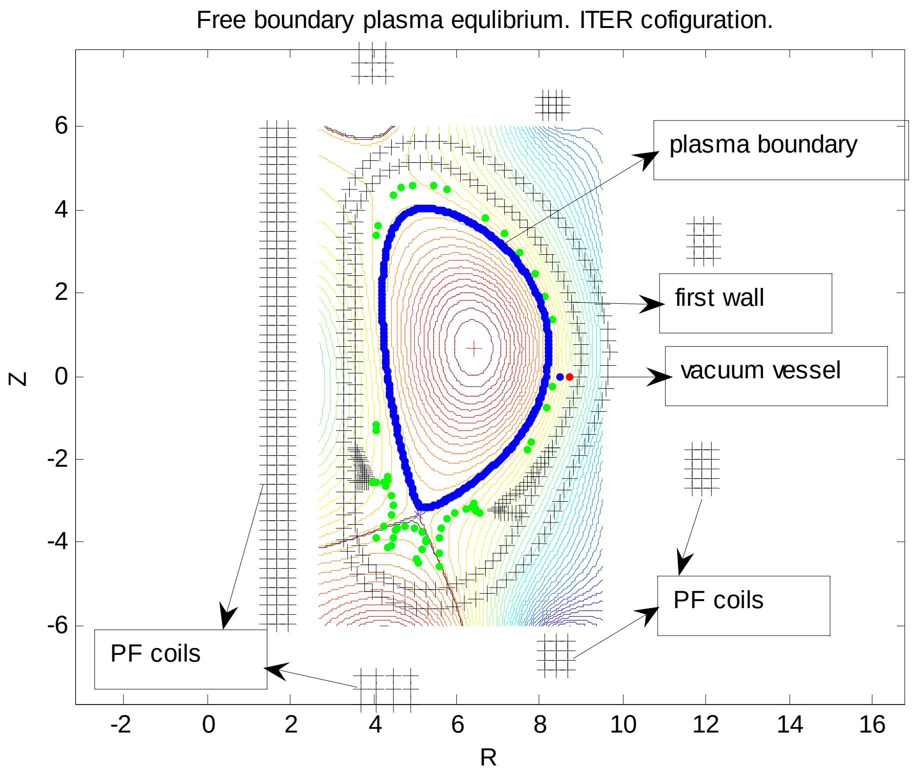

Fig.1 Free boundary plasma equilibrium on rectangular grid. ITER configuration.

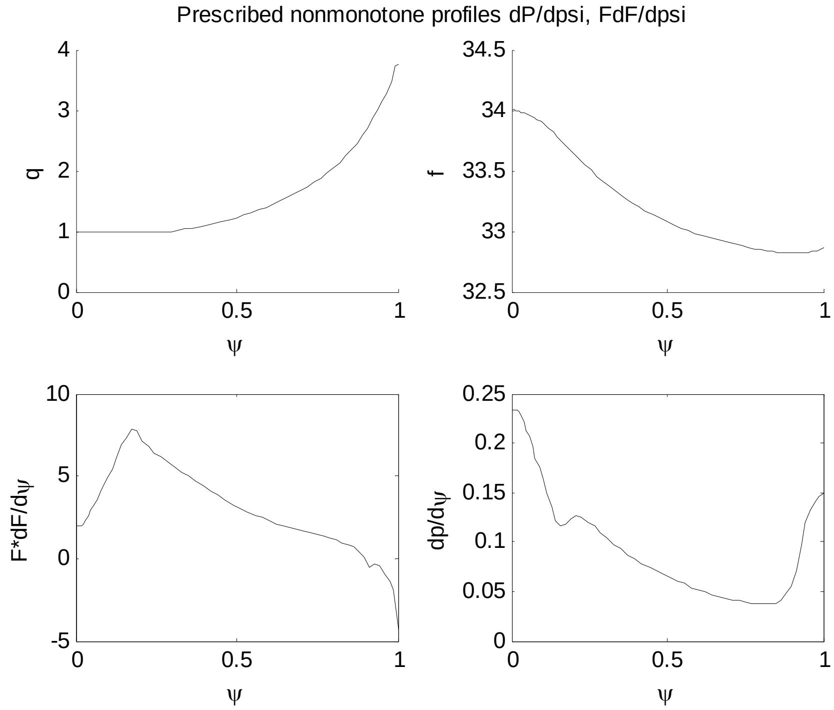

Fig.2 The plasma profiles versus normalized poloidal flux for the equilibrium from Fig.1: safety factor q; poloidal current f; input profiles $dP/d\Psi$ and $F\,dF/d\Psi$.

2.6. Equilibrium on magnetic surface adaptive grid

In case of free boundary plasma equilibrium computation on adaptive grid, difference problem is solved in the domain, which is geometrically similar to the plasma domain and necessarily covers plasma domain. This prescribed computational domain is covered by a computational grid, which is topologically equivalent to a radial-annular grid and it is used as initial guess for construction of a final magnetic surface adaptive grid.

During the process of iterations in the plasma domain computational grid is adapted to desired magnetic surfaces – solution level lines – on each iterative step.

Fig.3a demonstrates an example of computation of the ITER plasma equilibrium configuration with the same input parameters as for the case of Fig.1

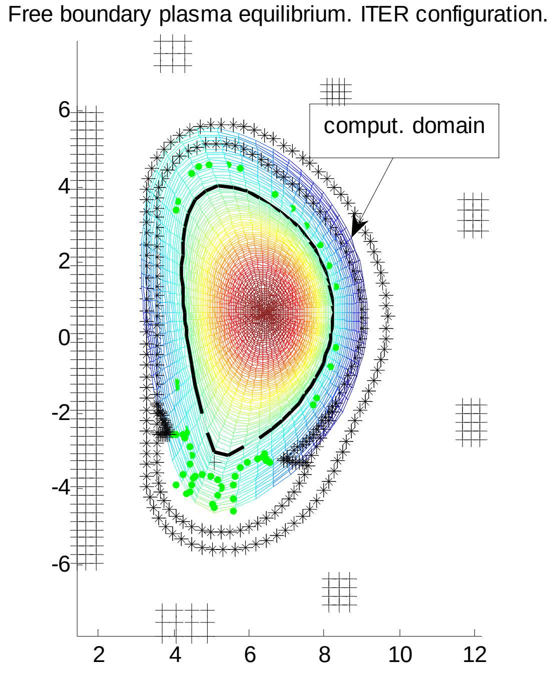

Fig.3a Free boundary plasma equilibrium on magnetic surface adaptive grid. ITER configuration.

A close-up view of the same equilibrium without PFC filaments is shown in Fig.3b. In this variant plasma boundary is determined by parameter $\alpha = 0.995$ and does not coincide with the separatrix. It is seen that the x-point of the separatrix is outside the plasma boundary.

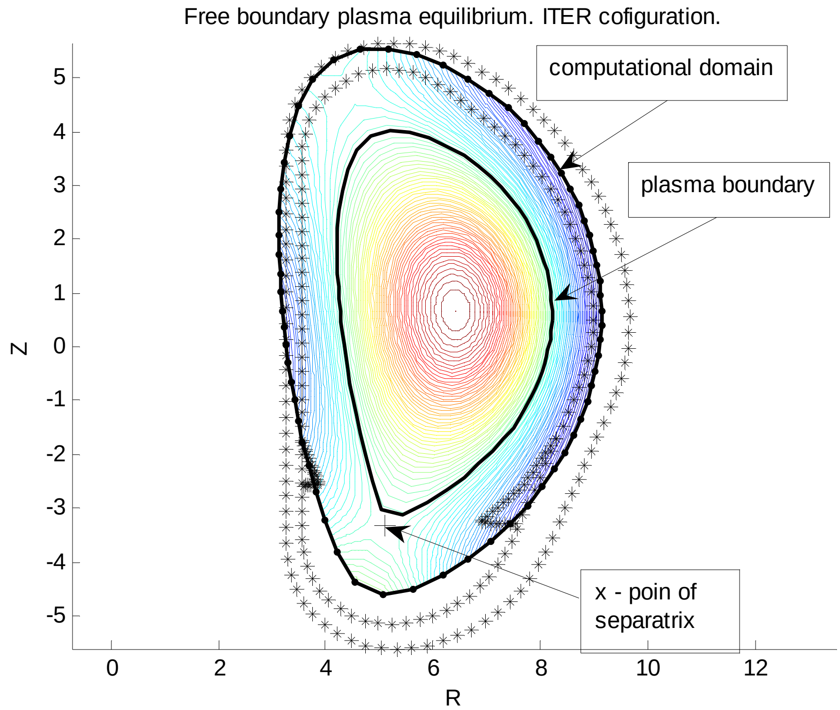

Fig.3b Free boundary plasma equilibrium on magnetic surface adaptive grid. ITER configuration.

3. Inverse problem. Reconstruction of PFC currents.

3.1. Formulation of inverse problem

The inverse free boundary plasma equilibrium problem of the PFC currents reconstruction, which form a desired free boundary plasma equilibrium, is stated as follows:

1. Plasma boundary location and shape are approximated by a finite set of so-called "fitting" points with prescribed coordinate values (r_l, z_l) in (R, Z) plane. During the iterations equilibrium plasma boundary will be fitted to these points by means of minimization of the functional which is given below.

2. The coordinates of a finite number of so-called "control" points, which must belong to plasma boundary, are prescribed.

3. The supporting values (approximations) of the PFC currents are prescribed. Desired currents (solution of the inverse problem) must be close to them.

4. The coordinates of the x-point where $|\nabla\Psi| = 0$ are prescribed. The x-point coordinates of the desired free boundary plasma equilibrium must be the same as prescribed values.

5. The following normalizing constants and plasma equilibrium profiles versus normalized poloidal flux $a = \overline{\Psi}$ are prescribed:

a) toroidal plasma current density profiles – $P'_\psi(a)$, $FF'_\psi(a)$; full toroidal plasma current $-I_{pl} = const$; toroidal vacuum magnetic field value $B_0 = const$ at the prescribed coordinate $R = R_0$

or

b) plasma pressure gradient profile $P'_\psi(a)$; safety factor profile $q(a) = -\dfrac{d\Phi}{d\Psi}$; and either full toroidal plasma current – $I_{pl} = const$ or poloidal flux value at the magnetic axis $\psi_{ax} = const$.

6. The value of the proximity parameter α (2.4) for desired free boundary plasma equilibrium is prescribed.

For given PF coils it is required to find the currents, which forms the desired free boundary plasma equilibrium, satisfying the above conditions (1)-(6).

3.2. Computational algorithm

The problem of reconstruction of the PFC currents, needed for confinement of the plasma with a boundary close to the fitting points (r_l, z_l), is formulated as minimization problem of the following functional

$$W = \sum_{l=1}^{L} \omega_l \left[\psi_p(r_l, z_l) + \sum_{k=1}^{K} J_k G(r_l, z_l; r_k, z_k) - \psi_{boun} \right]^2 + \sigma \sum_{k=1}^{K} d_k \left(J_k - J_k^{ref} \right)^2 \quad , \qquad (3.1)$$

$$\min_{J_k, \psi_{boun}} W \quad ,$$

under variations of the current values J_k and to the boundary poloidal flux $\psi = \psi_{boun}$ value. Parameter $\sigma > 0$ provides regularization of the problem. Coefficients ω_l, d_k are prescribed. They intend for corrections of the flux values in every fitting point and of the PFC equilibrium currents deviations from the prescribed supporting values. $\psi_p(r_l, z_l)$ are the poloidal flux values in the fitting points (r_l, z_l).

Condition (2) of the plasma boundary passing through the control points (r_m, z_m) is realized by means of the Lagrange multipliers λ_m and extension of the functional (3.1) to the following representation:

$$W' = \sum_{m=1}^{M} \lambda_m \left[\psi_p(r_m, z_m) + \sum_{k=1}^{K} J_k G(r_m, z_m; r_k, z_k) - \psi_{boun} \right] \quad , \qquad (3.2)$$

$$\min_{J_k, \psi_{boun}, \lambda_m} (W + W') \quad .$$

Conditions (3), (6) are realized by adding other Lagrange multipliers $\lambda_{Rx}, \lambda_{Zx}, \lambda_x$ to the functional (3.2):

$$W'' = \lambda_{Rx}\left[\left(\frac{\partial \psi_p}{\partial R}\right)_x + \sum_k J_k \frac{\partial G(\vec{r}_x, \vec{r}_k)}{\partial R_x}\right] + \lambda_{Zx}\left[\left(\frac{\partial \psi_p}{\partial Z}\right)_x + \sum_k J_k \frac{\partial G(\vec{r}_x, \vec{r}_k)}{\partial Z_x}\right] +$$

$$+\lambda_x\left[(\psi_{ax} - \psi_{boun}) - \alpha(\psi_{ax} - \psi_{separ})\right] , \tag{3.3}$$

$$\min_{J_k, \psi_{boun}, \lambda_m, \lambda_{Rx}, \lambda_{Zx}, \lambda_x} (W + W' + W'') \ .$$

Adding to the functional (3.3) the following part with the Lagrange multipliers $\lambda_{RZ}, \lambda_{ZZ}$

$$W''' = \lambda_{RZ}\left[\left(\frac{\partial^2 \psi_p}{\partial R \partial Z}\right)_x + \sum_{k=1}^{K} J_k \frac{\partial^2 G(\vec{r}_x, \vec{r}_k)}{\partial R \partial Z}\right] + \lambda_{ZZ}\left[\left(\frac{\partial^2 \psi_p}{\partial Z^2}\right)_x + \sum_{k=1}^{K} J_k \frac{\partial^2 G(\vec{r}_x, \vec{r}_k)}{\partial Z^2}\right] , \tag{3.4}$$

$$\min_{J_k, \psi_{boun}, \lambda_m, \lambda_{Rx}, \lambda_{Zx}, \lambda_x, \lambda_{RZ}, \lambda_{ZZ}} (W + W' + W'' + W''') \ .$$

makes possible solving of the inverse problem for the case of so-called «snowflake» [18] equilibrium with the second order x-point, which is defined by the usual x-point condition $|\nabla \Psi| = 0$ and the following conditions for the second order derivatives of the equilibrium problem solution Ψ :

$$\frac{\partial^2 \Psi}{\partial R^2} = \frac{\partial^2 \Psi}{\partial Z^2} = \frac{\partial^2 \Psi}{\partial R \partial Z} = 0 \ . \tag{3.5}$$

3.3 <u>Examples</u>

Fig.4 demonstrates an example of the ITER equilibrium configuration defined by the reconstructed PFC currents. Corresponding inverse problem input parameters (besides fitting point coordinates) were the following:

- the non-monotone current density profile is prescribed by the functions $dP/d\Psi$ and $F\, dF/d\Psi$ shown in Fig.5 versus normalized poloidal flux;
- full toroidal plasma current value $I_{pl} = 15 MA$;
- proximity of the plasma boundary and the separatrix is set by the parameter $\alpha = 0.995$;

- vacuum magnetic field value $B_0 = 5.3T$ at the prescribed coordinate $R_0 = 6.2m$;
- locations of the filaments, which approximate PFC cross-sections, are shown in Fig.4; the reconstructed currents for each PFC are distributed over these filaments;
- coordinates of the five control points are shown by the circles markers – three at the plasma boundary and two at the separatrix legs.

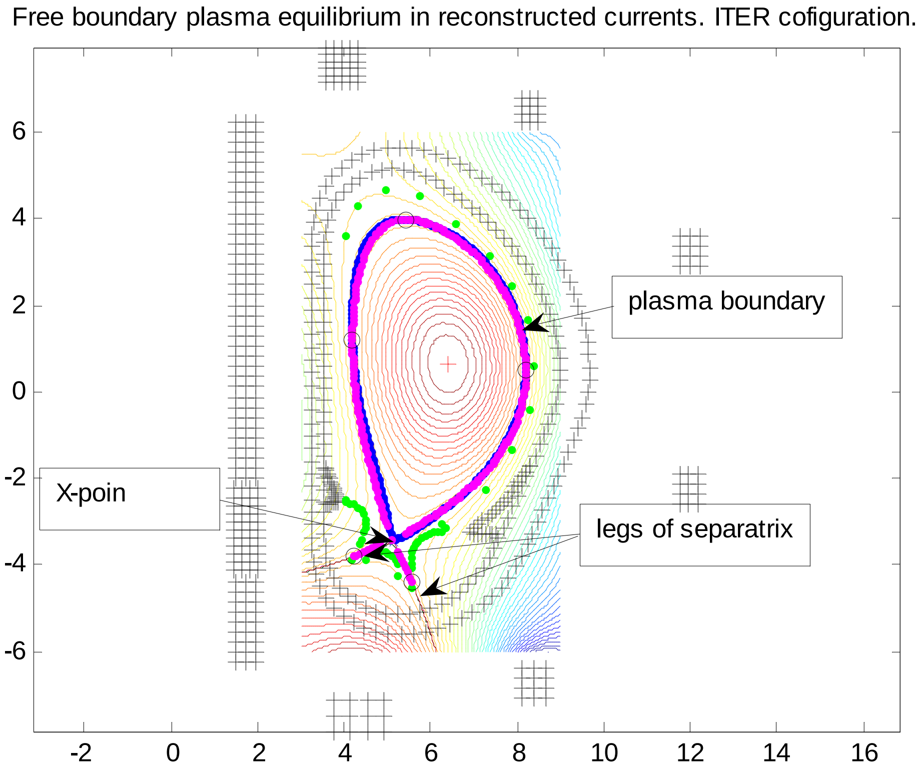

Fig.4 Free boundary plasma equilibrium on rectangular grid in reconstructed currents. ITER configuration.

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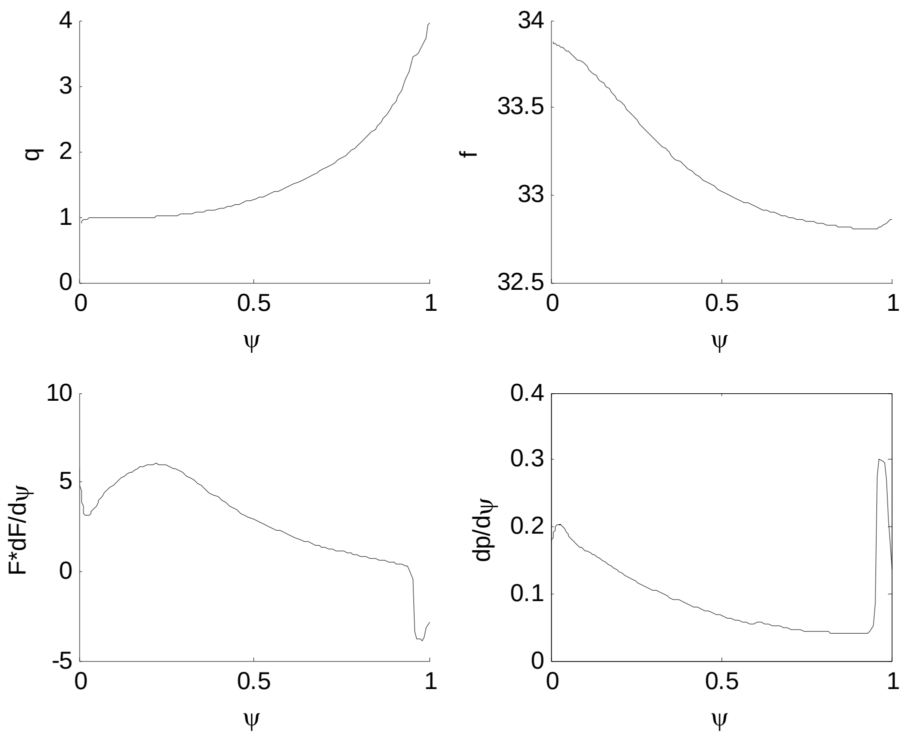

Fig.5 The plasma profiles versus normalized poloidal flux for the equilibrium from Fig.4: safety factor q; poloidal current f; input profiles $dP/d\Psi$ and $F\,dF/d\Psi$.

An example of the «snowflake» equilibrium configuration in TCV tokamak device in the reconstructed PFC currents is shown in Fig.6. The corresponding inverse problem input parameters (besides fitting and control points coordinates) were the following:

- the non-monotone current density profile is prescribed by the functions $dP/d\Psi$ and $F\,dF/d\Psi$ shown in Fig.7 versus normalized poloidal flux;
- full toroidal plasma current value $I_{pl} = 0.378MA$;
- proximity of the plasma boundary and the separatrix is set by parameter $\alpha = 0.995$;
- vacuum magnetic field value $B_0 = 1.44T$ at the prescribed coordinate $R_0 = 0.88m$;
- locations of the filaments, which approximate PFC cross-sections, are shown in Fig.6; the reconstructed currents for each PFC are distributed over these filaments;

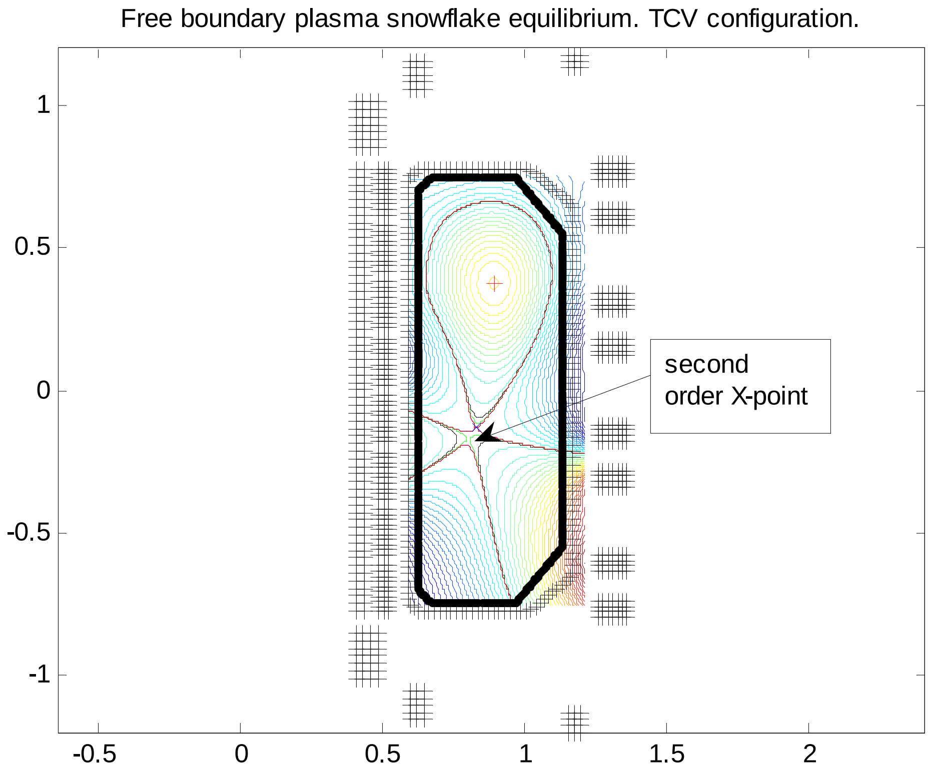

Fig.6 Free boundary plasma equilibrium on rectangular grid in reconstructed currents. TCV "snowflake" divertor configuration.

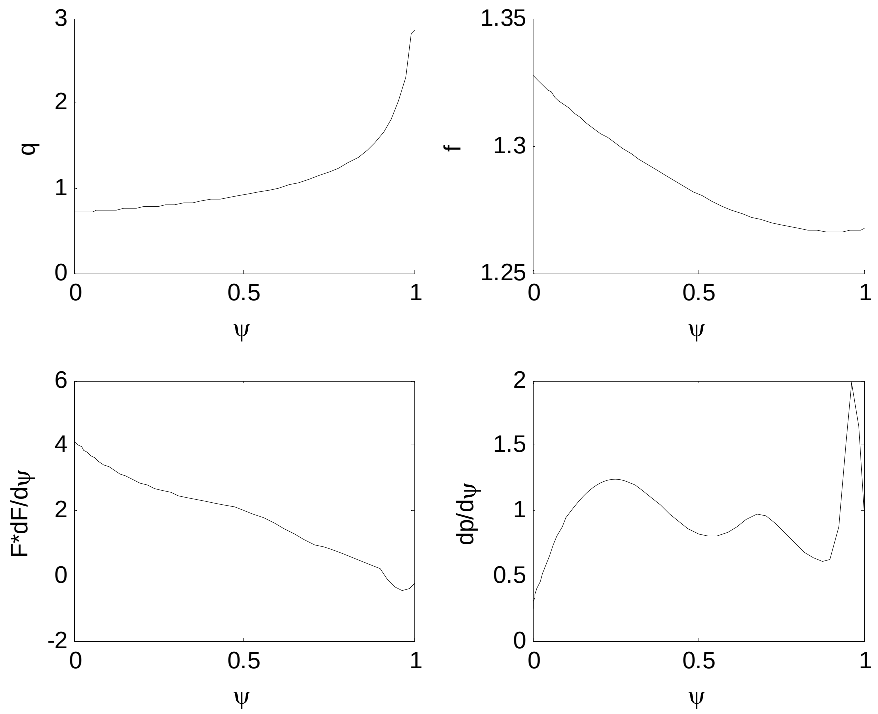

Fig.7 The plasma profiles versus normalized poloidal flux for the equilibrium from Fig.6: safety factor q; poloidal current f; input profiles $dP/d\Psi$ and $F\,dF/d\Psi$.

4. Appendix: Grad-Shafranov equlibrium equation

In MHD approach plasma equilibrium is described by the ideal magnetostatic equations [11]:

$$\vec{j}\times\vec{B}-\nabla p=0\ ,\quad \mu_0\vec{j}=\nabla\times\vec{B}\ ,\quad \nabla\cdot\vec{B}=0\ , \tag{A1}$$

where $\vec{B}$ – magnetic field, $\vec{j}$ – electric current density, p – plasma pressure.

1) For convenience of the presentation let us introduce the cylindrical coordinate system (R,φ,Z). An arbitrary vector $\vec{w}$ may be decomposed in the sum of mutually orthogonal poloidal and toroidal vectors:

$\vec{w}=\vec{w}_p+\vec{w}_\varphi$, where $\vec{w}_p$ lies in the (R,Z) plane, $\vec{w}_\varphi=(\vec{w},\vec{e}_\varphi)\vec{e}_\varphi$.

Under condition of axial symmetry, i.e. independence from φ-coordinate, the following relation will be always true:

$$\nabla\cdot\vec{w}_\varphi=0,$$

and, in addition, the rotor of any poloidal vector is a toroidal vector, and the rotor of any toroidal vector is a poloidal vector.

2) Let Ψ be the poloidal flux of the magnetic field $\vec{B}$, J_{pol} be the plasma poloidal current (poloidal flux of the plasma current density $\vec{j}$) and introduce the following associated functions:

$$\psi=\frac{\Psi}{2\pi}\ ,\quad F=\mu_0\frac{J_{pol}}{2\pi}\ .$$

3) <u>Magnetic field</u>

It follows from $\nabla\cdot\vec{B}_\varphi=0$ that $\nabla\cdot\vec{B}_p=0$, and the poloidal component of the magnetic field $\vec{B}_p$ may be written as $\vec{B}_p=\nabla\times\vec{A}_\varphi$. It is easy to see that for the functions ψ and $\vec{A}_\varphi$ the following relation holds: $\vec{A}_\varphi=\psi\nabla\varphi$. Whence it follows that

$$\vec{B}_p=\nabla\psi\times\nabla\varphi\ . \tag{A2}$$

The following relation

$$\vec{B}_\varphi = F\nabla\varphi \ . \tag{A3}$$

is true for magnetic field toroidal component $\vec{B}_\varphi$ and function F.

4) Current

The following relations for poloidal and toroidal components of the current density

$$\vec{j}_p = \frac{1}{\mu_0}\nabla\times\vec{B}_\varphi = \frac{1}{\mu_0}\nabla F\times\nabla\varphi \ , \tag{A4}$$

$$\vec{j}_\varphi = \frac{1}{\mu_0}\nabla\times\vec{B}_p = \frac{1}{\mu_0}\nabla\times(\nabla\psi\times\nabla\varphi) \ , \tag{A5}$$

may be derived from the equation $\mu_0\vec{j} = \nabla\times\vec{B}$ (A1).

5) Projection of the equation (A1) onto $\vec{e}_\varphi$ direction gives $\vec{j}_p\times\vec{B}_p = 0$, from where and relations (A2),(A4) we can written as

$$\nabla\psi\times\nabla F = 0 \, . \tag{A6}$$

Relation (A6) implies that vectors $\nabla\psi$ and ∇F are collinear and, respectively, functions $\psi(R,Z)$ and $F(R,Z)$ have the same set of the level lines.

Projection of the equation (A1) onto orthogonal $\vec{e}_\varphi$ direction gives

$$\vec{j}_p\times\vec{B}_p + \vec{j}_\varphi\times\vec{B}_\varphi - \nabla p = 0 \ ,$$

from which and from relations (A2)-(A5) after not complicated transformations we can derive

$$(\nabla\times(\nabla\psi\times\nabla\varphi),\nabla\varphi)\nabla\psi = \frac{1}{R^2}F\nabla F + \mu_0\nabla p \, . \tag{A7}$$

From equation (A7), in view of (A6), we can conclude that vectors $\nabla\psi$ and ∇p are collinear too. Thus we can conclude that the functions $F = F(\psi)$ and $p = p(\psi)$ depend on the poloidal flux ψ only so that

$$\nabla F = \frac{dF}{d\psi}\nabla\psi \ , \quad \nabla p = \frac{dp}{d\psi}\nabla\psi \, .$$

From which, in view of (A7), we derive two-dimensional scalar Grad-Shafranov equilibrium equation

$$\left(\nabla\times\left(\nabla\psi\times\nabla\varphi\right),\nabla\varphi\right)=\frac{1}{R^2}F\frac{dF}{d\psi}+\mu_0\frac{dp}{d\psi} \qquad \text{(A8)}$$

for the desired scalar poloidal flux function $\psi(R,Z)$. Differential operator in the left-hand side of (A8), using traditional Grad-Shafranov notation Δ^*, can be rewritten in the following form:

$$\left(\nabla\times\left(\nabla\psi\times\nabla\varphi\right),\nabla\varphi\right)=-\frac{1}{R^2}\Delta^*\psi=-\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right) \quad .$$

From (A5), (A8) the following expression for the toroidal current density component can be derived:

$$\mu_0\left(\vec{j},\nabla\varphi\right)=\frac{1}{R^2}F\frac{dF}{d\psi}+\mu_0\frac{dp}{d\psi} \quad . \qquad \text{(A9)}$$

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