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# Matrix Riemann-Hilbert Analysis for the Case of Higher Genus – Asymptotics of Polynomials Orthogonal on a System of Intervals

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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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# Analysis of the matrix Riemann-Hilbert problems for the case of higher genus - asymptotics of polynomials orthogonal on a system of intervals

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*Анализ матричных задач Римана-Гильберта в случае высокого рода — асимптотика многочленов ортогональных на системе интервалов* [1]

**Аннотация.** Рассматривается адаптация метода матричной задачи Римана-Гильберта для получения сильных асимптотик многочленов ортогональных на системе интервалов действительной оси. Основным моментом является привлечение тета-функций Римана для получения асимптотических формул. Работа мотивирована распространением обсуждаемой методики на краевые задачи для аналитических матриц функций высоких размерностей (больше чем 2x2). Именно такие задачи возникают при асимптотическом анализе аппроксимаций Эрмита-Паде. Работа продолжает серию методических разработок асимптотической техники матричной задачи Римана-Гильберта.

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## Abstract

The method of the matrix Riemann-Hilbert problem is adapted for obtaining the strong asymptotics of polynomials orthogonal on a system of intervals on the real axis. The use of the Riemann theta-functions for deriving the asymptotical formulas is the main ingredient of the approach. An extension of the technique under consideration to Boundary Values Problems for analytic matrix functions of higher dimensions (greater than 2x2) is the main motivation of the work. Precisely this type of problem arise under asymptotical analysis of the Hermite-Pade approximants. The paper is continuation of the series of the lecture notes devoted to exposition of the "Riemann-Hilbert matrix problem"asymptotical techniques.

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# 1 Introduction and the model problem statement

The present paper is methodical by nature. It is intended to adapt some techniques connected with the Riemann theta-functions for obtaining the asymptotic solutions of the matrix boundary value problems in multi-connected domains. One should note that the methods discussed here have already been developed in a renown paper [1] and used in a number of subsequent papers. Nevertheless, we have found it worthwhile to get back to the simplest model problem of such kind and to expound its solution in such a way that one could use these methods to obtain solutions for boundary value problems for matrix of order greater than (2 x 2) – exactly such problems arise in the asymptotic analysis of the Hermite-Pade approximants (see [2], [3]). As a model problem we have chosen the problem of obtaining asymptotics on a system of intervals on the real axis. Of course a great number of papers is devoted to the asymptotical formulas for this problem (including expressions of asymptotics in terms of Riemann theta-functions) – see, for instance, [4], [5], [6], [7]. Hence, the result which is proven here should not be regarded as a new one, and the present paper in fact offers the material for some advanced course, continuing the series of papers [8], [9], (see also [10]).

We start with the statement of the problem of finding orthogonal polynomials asymptotics. Let us consider a system of intervals

$$\triangle := \bigcup_{j=1}^{g+1} \triangle_j := \bigcup_{j=1}^{g+1} [a_j, b_j] \subset \mathbb{R}\,,$$

where the weight function is given by

$$w(z) := w_0(z)(z-a_j)^{\alpha_j}(z-b_j)^{\beta_j}\,,\ z \in \triangle_j\,,\ j = 1, \dots, g+1\,, \tag{1}$$

here the parameters satisfy $\alpha_j, \beta_j > -1$, and $w_0(z)$ is piecewise analytic in some neighborhood of $\triangle$:

$$w_0 \in H(\triangle)\,, \quad w_0 > 0 \text{ on } \triangle\,.$$

Let us consider the system of monic orthogonal polynomials:

$$P_n(z) = z^n + \dots \quad : \quad \int_{\triangle} P_n(z)\, z^\nu w(z)\, dz = 0\,, \quad \nu = 0, \dots, n-1\,, \tag{2}$$

and the corresponding functions of the second kind:

$$R_n(z) = \int_{\triangle} \frac{P_n(x) w(x)\, dx}{x-z} = \frac{1}{P_n(z)} \int_{\triangle} \frac{P_n^2(x) w(x)\, dx}{x-z} = \frac{m_n}{z_{n+1}} + \dots \quad . \tag{3}$$

Our goal is to find the asymptotics of $P_n(z)$ и $R_n(z)$ as $n \to \infty$.

## 2 Statement of the result

We start with an introduction of some basic notions which we shall use to state desired asymptotic formulas.

1°. Standard functions of multi-connected domain geometry.

Let us denote $G$ the complex Green function of the domain $\Omega := \overline{\mathbb{C}} \backslash \triangle$, (with singularity at the point $\infty$). Its derivative is analytic in $\Omega$

$$h(z) := G'(z) = \frac{1}{z} + \ldots \ \in H(\Omega) \ , \tag{4}$$

it has $g$ finite zeros

$$\{z_k^*\}_{k=1}^{g} \ : \ h(z_k^*) = 0 \ . \tag{5}$$

We denote

$$\Phi(z) := e^{G(z)} = \frac{z}{C_\triangle} + \ldots \ , \tag{6}$$

this function has single valued absolute value in $\Omega$; we denote the changes of its argument around the contours encircling the segments

$$\underset{\triangle_k}{\triangle} \arg \Phi =: 2\pi\omega_k \ , \quad k = 1, \ldots, g \ . \tag{7}$$

We consider also the harmonic measures of the domain $\Omega$

$$\{\omega_k(z)\}_{k=1}^{g+1} \ : \quad a)\, \omega_k \in Harm(\Omega) \ , \quad b)\, \omega_k|_{\triangle_l} = \delta_{k,\,l} \ , \quad k, l = 1, \ldots g+1 \ ,$$

then (see [4]) in (7) we have $\omega_k = \omega_k(\infty)$.

2°. Standard function related to the weight – Szegö function.

Let $F \in \mathcal{A}(\Omega)$, $F \neq 0$, is a solution of the boundary value problem

$$|F|^2 i \frac{w}{h_-} = 1 \quad \text{on } \triangle \ , \tag{8}$$

where $w$ is weight function (1), and $h_-$ is the boundary value of function (4) corresponding to approaching $\triangle$ from below. Function $F$ is called Szegö function for weight $w$. It has single valued absolute value and multi valued argument, change of which we denote by

$$\underset{\triangle_k}{\triangle} \arg F =: \ 2\pi c_w^{(k)} \ . \tag{9}$$

By the use of harmonic measures (see [4]) we have

$$c_w^{(k)} = -\frac{1}{4\pi} \oint\limits_{\triangle_k} \log\left( i \frac{w}{h_-}(\xi) \right) \frac{\partial \omega_k}{\partial n_\xi}(\xi) \, |d\xi| \ .$$

3°. Standard functions of Riemann surface.

Let

$$\mathfrak{R} := \overline{\mathfrak{R}_0 \bigcup \mathfrak{R}_1}\,, \quad \mathfrak{R}_i := \overline{\mathbb{C}} \setminus \triangle\,, \quad \partial\mathfrak{R}_{01} := \triangle_+ \bigsqcup \triangle_-\,, \tag{10}$$

be two-sheeted Riemann surface, and $\{\mathbf{a}_k\}_{k=1}^g$ , $\{\mathbf{b}_k\}_{k=1}^g$ be its homological cycles (cycle $\mathbf{a}_k$ starts and ends at a point of the segment $\triangle_{g+1}$, crossing the segment $\triangle_k$ through both sheets of the Riemann surface, accordingly, cycle $\mathbf{b}_k$ encircles the segment $\triangle_k$, i.e. $\mathbf{b}_k := \triangle_{k+} \bigsqcup \triangle_{k-}$). On $\mathfrak{R}$ vector of normalized Abel 1st kind integrals is defined:

$$\overrightarrow{\Omega}(\zeta) := \{\Omega_k(\zeta)\}_{k=1}^g\,, \quad \zeta \in \mathfrak{R} \quad : \quad (\underset{\mathbf{a}_l}{\triangle}\,\Omega_k = \delta_{k,\,l}\ ;\ \underset{\mathbf{b}_l}{\triangle}\,\Omega_k = B_{k,\,l})\,, \tag{11}$$

where the matrix $\{B_{k,\,l}\}$ has positive definite imaginary part.
(The function $\Omega_k$ could be considered as a continuation of the harmonic measure functions $\frac{1}{2}(\omega_k + \widetilde{\omega_k}) =: \Omega_k$ c $\mathfrak{R}_0$ on the whole Riemann surface (10))

Let

$$\theta(u_1, \ldots, u_g) := \sum_{m_1,\ldots,m_g}^{-\infty,\ldots,\infty} \exp\left\{\pi\, i \sum_{\mu=1}^{g} \sum_{\nu=1}^{g} B_{\mu\nu} m_\mu m_\nu + 2\pi\, i \sum_{\nu=1}^{g} m_\nu u_\nu \right\}\,,$$

be multiple series of $g$ variables, with parameter matrix $\{B_{\mu\nu}\}$.

For an arbitrary vector $\vec{e} \in \mathbb{C}^g$ theta-function of the Riemann surface $\mathfrak{R}$ (with the parameters $B_{\mu,\nu}$) is defined via substitution of the coordinates of vector $\vec{\Omega}(\zeta) - \vec{e}$ as multiple series variables $\theta$

$$\Theta^{(\vec{e})}(\zeta) := \theta\left(\vec{\Omega}(\zeta) - \vec{e}\right)\ . \tag{12}$$

Eventually, we obtain the function of one variable $\zeta \in \mathfrak{R}$, which has the following basic properties:

$$\begin{aligned} &A) \quad \Theta^{(\vec{e})} \in H(\mathfrak{R} \setminus \{\bigcup_{j=1}^{g} \mathbf{a}_j\})\,; \\ &B) \quad \exists \quad \{\dot{\zeta}_k\}_{k=1}^g\ :\ \Theta^{(\vec{e})}(\dot{\zeta}_k) = 0\ ,\ k = 1, \ldots, g\ ; \end{aligned} \tag{13}$$

at that if $\Theta^{(\vec{e})}$ does not identically equal zero , then it has no other zeros, and there exists isomorphism between the vectors $\vec{e}$ and $\{\dot{\zeta}_k\}_{k=1}^g$.

4°. Asymptotics of the orthogonal polynomials.

Let us define vector of constants with the condition

$$\vec{e} \quad : \quad \Theta^{(\vec{e})}(\pi_1^{-1}(z_k^*)) = 0\ , \quad k = 1, \ldots, g\ ,$$

where $\pi_1^{-1}(z)$ denotes a raising of a point $z$ from the complex plane to $\mathfrak{R}_1$ sheet of the Riemann surface (10), (correspondingly, $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$, and we recall (5)).

We take the vector of constants (see (7), (9))

$$\vec{c}_{nw} := \left( n\omega_1 + c_w^{(1)}, \ldots, n\omega_g + c_w^{(g)} \right) ,$$

and define the function

$$T^{(\vec{e},\, \vec{c}_{n,w})}(\zeta) := \frac{\Theta^{(\vec{e}+\vec{c}_{n,w})}(\zeta)}{\Theta^{(\vec{e})}(\zeta)} =: \, (T_0, T_1) ,$$

where $T_0$ and $T_1$ denotes the values of the function $T^{(\vec{e},\, \vec{c}_{n,w})}$, correspondingly on the zeroth and first sheets of Riemann surface (10).

Finally, let us designate (see (4), (8) )

$$X_P(z) \; := \frac{F(z)}{F(\infty)} \frac{T_0(z)}{T_0(\infty)} , \quad X_R(z) \; := \frac{ih(z)}{F(z)F(\infty)} \frac{T_1(z)}{T_0(\infty)} .$$

We prove

**Theorem 2.1** *Using introduced notations, polynomials* (2), *which are orthogonal with respect to weight* (1), *and the second kind functions* (3) *have the following asymptotics:*

$$\begin{cases} P_n(z) = (C_\triangle \Phi(z))^n X_P(z) \; (1 + O(\frac{1}{n})) \\[2ex] R_n(z) = \left( \dfrac{C_\triangle}{\Phi(z)} \right)^n X_R(z) \; (1 + O(\frac{1}{n})) \end{cases} ,$$

*uniformly on the compact sets* $\quad z \in K \Subset \Omega$ , *and*

$$\begin{cases} P_n(x) = \left( \{(C_\triangle \Phi(x))^n X_P(x)\}_+ + \{(C_\triangle \Phi(x))^n X_P(x)\}_- \right) \; (1 + O(\frac{1}{n})) \\[2ex] R_{n\pm}(x) = \left( \dfrac{C_\triangle}{\Phi(x)} \right)^n_{\pm} X_{R\pm}(x) \; (1 + O(\frac{1}{n})) \end{cases} ,$$

*uniformly on the compact sets* $\quad x \in K \Subset \triangle$ .

**Remark**. One should note, that the obtained formulas differs from the corresponding ones for the polynomials which are orthogonal on the one interval with presence of theta-functions $T_0$ and $T_1$ ratio. Besides, zeros of the theta-function $\Theta^{(\vec{e}+\vec{c}_{n,w})}$ are distributed between "spurious" zeros of $P_n$ and "spurious" additional interpolations – finite zeros of $R_n$.

# 3 Proof of the Theorem

## 3.1 Statement of the matrix boundary value problem

It is well-known (and can be easily shown on the base of (2),(3)), that the matrix

$$\begin{pmatrix} P_n & R_n \\ \frac{1}{m_{n-1}}P_{n-1} & \frac{1}{m_{n-1}}R_{n-1} \end{pmatrix} =: Y \quad , \tag{14}$$

is the unique solution of the following matrix Riemann-Hilbert problem:

$$\begin{cases} Y \in H^{2\times 2}(\mathbb{C}\backslash\triangle) \ , \ \exists\, Y \in C(\overset{\circ}{\triangle}) \ , \\ Y_+ = Y_- W \ \text{on} \ \overset{\circ}{\triangle} \ , \\ Y(z)\Big|_{z\to e} = \begin{pmatrix} O(1) & O(\varepsilon_e) \\ O(1) & O\varepsilon_e) \end{pmatrix} , \\ Y(z)\Big|_{z\to\infty} = \left(I + O\left(\frac{1}{z}\right)\right) \ diag\left\{z^n, z^{-n}\right\} \ , \end{cases} \tag{15}$$

Here $\overset{\circ}{\triangle}$ is the set of interior points of the segments $\triangle$, $e$ belongs to the set of endpoints $\triangle$ : $e \in \bigcup\limits_{j=1}^{g+1} \{a_j, b_j\}$, and $\varepsilon_e$ depends on the singularity exponent $\alpha_e$ of the weight function $w$ (see (1)) at the endpoint $e$

$$\varepsilon_e \ := \begin{cases} |z-a_e|^{\alpha_e} & \alpha_e \in (-1,0) \\ \log|z-a_e| & \alpha_e = 0 \\ 1 & \alpha_e > 0 \end{cases} \quad , \qquad \alpha_e \in \bigcup_{j=1}^{g+1} \{\alpha_j, \beta_j\} \, .$$

At last, the jump matrix is:

$$W := \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \, .$$

Our goal is to find asymptotics of the problem (15) solution for $n \to \infty$.

## 3.2 Geometry of the problem

On the Riemann surface $\mathfrak{R}$ there exists unique (up to additive constant) Abel integral $G$:

$$\begin{cases} a)\, G \in \mathcal{A}\left(\mathfrak{R}\backslash\{\infty^{(0)}, \infty^{(1)}\}\right) \\ \\ b)\, G(\zeta) = \begin{cases} \ \ \log\zeta + O(1)\, , \ \zeta \to \infty^{(0)} \\ -\log\zeta + O(1)\, , \ \zeta \to \infty^{(1)} \end{cases} \quad , \\ \\ c)\, g := Re\, G - \text{is single valued on } \mathfrak{R} \end{cases} \tag{16}$$

and the additive constant can be fixed by condition:

$$c')g\left(z^{(0)}\right)+g\left(z^{(1)}\right)=:g_0+g_1=0\ .$$

Let us note, that $g_0(z)$ is Green function of zeroth sheet of $\mathfrak{R}_0$, the function

$$h:=G'\in\mathfrak{M}(\mathfrak{R}) \tag{17}$$

is single valued meromorphic function on $\mathfrak{R}$ (it can be considered as continuation of the defined in (4) function $h$ from $\mathfrak{R}_0$ to the whole Riemann surface (10)), it has $g$ finite zeros on $\mathfrak{R}_0$ (see (5)),

$$\{z_k^*\}_{k=1}^g\ :\ h_0(z_k^*)=0\ ,\ k=1,\ldots,g\ . \tag{18}$$

We denote the splitted by $\mathbf{a}$-cycles Riemann surface as follows:

$$\widehat{\mathfrak{R}}:=\mathfrak{R}\backslash\bigcup_{k=1}^g\mathbf{a}_k\ . \tag{19}$$

By means of Abel integral (16) we define on $\mathfrak{R}$ the function

$$\Phi:=e^G\quad . \tag{20}$$

We have

$$1)\quad \Phi\in\mathfrak{M}(\widehat{\mathfrak{R}})\ ,$$

$$2)\quad \begin{cases}\Phi_0(z)=\frac{z}{c_0}+\ldots\\[2ex] \Phi_1(z)=\frac{1}{c_1z}+\ldots\end{cases}\quad ,\quad z\to\infty \tag{21}$$

$$3)\quad \Phi_0\Phi_1\equiv 1\quad\text{in}\quad \overline{\mathbb{C}}\ .$$

(Here the lower index represents the branch of the function, i.e., from which sheet of $\mathfrak{R}$ the values are being taken). It is clear that the values of $\Phi$ on the zeroth sheet of $\widehat{\mathfrak{R}}$ (i.e. $\Phi_0$), coincide with the values of the defined in (6) function $\Phi$, being considered at $\mathbb{C}\backslash a$, where $a$ is the projection of $\mathbf{a}$-cycles from the zeroth sheet of $\mathfrak{R}$ to $\mathbb{C}$:

$$a:=\bigcup_{k=1}^g a_k\ :\ a_k=\pi(\mathbf{a}_k^{(0)})\ ,\ k=1,\ldots,g\,. \tag{22}$$

At the same time (see (21),(6) and (7)) we have $c_0=C_\triangle$ and

$$\mathop{\triangle}_{\triangle_k}\Phi_0=2\pi\omega_k\ ,\ k=1,\ldots,g\ . \tag{23}$$

### 3.3 Normalization of the initial BVP at the point $\infty$

Let us proceed from the BVP (15) to the BVP for the function

$$Z := diag\,(c_0^{-n}, c_1^{-n})\, Y\, diag\,(\Phi_0^{-n}, \Phi_1^{-n})\ , \tag{24}$$

here $\{c_0, c_1\}$ are normalization constants from 2)-(21). We have

$$\begin{cases} Z \in H(\overline{\mathbb{C}} \backslash (\triangle \bigcup a)) \\ \\ Z_+ = Z_- J \text{ on } \triangle \bigcup a \\ \\ Z(z) = I + O\left(\frac{1}{z}\right) \quad , \quad z \to \infty \end{cases} , \tag{25}$$

where

$$J := \begin{pmatrix} \dfrac{\Phi_{0+}^{-n}}{\Phi_{0-}^{-n}} & \dfrac{\Phi_{1+}^{-n}}{\Phi_{0-}^{-n}} w \\ \\ 0 & \dfrac{\Phi_{1+}^{-n}}{\Phi_{1-}^{-n}} \end{pmatrix} \text{ on } \triangle\ , \quad J := diag\left(\frac{\Phi_{0+}^{-n}}{\Phi_{0-}^{-n}}, \frac{\Phi_{1+}^{-n}}{\Phi_{1-}^{-n}}\right) \text{ on } a\ .$$

Taking into account the properties of $\Phi$ (см. (21),(23)), we transform the jump

$$J\ := \begin{cases} \begin{pmatrix} \left(\dfrac{\Phi_{0+}}{\Phi_{0-}}\right)^{-n} & w \\ \\ 0 & \left(\dfrac{\Phi_{1+}}{\Phi_{1-}}\right)^{-n} \end{pmatrix} & \text{on } \triangle \\ \\ diag\left(e^{2\pi i n \omega_k},\ \ e^{-2\pi i n \omega_k}\right) & \text{on } a_k\ ,\ \ k = 1, \ldots, g \end{cases} . \tag{26}$$

### 3.4 Opening of the local lenses

Let $\triangle_k^{(+)}$ be a Jordan arc in the upper half-plane connecting the points $\{a_k, b_k\}$, $\triangle_k^{(-)}$ - be the similar arc in lower half-plane, $L_k^{(+)}$ and $L_k^{(-)}$ be lens-shaped domains, bounded with $\partial L_k^{(+)} = \triangle_k^{(+)} \bigsqcup \triangle_k$ and $\partial L_k^{(-)} = \triangle_k^{(-)} \bigsqcup \triangle_k$, $k = 1, \ldots, g+1$ correspondingly. Let us designate

$$\triangle^{(\pm)} := \bigsqcup_{i=1}^{g+1} \triangle_i^{(\pm)}\ , \quad L^{(\pm)} := \bigsqcup_{i=1}^{g+1} L_i^{(\pm)}\ ,$$

and introduce matrix-function

$$D := \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\left(\frac{\Phi_0}{\Phi_1}\right)^{-n} & 1 \end{pmatrix}. \tag{27}$$

Let us also define the function

$$\widehat{Z} := \begin{cases} z\,D^{-1} \text{ in } L^{(+)} \\ z\,D \quad \text{in } L^{(-)} \\ z \qquad \text{in } \overline{\mathbb{C}} \setminus \left\{L^{(+)} \bigcup L^{(-)}\right\} \end{cases}. \tag{28}$$

The formulas (25)-(26) yield that $\widehat{Z}$ satisfy the following BVP:

$$\begin{cases} \widehat{Z} \in H\left(\overline{\mathbb{C}} \setminus \{\triangle \bigcup \triangle^{(+)} \bigcup \triangle^{(-)} \bigcup a\}\right) \\ \widehat{Z}_+ = \widehat{Z}_-\,\widehat{J} \quad \text{on } \{\triangle \bigcup \triangle^{(+)} \bigcup \triangle^{(-)} \bigcup a\} \\ \widehat{Z}(z) = I + O\left(\frac{1}{z}\right)\ ,\ z \to \infty \end{cases}. \tag{29}$$

with the jump

$$\widehat{J} := \begin{cases} D \quad \text{on } \triangle^{(+)} \bigcup \triangle^{(-)} \\ \widetilde{W} \quad \text{on } \triangle \\ J \quad \text{on } a \end{cases}, \tag{30}$$

where

$$\widetilde{W} := \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix}. \tag{31}$$

The fact that the jumps on the contours $a$ are identical, both inside and outside the lenses, follows from the identity

$$J\,D_+ \;=\; D_-\,J \quad \text{on } a\,.$$

### 3.5 The limit external BVP

If we address to the explicit form (27) of the jump $D$ on the external boundary of the lenses $\triangle^{(+)} \bigcup \triangle^{(-)}$, we see that beyond the endpoints this jump tends to the identity matrix when $n \to \infty$. Because of this, for obtaining the asymptotics for the solutions of problems (29)-(30) we need to solve the following BVP:

$$X: \quad \begin{cases} X \in \ H\left(\overline{\mathbb{C}} \backslash \{\triangle \bigcup a\}\right) \\ \\ X_+ = X_- \begin{cases} \widetilde{W} \text{ at } \triangle \\ J \text{ at } a \end{cases} \\ \\ X(z) = I + O\left(\frac{1}{z}\right) \end{cases} , \tag{32}$$

where, $\widetilde{W}$ is given in (31)

$$\widetilde{W} := \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix} \quad \text{on } \triangle ,$$

and the jump on the projection **a**-cycle (see (26)) is

$$J := diag\left(e^{2\pi i n\omega_k}, e^{-2\pi i n\omega_k}\right) \quad \text{on } a_k \ , \quad k = 1, \ldots, g \ .$$

The solution of this problem constitutes indeed the main (methodical) content of this paper.

#### 3.5.1 Szegö function (from the viewpoint of Riemann surface)

By means of the function $h$ defined in (17), we modify the weight function $w$ (see (1)), defined on $\triangle$:

$$w_h := i\frac{w}{h_{0-}} \quad \text{on} \quad \triangle \tag{33}$$

($w_h$ is analogue of "trigonometrical" weight for the polynomials orthogonal on the segment.)

Let us define the function $w_h$ on the contour $\partial\mathfrak{R}_{01}$ separating the two sheets of $\mathfrak{R}$:

$$w_h := \begin{cases} w_h & \text{on} & \triangle_+ \\ w_h & \text{on} & \triangle_- \end{cases} .$$

Let $d\widehat{\omega}_{\xi_1\xi_2}(\zeta)$ be meromorphic, single valued on $\hat{\mathfrak{R}}$ differential with simple poles at the points $\xi_1$, $\xi_2$ and with residues $+1$ и $-1$ there correspondingly. Differential of such kind can be called Cauchy differential (see [11]).

$$\frac{d}{d\zeta}\widehat{\omega}_{\xi_1\xi_2}(\zeta) = \begin{cases} \frac{1}{\zeta-\xi_1} \ + \ O(1) \, , & \zeta \to \xi_1 \\ \\ \frac{-1}{\zeta-\xi_2} \ + \ O(1) \, , & \zeta \to \xi_2 \end{cases} . \tag{34}$$

Let us designate $\widetilde{\xi}$ the point of $\mathfrak{R}$ which has the same projection on $\overline{\mathbb{C}}$ as the point $\xi$, but belongs to the other sheet:

$$\xi, \widetilde{\xi} \in \mathfrak{R} \quad : \quad \pi(\xi) = \pi(\widetilde{\xi}) \; , \quad \xi \neq \widetilde{\xi} \; .$$

<u>Definition.</u> We shall call piecewise-holomorphic on $\widehat{\mathfrak{R}} \backslash \partial\mathfrak{R}_{01}$ function

$$\mathcal{F}(\xi) := \exp\left\{ \frac{1}{4\pi i} \int\limits_{\partial\mathfrak{R}_{01}} \ln w_h(\zeta)\, d\widehat{\omega}_{\xi\widetilde{\xi}}(\zeta) \right\} \; , \quad \xi \in \widehat{\mathfrak{R}} \backslash \partial\mathfrak{R}_{01} \tag{35}$$

Szegö function of the weight function $w$.

Let us note the main properties of $\mathcal{F}$:

$$\begin{cases} 1)\; \mathcal{F}(\xi)\mathcal{F}(\widetilde{\xi}) \equiv 1 \; , \; \forall \xi \in \mathfrak{R} \; , \\ \\ 2)\; \mathcal{F}_{+} = \mathcal{F}_{-} w_h \; \text{ on } \; \partial\mathfrak{R}_{01} \; , \\ \\ 3)\; \mathcal{F}_{+} = \mathcal{F}_{-}\, e^{2\pi i c_w^{(k)}} \; \text{ on } \; \mathbf{a}_k \; , \; \; k = 1, \ldots, g \; , \end{cases} \tag{36}$$

here the constants $c_w^{(k)}$ have the form:

$$c_w^{(k)} := -\frac{1}{2\pi i} \int\limits_{\partial\mathfrak{R}_{01}} \ln w_h(\zeta)\, d\Omega_k(\zeta) \; , \tag{37}$$

and $\{d\Omega_k(\zeta)\}_{k=1}^{g}$ is the basis of normalized holomorphic (of 1-st kind) Abel differentials (see (11)):

$$\int\limits_{\mathbf{a}_i} d\Omega_k(\zeta) = \delta_{i,k} \; , \quad \int\limits_{\mathbf{b}_i} d\Omega_k(\zeta) = B_{i,k} \; , \; i, k = 1, \ldots, g \; , \tag{38}$$

here $\delta_{i,k}$ is Kronecker symbol, and the matrix $\|B_{i,k}\|$ is symmetric and has positive defined imaginary part.

$\square$ Let us prove the properties (36) - (see details in [11]).

1) All the residues of the differential $d\widehat{\omega}_{\xi\widetilde{\xi}}(\zeta) + d\widehat{\omega}_{\widetilde{\xi}\xi}(\zeta)$ equals zero.

2) From (35), taking into account 1), we have:

$$\frac{1}{2\pi i} \int\limits_{\partial\mathfrak{R}_{01}} \ln w_h(\zeta)\, d\widehat{\omega}_{\xi\widetilde{\xi}}(\zeta) = 2 \ln \mathcal{F}(\xi) = \ln \mathcal{F}(\xi) - \ln \mathcal{F}(\widetilde{\xi}) \; ,$$

substituting into the left part

$$\ln w_h = \ln \mathcal{F}_{+} - \ln \mathcal{F}_{-} \; ,$$

by means of Sokhocky-Plemelj formulas (or Cauchy Residue Theorem) we obtain the identity proving 2).
3) Follows from the well-known Riemann relation:

$$t \in \mathbf{a}_k \Rightarrow d\widehat{\omega}_{t_+p}(\zeta) - d\widehat{\omega}_{t_-p}(\zeta) = -2\pi i \, d\Omega_k(\zeta) \ , \ k = 1, 2, \dots, g \ . \qquad \blacksquare$$

#### 3.5.2 Limiting external problem with weight-independent jumps (the statement)

Let us consider the branches of Szegö function (defining vector Szegö function on the plane):

$$F_l(z) := \mathcal{F}(z^{(l)}) \ , \ l = 0, 1 \ , \ z \in \mathbb{C} \ , \ z^{(l)} \in \mathfrak{R}_l \ .$$

We have $F_0, \, F_1 \in \ H\left(\overline{\mathbb{C}} \backslash \{\triangle \bigcup a\}\right) \ , \quad F_{0\pm} = F_{1\mp} w_h$ on $\triangle$ , и

$$\begin{cases} F_{0+} = F_{0-} e^{2\pi i c_w^{(k)}} \\ \\ F_{1+} = F_{1-} e^{-2\pi i c_w^{(k)}} \end{cases} , \quad \text{on} \quad a_k \ , \quad k = 1, 2, \dots, g \ .$$

Let us note that $F_0$ coincides with the standard definition of Szegö function (see (8)-(9)).

Let us have

$$F(z) := (F_0(z), \, F_1(z)) \ , \ F_\infty := F(\infty) \ . \tag{39}$$

We define (see (32))

$$\widetilde{X} := F_\infty^{-1} X \, F \ . \tag{40}$$

Then for this function we have the following BVP:

$$\begin{cases} \widetilde{X} \in \ H\left(\overline{\mathbb{C}} \backslash \{\triangle \bigcup a\}\right) \ , \\ \\ \widetilde{X}_+ = \widetilde{X}_- \, H \ \text{ on } \ \triangle \bigcup a \ , \\ \\ \widetilde{X}_\infty = I \qquad , \end{cases} \tag{41}$$

where for the jump matrix $H$, taking into account (33) and the boundary properties of the Szegö vector-function components, we get:

$$H := \begin{cases} \begin{pmatrix} 0 & -ih_{0-} \\ \frac{1}{ih_{0-}} & 0 \end{pmatrix} & \text{on } \triangle \\ \\ diag \left\{ e^{2\pi i (n\omega_k + c_w^{(k)})}, e^{-2\pi i (n\omega_k + c_w^{(k)})} \right\} & \text{on } a_k \ , \ k = 1, \dots, g \ . \end{cases} \tag{42}$$

Thus, we have transformed our problem (32) into the problem (41)-(42) with the jump on $\triangle$ (i.e. through **b**-cycle) being standard (independent of the weight $w$) function $h$, and the jump through the projection of $a$-cycle being *const* for all $a_k\ ,\ k = 1, \ldots, g$.

We begin solution of this problem with constructing the function with the jump $H$ on $\triangle$, and continuous passing through $a$. Next, by means of Riemann theta-function, which is holomorphic on $\widehat{\mathfrak{R}}$ (see (19)), i.e. continuous passing through $b$-cycles, we satisfy the boundary conditions on $a$.

#### 3.5.3 Limiting external problem with weight-independent jumps (preliminaries)

1°. One standard scalar rational function on $\mathfrak{R}$.

On the Riemann surface of zero genus ($\mathbb{C}$, for instance) rational function from $R_n$ class (single valued on this surface rational functions of the degree $n$) could be defined (up to multiplicative constant) via arbitrary setting of the position of $n$ poles and $n$ zeros. It is known (from Abel's theorem), that on the Riemann surface of genus $g$ arbitrary positions could be taken up by all the zeros and poles of rational function except $g$ ones, position of which could be uniquely determined from the position of others (this is why there are no single valued rational functions from the class $R_1$ on the Riemann surface of genus $> 1$).

Let us define on $\mathfrak{R}$ the rational function $\chi$, which is multiple of the divisor

$$\chi \in \mathcal{M}(\mathfrak{R}) \quad : \qquad \frac{(\infty^{(0)})}{(\infty^{(1)}\ \pi_1^{-1}(z_1^*) \ldots \pi_1^{-1}(z_g^*))} \quad \Bigg| \quad \chi\ , \tag{43}$$

i.e. we fix $g+1$ pole (one at the point $\infty^{(1)}$ and $g$ poles at the zeros of $h_0$ function projected on the first sheet (see (18)) and one zero at the point $\infty^{(1)}$. Other zeros ($g$ ones) take some determined positions so that $\chi$ is singlevalued on $\mathfrak{R}$.

Unique function of such kind exists, up to a multiplicative constant, which we fix by the following condition:

$$\chi(\xi) = -\xi + \ldots\ ,\ \xi \to \infty^{(1)}\ . \tag{44}$$

The other (not prescribed) zeros of $\chi$ function on $\mathfrak{R}$ we denote as

$$\{\zeta_j\}_{j=1}^{g}\ :\ \chi(\zeta_j) = 0\ ,\ j = 1, \ldots, g\ . \tag{45}$$

2°.Matrix function with the required jump on $\triangle$.

We start with constructing the solution of the problem (41) with $H\Big|_a := I$. I.e., we look for the function $\tilde{\tilde{X}}$, such that

$$\begin{cases} \tilde{\tilde{X}} \in H(\mathbb{C}\backslash\triangle)\ , \\ \\ \tilde{\tilde{X}}_+ = \tilde{\tilde{X}}_- H \text{ on } \triangle\ , \\ \\ \tilde{\tilde{X}}\Big|_\infty = I\ . \end{cases} \tag{46}$$

Let us consider the function

$$\tilde{\tilde{X}} := \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} = \begin{pmatrix} 1 & ih_0 \\ i\chi_0 & -h_0\chi_1 \end{pmatrix} \quad \text{in } \mathbb{C}\backslash\triangle\ . \tag{47}$$

Holomorphicity condition and normalization at the point $\infty$ for (44) follows from the definitions of the functions $h$ (см. (17)) и $\chi = \{\chi_0, \chi_1\}$, (see (43)-(44)).

Let us check the boundary conditions on $\triangle$:

$$\tilde{\tilde{X}}_+ = \begin{pmatrix} x_{00+} & x_{01+} \\ x_{10+} & x_{11+} \end{pmatrix} = \tilde{\tilde{X}}_- \begin{pmatrix} 0 & ih_{0-} \\ -i\frac{1}{h_{0-}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{ih_{0-}}x_{01-} & -ih_{0-}x_{00-} \\ \frac{1}{ih_{0-}}x_{11-} & -ih_{0-}x_{10-} \end{pmatrix}. \tag{48}$$

We advert to the explicit form (47) of the function $\tilde{\tilde{X}}$ and (taking into account that $h_{0-} = -h_{0+}$ on $\triangle$) we get the identity.

3°. Some minimal information about Riemann theta-function.

Let us remind the basic facts about Riemann theta-function (some of them were already listed in the item 3° of the section 2). Theta-function is the entire function of $g$ complex variables $(u_1, u_2, \ldots, u_g) =: \vec{u}$ :

$$\theta(\vec{u}) := \sum_{m_1,\ldots,m_g}^{-\infty,\ldots,\infty} \exp\left\{\pi\, i \sum_{\mu=1}^{g}\sum_{\nu=1}^{g} B_{\mu\nu}m_\mu m_\nu + 2\pi\, i \sum_{\nu=1}^{g} m_\nu u_\nu\right\}\ ,$$

where all the summation indices $\{m_j\}_{j=1}^g$ changes independently from $-\infty$ to $\infty$, and the matrix $\|B_{\mu\nu}\|$ is symmetric and has positive defined imaginary part. If we take as variables of $\theta(\vec{u})$ $g$ Abel 1-st kind integrals with normalized holomorphic differentials (see (38)):

$$u_1 := \int_{a_{g+1}}^{\zeta} d\Omega_1(t) - e_1 =: \Omega_1(\zeta) - e_1\, , \ldots,\, u_g := \Omega_g(\zeta) - e_g\, ,$$

with arbitrary vector of complex constants

$$\vec{e} := (e_1, \dots, e_g)\ ,$$

and as matrix $B$ – the matrix of $b$-periods of these integrals (38), then we obtain the function of one variable $\zeta \in \mathfrak{R}$ which is called Riemann theta-function of the surface $\mathfrak{R}$:

$$\Theta^{(\vec{e})}(\zeta) := \theta\left(\vec{\Omega}(\zeta) - \vec{e}\right)\ . \tag{49}$$

Let us recall, basic properties of $\Theta^{(\vec{e})}(\zeta)$, (see details in, for instance, [11]) :

1) $\Theta^{(\vec{e})}(\zeta) \in H(\widehat{\mathfrak{R}})\ , \quad \widehat{\mathfrak{R}} := \mathfrak{R} \backslash\, \mathbf{a}\ .$

2) The function $\Theta^{(\vec{e})}(\zeta)$ has precisely $g$ zeros on $\mathfrak{R}$ (if it is not identically equal to zero):

$$\{\zeta_\nu\}\ :\ \Theta^{(\vec{e})}(\zeta_\nu) = 0\ ,\ \nu = 1, \dots, g\ .$$

These zeros are connected with the vector of constants $e$ with the following relation:

$$\sum_{k=1}^{g} \Omega_k(\zeta_\nu) \equiv e_\nu - k_\nu\ ,\ \ mod\,(\text{periods})\ ,\ \ \nu = 1, \dots, g\ , \tag{50}$$

where $k_\nu$ are called *Riemann constants* and have the form (for arbitrary Riemann surface of genus $g$):

$$k_\nu = -\frac{1}{2} + \frac{1}{2} B_{\nu\nu} - \sum_{\substack{j=1 \\ j\neq\nu}}^{k} \int\limits_{a_j} \Omega_{\nu-}(t)\, d\Omega_j(t)\ , \quad \nu = 1, \dots, g\ .$$

Thus, to obtain $\Theta_e(\zeta)$ with the fixed zeros in $\{\zeta_\nu\}_{\nu=1}^{g}$, one should choose the vector of constants by means of (50). Inverse problem of the search for $\{\zeta_\nu\}_{\nu=1}^{g}$ satisfying the system (50) with the given right-hand sides, is called *Jacobi problem of Abel integrals inversion.*

3) On the $\mathbf{a}$-cycles the function $\Theta^{(\vec{e})}(\zeta)$ satisfies the boundary condition

$$\Theta_+^{(\vec{e})}(\zeta) = \Theta_-^{(\vec{e})}(\zeta)\, \exp\{\pi\, i\, B_{jj} + 2\pi\, i\, (\Omega_j^+(\zeta) - e_j\},\ \zeta \in a_j\ ,\ j = 1, \dots, g\ . \tag{51}$$

Let us note that the problems (50) and (51) are equivalent in the following sense: the aggregate of zeros of each nontrivial solution for BVP (51) on $\mathfrak{R}$ forms the solution for Jacobi problem (50) and, conversely, one can consider each solution for (50) as the aggregate of zeros of some nontrivial solution for problem (51). Let us also emphasize the following fact that is useful for applications: for arbitrary vector of constants

$$\vec{c} := (c_1, \ldots, c_g) \ ,$$

the meromorphic on $\widehat{\mathfrak{R}}$ function (which is multiple of the $(\{\zeta_\nu\})^{-1}$ see (50)) :

$$T^{(\vec{e},\,\vec{c})}(\zeta) := \frac{\Theta^{(\vec{e}+\vec{c})}(\zeta)}{\Theta^{(\vec{e})}(\zeta)} \quad \Big| \quad (\{\zeta_\nu\})^{-1} \tag{52}$$

on the **a**-cycles has the constant (independent of $\zeta$) jumps:

$$T_+^{(\vec{e},\,\vec{c})} = T_-^{(\vec{e},\,\vec{c})}\, e^{-2\pi\, i\, c_\nu} \quad \text{on} \quad \mathbf{a}_\nu \ , \quad \nu = 1, \ldots, g \ . \tag{53}$$

#### 3.5.4 Solution of the limiting external BVP

Let us modify the function $\widetilde{\widetilde{X}}$ (see (47))

$$\widetilde{\widetilde{X}} := \begin{pmatrix} 1 & ih_0 \\ i\chi_0 & -h_0\chi_1 \end{pmatrix} ,$$

keeping all the properties of the problem (46), except continuity on $a$-cycles, so that that modified function acquires the constant jumps $H$ on the projections of **a**-cycles (see (42))

$$H := diag\left\{ e^{2\pi i (n\omega_k + c_w^{(k)})}, e^{-2\pi i (n\omega_k + c_w^{(k)})} \right\} \quad \text{on } a_k \ , \ k = 1, \ldots, g \ .$$

Let us remind the notations. Projections of zeros of the function $h$ (see (18)) are

$$\{\pi_1^{-1}(z_k^*)\}_{k=1}^g \ ,$$

and we denote the zeros $\chi$ on $\mathfrak{R}_1$ as

$$\{\zeta_k\}_{k=1}^g \ .$$

By means of (50) we define two vectors of constants

$$\begin{array}{lll} e^* & : \ \Theta^{(e^*)}(\pi_1^{-1}(z_k^*)) & = 0 \\ \widetilde{e} & : \ \Theta^{(\widetilde{e})}(\zeta_k) & = 0 \end{array} , \quad k = 1, \ldots, g \ . \tag{54}$$

Let us denote also (see (42)) vector of constants

$$\vec{c}_{n,w} := \left( n\omega_1 + c_w^{(1)}, \ldots, n\omega_g + c_w^{(g)} \right) =: (c_{n,w}^{(1)}, \ldots, c_{n,w}^{(g)}) \ .$$

We consider two meromorphic functions on $\widehat{\mathfrak{R}}$ (see (52)):

$$\begin{array}{lll} T^{(e^*,\,\vec{c}_{n,w})}(\zeta) & := & \dfrac{\Theta^{(e^*+\vec{c}_{n,w})}(\zeta)}{\Theta^{(e^*)}(\zeta)} \\ T^{(\widetilde{e}\,,\,\vec{c}_{n,w})}(\zeta) & := & \dfrac{\Theta^{(\widetilde{e}+\vec{c}_{n,w})}(\zeta)}{\Theta^{(\widetilde{e})}(\zeta)} \end{array} , \quad \zeta \in \widehat{\mathfrak{R}} \ .$$

In view of (53) both of these functions have on **a**-cycle of $\mathfrak{R}$ the jump

$$T_+ = T_- e^{-2\pi i\,(n\,\omega_k + c_w^{(k)})} \quad \text{on} \quad \mathbf{a}_k, \quad k = 1, \ldots, g\ . \tag{55}$$

Finally, let us denote the diagonal matrix of constants:

$$T_\infty^{-1} := \left\{ \left( T_0^{(e^*,\ \vec{c}_{n,w})}(\infty) \right)^{-1},\ \left( T_1^{(\widetilde{e},\ \vec{c}_{n,w})}(\infty) \right)^{-1} \right\}\ , \tag{56}$$

here lower index denotes the branch of the function (i.e. from which sheet of $\mathfrak{R}$ the values are being taken).

Let us consider the function

$$\widetilde{X}(z) := T_\infty^{-1} \begin{pmatrix} T_0^{(e^*,\ \vec{c}_{n,w})}(z) & ih_0(z) T_1^{(e^*,\ \vec{c}_{n,w})}(z) \\ i\chi_0(z)\, T_0^{(\widetilde{e},\ \vec{c}_{n,w})}(z) & -h_0(z)\chi_1(z)\, T_1^{(\widetilde{e},\ \vec{c}_{n,w})}(z) \end{pmatrix}\ . \tag{57}$$

Now let us check that it is indeed the solution of the problem (41). First condition (holomorphicity) follows from (54), (43). Third condition (normalization) is consequence of (56) and 3) in (46). Let us check the jumps.

First, we look at the jump for the diagonal matrix $H$ on the projection of **a**-cycle from $\mathfrak{R}_0$ to $\mathbb{C}$. We have for $a_k$

$$\widetilde{X}_+ := T_\infty^{-1} \begin{pmatrix} T_{0+} & ih_0 T_{1+} \\ i\chi_0\, T_{0+} & -h_0\chi_1\, T_{1+} \end{pmatrix} =$$

$$= T_\infty^{-1} \begin{pmatrix} T_{0-} e^{-2\pi i\, c_{n,w}^{(k)}} & ih_0 T_{1-} e^{2\pi i\, c_{n,w}^{(k)}} \\ i\chi_0\, T_{0-} e^{-2\pi i\, c_{n,w}^{(k)}} & -h_0\chi_1\, T_{1-} e^{2\pi i\, c_{n,w}^{(k)}} \end{pmatrix}.$$

Since that the projection of $a$-cycle from the sheet $\mathfrak{R}_1$ changes its orientation, (55) yields the identity.

Now we look at the jump on $\triangle$ (projection of **b**-cycle). The functions $T$ (see (52)), as well as $\chi$, are continuous on **b**-cycles, i.e.,

$$T_{0\pm} = T_{1\mp}\ , \quad \chi_{0\pm} = \chi_{1\mp} \quad \text{on} \quad \triangle\ ;$$

since that the role of $T$ (57) is analogical to the role of $\chi$, the jump of the matrix (57) on $\triangle$ is the same as one of the matrix $\widetilde{\widetilde{X}}$ (see (48)).

Thus, (57) is indeed solution of (41). Summing up, we see that the function

$$X := F_\infty \widetilde{X}\, F^{-1}\ , \tag{58}$$

where $F_\infty, F$ are defined in (39), and $\widetilde{X}$ is defined in (57), is the solution of external limiting BVP (32).

### 3.6 Local BVPs

Let us get back to the BVP (29)-(31), (27). Since the jump $D$ (see (27)) does not tend uniformly to the identity matrix at the neighborhood of the $\triangle$ endpoints, we should consider the local BVPs at the neighborhood of each endpoint. Let $e \in \{a_j, b_j\}_{j=1}^{g}$; we consider the neighborhood $O_e$ of the point $e$. We look for the solution of the following local BVP:

$$\begin{cases} U_e \in H(O_e \backslash \{\triangle_e \bigcup \triangle_e^{(+)} \bigcup \triangle_e^{(-)}\}) , \\ \\ U_{e+} = U_{e-} \begin{cases} D & \text{on } \triangle_e^{(+)} \bigcup \triangle_e^{(-)} \\ \widetilde{W} & \text{on } \triangle_e \end{cases} , \\ \\ U_e = (I + o(1))X \quad \text{on} \quad \partial O_e \quad . \end{cases} \tag{59}$$

Let us note that the problem (59) does not differ significantly from the corresponding local BVP for the case when the set $\triangle$ consists of the single segment. The only distinction is that in the boundary condition on the external boundary function $X$ (solution of the external limit BVP) is different. For the case of one segment and orthogonality weight (1) local BVP is solved in [12]. In order to obtain solution of the problem (59), we take the corresponding solution from [12] and change $X$ in it to $X$ from (58).

We have

$$U_e = E_e V_e A_e \ ,$$

where

$$A_e = diag \left\{ \left( \Phi_0^{-n} w^{1/2} \right)^{-1} , \ \Phi_0^{-n} w^{1/2} \right\} ,$$

$$E_e := \frac{1}{2} X \, diag \, (w^{1/2}, w^{-1/2}) \, M_e \, diag \left( \sqrt{\pi \, n \, \varphi}, \ \frac{1}{\sqrt{\pi \, n \, \varphi}} \right) \ ;$$

here

$$M_{a_j} := \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \ M_{b_j} := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \ j = 1, \ldots, g+1, \ \varphi(z) = -2 \log \Phi_0(z),$$

and in order to give the expression for $V_e$, we introduce the matrices

$$\Psi_{a_i} := \begin{pmatrix} I_{\alpha_i}\left(\frac{n\varphi}{2}\right) & \frac{i}{\pi} K_{\alpha_i}\left(\frac{n\varphi}{2}\right) \\ \\ n\pi \, i \, \varphi_1 I'_{\alpha_i}\left(\frac{n\varphi}{2}\right) & -n \, \varphi_1 \, K'_{\alpha_i}\left(\frac{n\varphi}{2}\right) \end{pmatrix} ,$$

and $\Psi_{b_i}$ (the same matrix, as for $\Psi_{a_i}$, but with $\alpha_i$ changed to $\beta_i$ second column multiplied on -1). Now one can define $V_e$ in the sectors $O_e^* \cup O_e^{(+)} \cup O_e^{(-)}$ by the use of the following formulas:

$$V_{a_i} := \begin{cases} \Psi_{a_i} & \text{in } O^*_{a_i} \\ \Psi_{a_i}\begin{pmatrix} 1 & 0 \\ e^{\alpha_i \pi i} & 1 \end{pmatrix} & \text{in } O^{(+)}_{a_i} \\ \Psi_{a_i}\begin{pmatrix} 1 & 0 \\ -e^{-\alpha_i \pi i} & 1 \end{pmatrix} & \text{in } O^{(-)}_{a_i} \end{cases}, \quad V_{b_i} := \begin{cases} \Psi_{b_i} & \text{in } O^*_{b_i} \\ \Psi_{b_i}\begin{pmatrix} 1 & 0 \\ -e^{-\beta_i \pi i} & 1 \end{pmatrix} & \text{in } O^{(+)}_{b_i} \\ \Psi_{b_i}\begin{pmatrix} 1 & 0 \\ e^{\beta_i \pi i} & 1 \end{pmatrix} & \text{in } O^{(-)}_{b_i} \end{cases},$$

here $O^{(\pm)}_e := L^{(\pm)} \cap O_e$, $O^*_e := O_e \setminus \overline{\{O^{(+)}_e \cup O^{(-)}_e\}}$.

### 3.7 The final transform and the asymptotic formulas

Let us consider the function

$$J := \begin{cases} \widehat{Z}X^{-1} & \text{in } \overline{\mathbb{C}}\setminus\left\{\bigcup\limits_{j=1}^{g+1}(O_{a_j}\bigcup O_{b_j})\right\} \\ \widehat{Z}U^{-1}_{e_j} & \text{in } O_{e_j}\,,\ e_j \in \{a_j, b_j\}_{j=1}^{g+1} \end{cases}. \tag{60}$$

Analysis of the problems (59), (32) and (29) yields that

$$J \in H\left(\overline{\mathbb{C}} \setminus \left\{(\triangle^{(+)} \cup \triangle^{(-)}) \setminus \left\{\bigcup_{j=1}^{g+1} O_{a_j} \cup \bigcup_{j=1}^{g+1} O_{b_j}\right\}\right\} \setminus \left\{\bigcup_{j=1}^{g+1} \partial O_{a_j} \cup \bigcup_{j=1}^{g+1} \partial O_{b_j}\right\}\right).$$

Jump on the break lines uniformly tends to the unitary matrix when $n \to \infty$

$$J_+ = J_- \widetilde{I}_n \text{ on } \sum_0, \quad \widetilde{I}_n \rightrightarrows I\,,\ n \to \infty\,,$$

and, eventually,

$$J(\infty) = 1.$$

The standard arguments (see [12]) make it possible to conclude:

$$J \rightrightarrows I \quad \text{in } \overline{\mathbb{C}}, \quad \text{when } n \to \infty\,. \tag{61}$$

Substituting (14) into (24) and further into (28), we obtain

$$\widehat{Z} = \begin{cases} \begin{pmatrix} \dfrac{P_n}{(C_1\Phi_1)^{-n}} & \dfrac{\Phi_1^{-n}}{C_1^{-n}}R_n \\ * & * \end{pmatrix} & \text{on } K \Subset \overline{\mathbb{C}}\setminus\{L^{(+)}\bigcup L^{(-)}\} \\ \begin{pmatrix} \dfrac{P_n}{(C_1\Phi_1)^{-n}} \mp \dfrac{R_n}{(C_1\Phi_1)^{-n}w} & \dfrac{\Phi_1^{-n}}{C_1^{-n}}R_n \\ * & * \end{pmatrix} & \text{on } K \Subset \{L^{(\pm)}\} \end{cases}, \tag{62}$$

and, in view of (60), (61), we get

$$\widehat{Z} = (I + O(\frac{1}{n}))X\,, \quad n \to \infty\,,$$

which yields the following asymptotic formulas:

$$\begin{cases} P_n(z) = (C_1\Phi_1(z))^{-n} X_{11}(z)\,(1 + O(\frac{1}{n})) \\ \\ R_n(z) = \left(\dfrac{C_1}{\Phi_1(z)}\right)^{-n} X_{12}(z)\,(1 + O(\frac{1}{n})) \end{cases} , \quad z \in K \Subset \overline{\mathbb{C}} \backslash \triangle\,,$$

as well as on $\triangle$ (62) easily yields

$$R_{n\pm}(x) = \left(\frac{C_1}{\Phi_{1\pm}(x)}\right)^{-n} X_{12\pm}(z)\,(1 + O(\frac{1}{n}))\,,$$

besides, for $P_n$ from (62) on $\triangle$ we get

$$P_n = \left(\frac{R_{n+}}{w} + (C_1\Phi_{1+})^{-n} X_{11+}\right)\,(1 + O(\frac{1}{n})) =$$

$$= \left(\left(\frac{C_1}{\Phi_{1+}}\right)^{-n} \frac{X_{12+}}{w} + (C_1\Phi_{1+})^{-n} X_{11+}\right)\,(1 + O(\frac{1}{n}))\,.$$

From the explicit form of the matrix $X$ it is clear that

$$X_{12+} = wX_{11-} \text{ on } \triangle\,,$$

which yields that uniformly for $x \in K \Subset \triangle$

$$P_n(x) = \left\{(C_1\Phi_{1+}(x))^{-n} X_{11+}(x) + C_1\Phi_{1-}(x))^{-n} X_{11-}(x)\right\}\,(1 + O(\frac{1}{n}))\,.$$

Let us give the explicit form of the first row of the matrix $X$ (see (57), (58)):

$$X_{11}(z) := \frac{F_0(\infty)}{F_0(z)}\,\frac{T_0^{(e^*,\,\vec{c}_{nw})}(z)}{T_0^{(e^*,\,\vec{c}_{nw})}(\infty)}\,;\; X_{12}(z) := ih_0\frac{F_0(\infty)}{T_0^{(e^*,\,\vec{c}_{nw})}(\infty)}\,\frac{T_1^{(e^*,\,\vec{c}_{nw})}(z)}{F_1(z)}\,,$$

and remind the boundary properties of the functions $F_0, F_1$:

$$F_{0\pm} = F_{1\mp} i\frac{w}{h_{0-}} \text{ on } \triangle\,,$$

as well as explicit form of the function $T^{(e^*,\,\vec{c}_{n,w})}$

$$T^{(e^*,\,\vec{c}_{n,w})}(\zeta) := \frac{\Theta^{(e^* +\,\vec{c}_{n,w})}(\zeta)}{\Theta^{e^*}(\zeta)}\,.$$

This completes the proof of the Theorem. $\blacksquare$

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