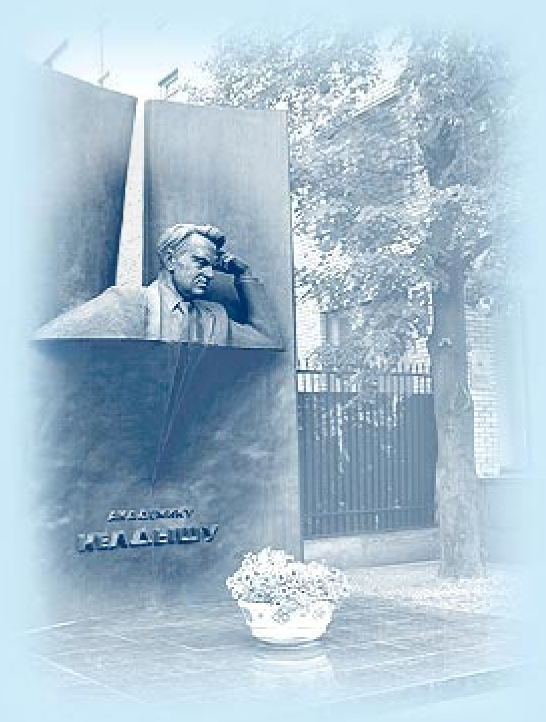

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POWER GEOMETRY
AS A NEW MATHEMATICS

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Abstract. Here we present basic ideas and algorithms of Power Geometry and give a survey of some of its applications. In Section 1, we consider one generic ordinary differential equation and demonstrate how to find asymptotic forms and asymptotic expansions of its solutions. In Section 2, we demonstrate how to find expansions of solutions to Painlevé equations by this method, and we analyze singularities of plane oscillations of a satellite on an elliptic orbit. In Section 3, we expound the space generalizations of constructions of Section 1 and give some applications to Mechanics and Hydromechanics.

Introduction

Traditional differential calculus is effective for linear and quasilinear problems. It is less effective for essentially nonlinear problems. A linear problem is the first approximation to a quasilinear problem. The linear problem is usually solved by methods of functional analysis, then the solution to the quasilinear problem is found as a perturbation of the solution to the linear problem. For an essentially nonlinear problem, we need to isolate its first approximations, to find their solutions, and to construct perturbations of these solutions. This is what Power Geometry (PG) is aimed at. For equations and systems of equations (algebraic, ordinary differential, and partial differential), PG allows to compute asymptotic forms of solutions as well as asymptotic and local expansions of solutions at infinity and at any singularity of the equation (including boundary layers and singular perturbations) [1].

Elements of plane PG were proposed by Newton for algebraic equations (1680); and by Briot and Bouquet for ordinary differential equations (1856). Space PG for a nonlinear autonomous system of ODEs were proposed by the author (1962), and for a linear PDE, by Mikhailov (1963).

In this talk we intend to give basic notions of PG, present some of its algorithms, results, and applications. It is clear that this calculus cannot be mastered during this presentation. This talk consists of two parts: (1) plane PG (Sections 1 and 2), and (2) space PG (Section 3). In each part, we first outline the theory, then describe some applications.

1. Plane Power Geometry. The theory [2]

First, consider one differential equation and power-logarithmic expansions of its solutions (although there possible more complex expansions).

1.1. Statement of the problem. Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A *differential monomial* $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$. A sum of differential monomials

$$f(x, y) = \sum a_i(x, y) \tag{1.1}$$

is called the *differential sum*.

The main problem. Let a differential equation be given

$$f(x, y) = 0, \tag{1.2}$$

where $f(x, y)$ is a differential sum. As $x \to 0$, or as $x \to \infty$, for solutions $y = \varphi(x)$ to the equation (1.2), find all expansions of the form

$$y = c_r x^r + \sum c_s x^s, \quad c_r = \text{const} \in \mathbb{C}, \quad c_r \neq 0, \tag{1.3}$$

where c_s are polynomials in $\log x$, and power exponents $r, s \in \mathbb{R}$,

$$\omega r > \omega s, \tag{1.4}$$

$$\omega = \begin{cases} -1, & \text{if } \ x \to 0, \\ \ \ 1, & \text{if } \ x \to \infty. \end{cases} \tag{1.5}$$

The procedure to compute expansions (1.3) consists of two steps: computation of the first approximations

$$y = c_r x^r, \quad c_r \neq 0 \tag{1.6}$$

and computation of further expansion terms in (1.3).

1.2. Computation of truncated equations. To each differential monomial $a(x, y)$, we assign its (vector) *power exponent* $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules:

$$Q(cx^{r_1}y^{r_2}) = (r_1,\ r_2); \quad Q(d^l y/dx^l) = (-l, 1);$$

when differential monomials are multiplied, their power exponents must be added as vectors

$$Q(a_1 a_2) = Q(a_1) + Q(a_2).$$

The set $\mathbf{S}(f)$ of power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$ present in the differential sum (1.1) is called the *support of the sum* $f(x, y)$.

Obviously, $\mathbf{S}(f) \in \mathbb{R}^2$. The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polygon of the sum* $f(x,y)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of the vertices $\Gamma_j^{(0)}$ and the edges $\Gamma_j^{(1)}$. They are called (generalized) *faces* $\Gamma_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\Gamma_j^{(d)}$ corresponds to the *truncated sum*

$$\hat{f}_j^{(d)}(x,y) = \sum a_i(x,y) \ \text{ over } \ Q(a_i) \in \Gamma_j^{(d)} \cap \mathbf{S}(f). \tag{1.7}$$

Example. Consider the third Painlevé equation

$$f(x,y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{1.8}$$

assuming the complex parameters $a, b, c, d \neq 0$. Here the first three differential monomials have the same power exponent $Q_1 = (-1,2)$, then $Q_2 = (0,3)$, $Q_3 = (0,1)$, $Q_4 = (1,4)$, $Q_5 = (1,0)$. They are shown in Fig. 1 in coordinates q_1, q_2. Their convex hull $\Gamma(f)$ is the triangle with three vertices $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_4$, $\Gamma_3^{(0)} = Q_5$, and with three edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$. The vertex $\Gamma_1^{(0)} = Q_1$ corresponds to the truncation

$$\hat{f}_1^{(0)}(x,y) = -xyy'' + xy'^2 - yy',$$

and the edge $\Gamma_1^{(1)}$ corresponds to the truncation

$$\hat{f}_1^{(1)}(x,y) = \hat{f}_1^{(0)}(x,y) + by + dx. \ \blacksquare$$

Let the plane $\mathbb{R}^2_*$ be dual to the plane $\mathbb{R}^2$ such that for $P = (p_1, p_2) \in \mathbb{R}^2_*$ and $Q = (q_1, q_2) \in \mathbb{R}^2$, the scalar product

$$\langle P, Q \rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$$

is defined. Each face $\Gamma_j^{(d)}$ in $\mathbb{R}^2_*$ corresponds to its own *normal cone* $\mathbf{U}_j^{(d)}$ formed by the outward normal vectors P to the face $\Gamma_j^{(d)}$. For the edge $\Gamma_j^{(1)}$, the normal cone $\mathbf{U}_j^{(1)}$ is the ray orthogonal to the edge $\Gamma_j^{(1)}$ and directed outward the polygon $\Gamma(f)$. For the vertex $\Gamma_j^{(0)}$, the normal cone $\mathbf{U}_j^{(0)}$ is the open sector (angle) in the plane $\mathbb{R}^2_*$ with the vertex at the origin $P = 0$ and limited by the rays which are the normal cones of the edges adjacent to the vertex $\Gamma_j^{(0)}$.

Example. For the the equation (1.8), the normal cones $\mathbf{U}_j^{(d)}$ of the faces $\Gamma_j^{(d)}$ are shown in Fig. 2. $\blacksquare$

Thus, each face $\Gamma_j^{(d)}$ corresponds to the normal cone $\mathbf{U}_j^{(d)}$ in the plane $\mathbb{R}^2_*$ and to the truncated equation

$$\hat{f}_j^{(d)}(x,y) = 0. \tag{1.9}$$

Theorem 1. *If the expansion* (1.3) *satisfies the equation* (1.2), *and* $\omega(1,r) \in \mathbf{U}_j^{(d)}$, *then the truncation* $y = c_r x^r$ *of the solution* (1.3) *is the solution to the truncated equation* $\hat{f}_j^{(d)}(x,y) = 0$.

Hence, to find all truncated solutions $y = c_r x^r$ to the equation (1.2), we need to compute: the support $\mathbf{S}(f)$, the polygon $\Gamma(f)$, all its faces $\Gamma_j^{(d)}$, and their normal cones $\mathbf{U}_j^{(d)}$. Then for each truncated equation $\hat{f}_j^{(d)}(x,y) = 0$, we need to find all its solutions $y = c_r x^r$ which have one of the vectors $\pm(1,\ r)$ lying in the normal cone $\mathbf{U}_j^{(d)}$.

1.3. Solution of the truncated equation. The vertex $\Gamma_j^{(0)} = \{Q\}$ corresponds to the truncated equation $\hat{f}_j^{(0)}(x,y) = 0$ the support of which consists of one point $Q = (q_1, q_2)$. Take $g(x,y) = x^{-q_1} y^{-q_2} \hat{f}_j^{(0)}(x,y)$, then $g(x, cx^r)$ does not depend on x and c, and it is a polynomial in r. Consequently, for the solution $y = c_r x^r$ to the equation $\hat{f}_j^{(0)}(x,y) = 0$, the power exponent r is the root of *characteristic equation*

$$\chi(r) \stackrel{\text{def}}{=} g(x,\ x^r) = 0, \tag{1.10}$$

with an arbitrary coefficient c_r. We need only those roots r of the equation (1.10) for which the vector $\omega(1,r)$ lies in the normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$.

Example. For the equation (1.8), the vertex $\Gamma_1^{(0)} = Q_1 = (-1,2)$ corresponds to the truncated equation

$$\hat{f}_1^{(0)}(x,y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' = 0, \tag{1.11}$$

and $\hat{f}_1^{(0)}(x, x^r) = x^{2r-1}[-r(r-1) + r^2 - r] \equiv 0$, i.e. any expression $y = cx^r$ is a solution to the equation (1.11). Here $\omega = -1$, and we are interested only in those solutions which have the vector $-(1,r) \in \mathbf{U}_1^{(0)}$. According to Fig. 2, this means that $r \in (-1,1)$. Thus, the vertex $\Gamma_1^{(0)}$ corresponds to the two-parameter family of power asymptotic forms of solutions

$$y = cx^r, \ \text{ arbitrary } \ c \neq 0, \ \ r \in (-1,1). \ \blacksquare \tag{1.12}$$

The edge $\Gamma_j^{(1)}$ corresponds to the truncated equation $\hat{f}_j^{(1)}(x,y) = 0$, the normal cone $\mathbf{U}_j^{(1)}$ of the edge is the ray $\{P = \lambda\omega'(1,r'),\ \lambda > 0\}$. The inclusion $\omega(1,r) \in \mathbf{U}_j^{(1)}$ means the equalities $\omega = \omega'$ and $r = r'$. This determines uniquely the power exponent r of the truncated solution $y = c_r x^r$ and the value ω. To determine the coefficient c_r, we need to substitute the expression $y = c_r x^r$ into the truncated equation $\hat{f}_j^{(1)}(x,y) = 0$. After cancelation of some power of x, we obtain an algebraic equation for the coefficient c_r

$$\tilde{\tilde{f}}(c_r) \stackrel{\text{def}}{=} x^{-s}\hat{f}_j^{(1)}(x, c_r x^r) = 0. \tag{1.13}$$

Each root $c_r \neq 0$ of this equation corresponds to its own asymptotic form $y = c_r x^r$.

Example. For the equation (1.8), the edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1(x, y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + by + dx = 0. \tag{1.14}$$

Since $\mathbf{U}_1^{(1)} = \{P = -\lambda(1,1),\ \lambda > 0\}$, then $\omega = -1$ and $r = 1$. Substituting $y = c_1 x$ into the truncated equation (1.14) and canceling x, we obtain the equation $bc_1 + d = 0$ for c_1, whence $c_1 = -d/b$. Thus, the edge $\Gamma_1^{(1)}$ corresponds to a unique power asymptotic form of solutions

$$y = -(d/b)x. \ \blacksquare \tag{1.15}$$

The truncated equation $\hat{f}_j^{(d)}(x, y) = 0$ may have non-power solutions $y = \varphi(x)$ which are the asymptotic forms for solutions to the original equation $f(x, y) = 0$. These non-power solutions $y = \varphi(x)$ may be found using power and logarithmic transformations. *Power transformation* is linear in logarithms

$$\begin{aligned} \log x &= \alpha_{11}\log u + \alpha_{12}\log v, \\ \log y &= \alpha_{21}\log u + \alpha_{22}\log v, \end{aligned} \quad \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \alpha_{ij} \in \mathbb{R}, \quad \det \alpha \neq 0.$$

It induces linear dual transformations in spaces $\mathbb{R}^2$ and $\mathbb{R}^2_*$. *Logarithmic transformation* has the form

$$\xi = \log u \ \text{ or } \ \eta = \log v.$$

Example. For the truncated equation (1.14) corresponding to the edge $\Gamma_1^{(1)}$ with the normal vector $-(1,1)$, we make power transformation

$$\begin{aligned} \log x &= \log u \\ \log y &= \log u + \log v, \end{aligned} \quad \alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

i.e. $x = u$, $y = uv$. Since $y' = xv' + v$, $y'' = xv'' + 2v'$, then, canceling x and collecting similar terms, the equation (1.14), takes the form

$$-x^2 vv'' + x^2 v'^2 - xvv' + bv + d = 0. \tag{1.16}$$

Its support consists of three points $\tilde{Q}_1 = (0, 2)$, $\tilde{Q}_2 = (0, 1)$, $\tilde{Q}_3 = 0$ placed on the axis $\tilde{q}_1 = 0$. Now we make the logarithmic transformation $\xi = \log x$. Since

$v' = \dot{v}/x$, $v'' = (\ddot{v} - \dot{v})/x^2$, where $\dot{} = d/d\xi$, then, collecting similar terms, the equation (1.16) takes the form

$$-v\ddot{v} + \dot{v}^2 + bv + d = 0.$$

Applying the technique described below to this equation, we obtain the expansion of its solutions

$$v = -(b/2)\xi^2 + \tilde{c}\xi + \sum_{k=0}^{\infty} c_k \xi^{-k},$$

where $\tilde{c}$ is an arbitrary constant, and the constants c_k are uniquely determined. In original variables, we obtain the family of non-power asymptotic forms of solutions to the original equation (1.8)

$$y \sim x\,[-(b/2)(\log x)^2 + \tilde{c}\log x + \sum_{k=0}^{\infty} c_k\,(\log x)^{-k}], \quad x \to 0.$$

1.4. Critical numbers of a truncated solution. If a truncated solution $y = c_r x^r$ is found, then the substitution $y = c_r x^r + z$ reduces the equation $f(x, y) = 0$ to the form

$$f(x, cx^r + z) \stackrel{\text{def}}{=} \tilde{f}(x, z) \stackrel{\text{def}}{=} \mathcal{L}(x)z + h(x, z) = 0, \tag{1.17}$$

where $\mathcal{L}(x)$ is a linear differential operator, and the support $\mathbf{S}(\mathcal{L}z)$ consists of only one point $(v, 1)$ that is the vertex $\tilde{\Gamma}_1^{(0)}$ of the polygon $\Gamma(\tilde{f})$; the point $(v, 1)$ is not in the support $\mathbf{S}(h)$. The operator $\mathcal{L}(x)$ is computed as the first variation $\delta\hat{f}_j^{(d)}/\delta y$ on the curve $y = c_r x^r$. Let $\nu(k)$ be characteristic polynomial of the differential sum $\mathcal{L}(x)z$, i.e.

$$\nu(k) = x^{-v-k}\mathcal{L}(x)x^k. \tag{1.18}$$

The real roots $k_1, ..., k_{æ}$ of the polynomial $\nu(k)$ that satisfy the inequality $\omega r > \omega k_i$ are called the *critical numbers of the truncated solution* $y = c_r x^r$.

Example. For the truncated equation (1.11), the first variation is

$$\frac{\delta\hat{f}_1^{(0)}}{\delta y} = -xy'' - xy\frac{d^2}{dx^2} + 2xy'\frac{d}{dx} - y' - y\frac{d}{dx}.$$

On the curve $y = c_r x^r$, this variation gives the operator

$$\mathcal{L}(x) = c_r x^{r-1}\left[-r(r-1) - x^2\frac{d^2}{dx^2} + 2rx\frac{d}{dx} - r - x\frac{d}{dx}\right].$$

The characteristic polynomial of the sum $\mathcal{L}(x)z$, i.e. $\mathcal{L}(x)x^k$, is

$$\nu(k) = c_r[-r(r-1) - k(k-1) + 2rk - r - k] = -c_r(k-r)^2.$$

It has one double root $k_1 = r$, which is not a critical number, since it does not satisfy the inequality $\omega r > \omega k_1$. Consequently, truncated solutions (1.12) have no critical numbers.

For the truncated equation (1.14), the first variation is

$$\frac{\delta \hat{f}^{(1)}}{\delta y} = \frac{\delta \hat{f}_1^{(0)}}{\delta y} + b.$$

On the curve (1.15), i.e. $y = c_1 x$, $c_1 = -d/b$, this variation gives the operator

$$\mathcal{L}(x) = c_1 \left[-x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 1 - x \frac{d}{dx} - \frac{b^2}{d} \right]$$

and the characteristic polynomial

$$\nu(k) = -c_1[k^2 - 2k + 1 + b^2/d].$$

Its roots are $k_{1,2} = 1 \pm b/\sqrt{-d}$. If $\operatorname{Im}(b/\sqrt{-d}) \neq 0$, then real critical numbers are absent. If $\operatorname{Im}(b/\sqrt{-d}) = 0$, then the inequality $\omega r > \omega k_i$ is satisfied by only one root $k_1 = 1 + |b/\sqrt{-d}|$ which is a unique critical number of the power asymptotic form (1.15). ■

1.5. Computation of asymptotic expansion (1.3). Using support $\mathbf{S}(\tilde{f})$ of the equation (1.17) and numbers $k_1, \ldots, k_{æ}$ with $\omega r > \omega k_i$, we can find the set of numbers $\mathbf{K}(k_1, \ldots, k_{æ}) \subset \mathbb{R}$. Its elements s satisfy the inequality $\omega r > \omega s$.

Theorem 2. The *equation* (1.17) *has an expansion of solutions of the form*

$$z = \sum c_s(\log x)x^s \quad over \quad s \in \mathbf{K}(k_1, \ldots, k_{æ}), \tag{1.19}$$

where $k_1, \ldots, k_{æ}$ are critical numbers of the truncated solution $y = c_r x^r$; c_s are polynomials in $\log x$, which are uniquely defined for $s \neq k_i$. If all critical numbers $k_1, \ldots, k_{æ}$ are simple roots, and each k_i does not lie in the set $\mathbf{K}(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{æ})$, then all coefficients c_s are constant; for $s \neq k_i$, they are uniquely determined; and for $s = k_i$, they are arbitrary.

Example. For the truncated solution (1.12)

$$\begin{gathered} \mathbf{K} = \{s = r + l(1-r) + m(1+r), \\ \text{integers } \ l, m \geq 0, \quad l + m > 0\}. \end{gathered} \tag{1.20}$$

Since there are no critical numbers, then all c_s are constant and uniquely determined in the expansion (1.19).

For the truncated solution (1.15)

$$\mathbf{K} = \{s = 1 + 2l, \ \ \text{integer} \ \ l > 0\}.$$

If $\mathrm{Im}\,(b/\sqrt{-d}) \neq 0$, then there are no critical numbers, and all power exponents s are odd integers greater than 1 in the expansion (1.19), and coefficients c_s are constant and uniquely determined. If $\mathrm{Im}\,(b/\sqrt{-d}) = 0$, then there is a unique critical number $k_1 = 1 + |b/\sqrt{-d}|$, and

$$\begin{gathered}\mathbf{K}(k_1) = \{s = 1 + 2l + m(k_1 - 1), \\ \text{integers} \ \ l, m \geq 0, \ \ l + m > 0\}.\end{gathered} \tag{1.21}$$

Consequently, if the number k_1 is not odd, then all c_s are constant and uniquely determined in the expansion (1.19) for $s \neq k_1$, and c_{k_1} is arbitrary. Finally, if k_1 is odd, then $\mathbf{K}(k_1) = \mathbf{K}$, and c_s is a uniquely determined constant in the expansion (1.19) if $s < k_1$; c_{k_1} is a linear function of $\log x$ with an arbitrary constant term; c_s is a uniquely determined polynomial in $\log x$ if $s > k_1$. ■

1.6. Complex power exponents. Expansions of solutions (1.3) with complex power exponents r and s, where $\omega\mathrm{Re}\, r > \omega\mathrm{Re}\, s$, are found in a similar way.

Example. In the equation (1.8), for the truncated solution (1.12) with complex r, $\mathrm{Re}\, r \in (-1, 1)$, the expansions (1.19) are also found by the set (1.20). And for the truncated solution (1.15) with $\mathrm{Im}\,(b/\sqrt{-d}) \neq 0$ and $\mathrm{Re}\, k_1 > 1$, we obtain the expansion (1.19) by the set (1.21). ■

Thus, in classical analysis, we encounter expansions in fractional powers and with constant coefficients, but here we obtain expansions in rather arbitrary complex powers of the independent variable with coefficients that are polynomials in logarithms of this variable. However, there are possible even more complicated expansions of solutions.

1.7. Types of expansions. As $x \to 0$, consider asymptotic expansions of solutions to the equation (1.2) of the form

$$y = c_r x^r + \sum_s c_s x^s, \tag{1.22}$$

where power exponents r and s are complex numbers without points of accumulation, $\mathrm{Re}\, s \geq \mathrm{Re}\, r$, $\mathrm{Re}\, s$ increase.

We define four types of expansions (1.22); the first three of which have finite number of power exponents s with the same real part $\mathrm{Re}\, s$ and $\mathrm{Re}\, s > \mathrm{Re}\, r$ (Fig. 3).

Type 1. c_r and c_s are constant (*power expansions*);

Type 2. c_r is constant, c_s are polynomials in $\log x$ (*power-logarithmic expansions*);

Type 3. c_r and c_s are power series in decreasing powers of $\log x$ (*complicated expansions*).
Type 4. There are infinitely many power exponents s with a fixed $\operatorname{Re} s$, and the convex hull of the points r and s from (1.22) in the complex plane lies in the angle with the vertex at the point r, one of the limiting rays of the angle parallel to the imaginary axis, and the span of the angle being less than π (*exotic expansions*) (Figs. 4 and 5).

In addition, we assume that the argument of the complex variable x is limited at one side. The types of asymptotic expansions as $x \to \text{const}$ and as $x \to \infty$ are defined in a similar way.

Similar technique is used for equations having small or big parameters. The power exponents of these parameters are accounted for in the same way as power exponents of variables tending to zero or infinity. Such parameter ε can be considered as a dependent variable, satisfying the equation $\varepsilon' = 0$.

2. Plane Power Geometry. Applications

2.1. The sixth Painlevé equation [3]**.** It has the form

$$y'' = \frac{(y')^2}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) - y'\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) +$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2}\right], \tag{2.1}$$

where a, b, c, d are complex parameters, x and y are complex variables, $y' = dy/dx$. The equation (2.1) has three singular points $x = 0$, $x = 1$, and $x = \infty$. After multiplying by common denominator, we obtain the equation as a differential sum. Its support and its polygon, in the case $a \neq 0$, $b \neq 0$, are shown in Fig. 6. We found all asymptotic expansions of solutions to the equation (2.1). They comprise 111 families. Among them, there are expansions of all four types. In particular, for $a = 1$ and $c = 0$, there is an expansion of the fourth type of the form

$$y = -\frac{1}{\cos\left[\log\,(C_1 x)\right]} + \sum_{\operatorname{Re} s \geq 1} c_s x^s, \tag{2.2}$$

where C_1 is an arbitrary constant, the coefficients c_s are constant and uniquely determined. The support of the expansion (2.2) is shown in Fig. 5, where $r = i$. For $C_1 = 1$ and real $x > 0$, the solution (2.2) has infinitely many poles accumulating at the point $x = 0$.

2.2. The Beletsky equation (1956) [4]

$$(1 + e\,\cos\nu)\delta'' - 2e\,\sin\nu\delta' + \mu\,\sin\delta = 4e\,\sin\nu \tag{2.3}$$

describes plane motions of a satellite around its mass center which is moving along an elliptic orbit with an eccentricity $e = \text{const} \in [0,1]$. In the equation, ν is the independent and δ is dependent variables, inertial parameter $\mu = \text{const} \in [-3,3]$. The equation (2.3) is singular at $e = 1$, $\nu = \pi$, since the coefficient at the higher derivative vanishes at this point. We introduce local coordinates $x = \nu - \pi$ and $\varepsilon = 1 - e$ at the singularity. Then the equation (2.3) takes the form

$$\left[\varepsilon + \frac{1}{2}x^2 + o(x^2,\varepsilon)\right]\frac{d^2\delta}{dx^2} + 2\left[x + o(x,\varepsilon)\right]\frac{d\delta}{dx} + \mu\sin\delta = -4[x + o(x,\varepsilon)]. \quad (2.4)$$

The support and the polygon of this equation for small coordinates x, ε is shown in Fig. 7. The boundary of the polygon Γ consists of three edges and two vertices. The unit vector along the edge $\Gamma_1^{(1)}$ is $(1,0)$, which corresponds to the variable x. The unit vector along the edge $\Gamma_2(1)$ is $(1,-1/2)$, which corresponds to the variable $x/\sqrt{\varepsilon}$. Using a variable with this type of behavior, we can regularize the equation (2.3) at the singularity and compute its solutions as relaxation oscillations. In 1997, we studied the limit equations corresponding to the vertex $\Gamma_1^{(0)}$ and to the edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$. Using their solutions, the limits of solutions to the equation (2.3) are matched as $e \to 1$. We found that for $e = 1$, the limit families of 2π-periodic solutions form a complicated structure: the family of symmetric solutions is twisted into the spiral with infinite number of revolutions around the solution $\mathbf{C} = \{\delta = -\nu, \quad \mu = -2\}$ (Fig. 8, schematically), and each convolution of the spiral corresponds to its own family of asymmetric 2π-periodic solutions having 4 spirals (2002) (Fig. 9, schematically). Apparently, the solution $\mathbf{C}$ is an accumulating point of infinitely many families of 2π-periodic solutions and of infinitely many their spirals.

3. The space Power Geometry

3.1. Theory [1]. Let $X \in \mathbb{C}^m$ be independent and $Y \in \mathbb{C}^n$ be dependent variables. Suppose $Z = (X,Y) \in \mathbb{C}^{m+n}$. A *differential monomial* $a(Z)$ is the product of an ordinary monomial $cZ^R = cz_1^{r_1}\dots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, $R = (r_1,\dots,r_{m+n}) \in \mathbb{R}^{m+n}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1}\dots\partial x_m^{l_m}} \stackrel{\text{def}}{=} \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geq 0, \quad \sum_{j=1}^{m} l_j = l, \quad L = (l_1,\dots,l_m).$$

A differential monomial $a(X)$ corresponds to its vector *power exponent* $Q(a) \in \mathbb{R}^{m+n}$ formed by the following rules

$$Q(cZ^R) = R, \quad Q(\partial^l y_j/\partial X^L) = (-L, E_j),$$

where E_j is unit vector. A product of monomials $a \cdot b$ corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

A *differential sum* is a sum of differential monomials

$$f(Z) = \sum a_k(Z).$$

A set $\mathbf{S}(f)$ of vector power exponents $Q(a_k)$ is called the *support* of the sum $f(Z)$. The closure of the convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(Z)$. Consider a system of equations

$$f_i(X, Y) = 0, \quad i = 1, \ldots, n, \tag{3.1}$$

where f_i are differential sums. Each equation $f_i = 0$ corresponds to: its support $\mathbf{S}(f_i)$, its polyhedron $\Gamma(f_i)$ with the set of faces $\Gamma_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$, the set of their normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$, and the set of truncated equations $\hat{f}_{ij}^{(d_i)}(X, Y) = 0$. The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X, Y) = 0, \quad i = 1, \ldots, n \tag{3.2}$$

is the *truncated system* if the intersection

$$\mathbf{U}_{1j_1}^{(d_1)} \cap \ldots \cap \mathbf{U}_{nj_n}^{(d_n)} \tag{3.3}$$

is not empty. A solution

$$y_i = \varphi_i(X), \quad i = 1, \ldots, n$$

to the system (3.1) is associated to its normal cone $\mathbf{u} \subset \mathbb{R}^{m+n}$. If the normal cone $\mathbf{u}$ intersects with the cone (3.3), then the asymptotic form $y_i = \hat{\varphi}_i(X)$, $i = 1, \ldots, n$ of this solution satisfies the truncated system (3.2), which is quasihomogeneous.

3.2. The Euler-Poisson equations [5]

$$\begin{aligned} Ap' + (C - B)qr &= Mg(y_0\gamma_3 - z_0\gamma_2), \ {\gamma'}_1 = r\gamma_2 - q\gamma_3, \\ Bq' + (A - C)pr &= Mg(z_0\gamma_1 - x_0\gamma_3), \ {\gamma'}_2 = p\gamma_3 - r\gamma_1, \\ Cr' + (B - A)pq &= Mg(x_0\gamma_2 - y_0\gamma_1), \ {\gamma'}_3 = q\gamma_1 - p\gamma_2, \end{aligned} \tag{3.4}$$

where $' = d/dt$, describes the motion of a rigid body with a fixed point. In (3.4), A, B, C, x_0, y_0, z_0, and Mg are real constants. The system (3.4) has three general first integrals. In the case

$$B \neq C, \quad x_0 \neq 0, \quad y_0 = z_0 = 0$$

N. Kowalewski (1908) reduced the system (3.4) to the system of two equations

$$\begin{aligned} f_1 &\overset{\text{def}}{=} \ddot{\sigma}\tau + \dot{\sigma}\dot{\tau}/2 + a_1 + a_2\sigma + a_3\dot{\tau}p + a_4\tau + a_5p^2 = 0, \\ f_2 &\overset{\text{def}}{=} \sigma\ddot{\tau} + \dot{\sigma}\dot{\tau}/2 + b_1 + b_2\dot{\sigma}p + b_3\sigma + b_4\tau + b_5p^2 = 0, \end{aligned} \tag{3.5}$$

where the dot means differentiation with respect to the new independent variable p, σ and τ are new dependent variables, $a_i, b_i = \text{const}$. This system has two general first integrals. Generically, the supports $\mathbf{S}(f_i)$ and polyhedrons $\Gamma(f_i)$ of both equations (3.5) coincide; they are shown in Fig. 10. We found all power-logarithmic expansions of solutions to the system (3.5) as $p \to 0$ and as $p \to \infty$ (they comprise 24 families) and 4 families of complicated expansions of solutions. This system does not have expansions of the 4-th type. Using power expansions we obtained all exact solutions of the form of finite sums of real powers of the variable p with complex coefficients. They comprise 12 families. Among them, 7 families were known. All new families are complex.

In the case

$$A = B, \ \ Mgx_0/B = 1, \ \ y_0 = z_0 = 0, \ \ C/B = c$$

the system (3.4) has a unique parameter $c \in (0, 2]$. The system (3.4) has 4 two-parameter families of stationary solutions. On each of these families there are sets D_j of real stationary solutions near which the system (3.4) is locally integrable as well as the sets R_j of stationary solutions near which the system (3.4) is locally nonintegrable. In Fig. 11, there shown the sets D_1, D_2, D_3 and the curves R_1–R_4 for one of these four families with $x = 1/c$ and $y = p^0\gamma_1^0$, where $(p, q, r, \gamma_1, \gamma_2, \gamma_3) = (p^0, 0, 0, \pm 1, 0, 0)$ is a stationary solution.

3.3. Boundary layer on the needle [6]. The theory of the boundary layer on the plate for a stream of viscous incompressible fluid was developed by Prandtl (1904) and Blasius (1908). However a similar theory for the boundary layer on the needle was not known untill recently, since no-slip conditions on the needle correspond to a more strong singularity as for the plate. This theory was developed with the help of Power Geometry (2004).

Let x be an axis in three-dimensional space, r be the distance from the axis, and semi-infinite needle be placed on the half-axis $x \geq 0$, $r = 0$. We studied stationary axisymmetric flows of viscous fluid which had constant velocity at $x = -\infty$ parallel to the axis x, and which satisfied no-slip conditions on the needle (Fig. 12). We considered two cases.

First case: incompressible fluid. For it, the Navier-Stokes equations in independent variables x, r are equivalent to the system of two equations for the stream function ψ and the pressure p

$$g_1 \overset{\text{def}}{=} -\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} -$$

$$-\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)=0,$$

$$g_2\stackrel{\text{def}}{=}\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial x}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial x}\right)+\frac{1}{\rho}\frac{\partial p}{\partial r}+$$

$$+\nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2\psi}{\partial x\partial r}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\frac{\partial\psi}{\partial x}\right)\right)=0, \tag{3.6}$$

where $\rho,\nu=\text{const}$, with the boundary conditions

$$\psi=\psi_0 r^2 \text{ for } x=-\infty,\ \psi_0=\text{const}; \tag{3.7}$$

$$\begin{gathered}\partial\psi/\partial x=\partial\psi/\partial r=\partial^2\psi/\partial x\partial r=\partial^2\psi/\partial r^2=0\\ \text{for } x\geq 0,\ r=0.\end{gathered} \tag{3.8}$$

The system (3.6) has the form (3.1) with $m=n=2$ and $m+n=4$. Hence the supports of the equations (3.6) must be considered in $\mathbb{R}^4$. It turned out that polyhedrons $\Gamma(g_1)$ and $\Gamma(g_2)$ of the equations (3.6) are three-dimensional tetrahedrons, which can be moved by translation in one linear three-dimensional subspace, that simplified the isolation of the truncated systems. An analysis of truncated systems and of the results of their matching revealed (2002) that the system (3.6) had no solution with $p\geq 0$ satisfying both boundary conditions (3.7), (3.8).

Second case: compressible heat-conducting gas. For this case, the Navier-Stokes equations in independent variables x,r are equivalent to the system of three equations for the stream function ψ, the density ρ, and the enthalpy h (an analog of the temperature)

$$f_1\stackrel{\text{def}}{=}-\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)+\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)-A\frac{\partial}{\partial r}(\rho h)+$$

$$+\frac{2}{3}C^n\frac{\partial}{\partial r}\left(\frac{h^n}{r}\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\right)\right)-\frac{2}{3}C^n\frac{\partial}{\partial r}\left(\frac{h^n}{r}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\right)\right)-$$

$$-\frac{2C^n}{r}\frac{\partial}{\partial r}\left(h^n r\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\right)+C^n\frac{\partial}{\partial x}\left(h^n\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)-$$

$$-C^n\frac{\partial}{\partial x}\left(h^n\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\right)+\frac{2C^n h^n}{\rho r^3}\frac{\partial\psi}{\partial x}=0,$$

$$f_2\stackrel{\text{def}}{=}\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)-A\frac{\partial}{\partial x}(\rho h)+$$

$$+\frac{2}{3}C^n\frac{\partial}{\partial x}\left(\frac{h^n}{r}\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\right)\right)-\frac{2}{3}C^n\frac{\partial}{\partial x}\left(\frac{h^n}{r}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\right)\right)+$$

$$+\frac{C^n}{r}\frac{\partial}{\partial r}\left(h^n r\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)-\frac{C^n}{r}\frac{\partial}{\partial r}\left(h^n r\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\right)+$$

$$+2C^n\frac{\partial}{\partial x}\left(h^n\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)=0,$$

$$f_3 \stackrel{\text{def}}{=} \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial h}{\partial x}-\frac{A}{\rho r}\frac{\partial\psi}{\partial x}\frac{\partial(\rho h)}{\partial r}+\frac{A}{\rho r}\frac{\partial\psi}{\partial r}\frac{\partial(\rho h)}{\partial x}+$$
$$+2C^nh^n\left(\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\right)^2+2C^nh^n\left(\frac{1}{r^2\rho}\frac{\partial\psi}{\partial x}\right)^2+$$
$$+2C^nh^n\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)^2+$$
$$+C^nh^n\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\right)^2-C^nh^n\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)+$$
$$+C^nh^n\left(\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)^2-\frac{2}{3}C^nh^n\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\right)\right)^2+$$
$$+\frac{4C^nh^n}{3r}\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)-\frac{2}{3}C^nh^n\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)^2+$$
$$+\frac{C^n}{\sigma r}\frac{\partial}{\partial r}\left(rh^n\frac{\partial h}{\partial r}\right)+\frac{C^n}{\sigma}\frac{\partial}{\partial x}\left(h^n\frac{\partial h}{\partial x}\right)=0, \qquad (3.9)$$

where parameters $A, C, \sigma > 0$ and $n \in [0,1]$, with the boundary conditions

$$\begin{gathered}\psi=\psi_0r^2,\ \rho=\rho_0,\ h=h_0 \text{ for } x=-\infty,\\ \psi_0,\rho_0,h_0=\text{const}\end{gathered} \qquad (3.10)$$

and (3.8). Here $m = 2$, $n = 3$, and $m + n = 5$. In the space $\mathbb{R}^5$, all polyhedrons $\Gamma(f_1)$, $\Gamma(f_2)$, $\Gamma(f_3)$ of the equations (3.9) are three-dimensional, and they can be moved into one linear subspace. In coordinates $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3)$ of this three-dimensional space, they are shown in Fig. 13, 14, 15 respectively. This simplified the isolation of the truncated system corresponding to the boundary layer on the needle

$$\hat{f}_{12}^{(0)} \stackrel{\text{def}}{=} -A\partial(\rho h)/\partial r=0 \ \text{ (or } \ \partial(\rho h)/\partial r=0),$$

$$\hat{f}_{22}^{(2)} \stackrel{\text{def}}{=} \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)-A\frac{\partial}{\partial x}(\rho h)+$$
$$\frac{C^n}{r}\frac{\partial}{\partial r}\left(h^nr\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)=0,$$

$$\hat{f}_{32}^{(2)} \stackrel{\text{def}}{=} \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial h}{\partial x}-\frac{A}{\rho r}\frac{\partial\psi}{\partial x}\frac{\partial(\rho h)}{\partial r}+\frac{A}{\rho r}\frac{\partial\psi}{\partial r}\frac{\partial(\rho h)}{\partial x}+$$
$$+C^nh^n\left(\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right)\right)^2+\frac{C^n}{\sigma r}\frac{\partial}{\partial r}\left(rh^n\frac{\partial h}{\partial r}\right)=0, \qquad (3.11)$$

with self-similar variables

$$\psi = xG(\xi),\ \rho = \mathrm{P}(\xi),\ h = H(\xi),\ \xi = r^2/x, \tag{3.12}$$

and with the boundary conditions

$$\psi = \psi_0 r^2,\ \rho = \rho_0,\ h = h_0;\ \psi_0, \rho_0, h_0 = \text{const},\ r \to \infty \tag{3.13}$$

and (3.8). In Figs. 13–15, the faces corresponding to the truncated system (3.11) are shown in bold. According to the first equation (3.11) and the equalities (3.12), (3.13), the product $P(\xi)H(\xi) = \text{const} = C_0 \overset{\text{def}}{=} \rho_0 h_0$. Hence $P(\xi) = C_0/H(\xi)$, and the system (3.11), for the variables (3.12), is equivalent to the system of two ordinary equations

$$\begin{gathered} F_2 \overset{\text{def}}{=} G\,(G'H)' + 2C^n\,[\xi H^n (G'H)']' = 0, \\ F_3 \overset{\text{def}}{=} 2GH' + 16C^n C_0^{-2} \xi H^n ((G'H)')^2 + \\ +4C^n \sigma^{-1} (\xi H^n H')' = 0, \end{gathered} \tag{3.14}$$

where $' \overset{\text{def}}{=} d/d\xi$, with the boundary conditions

$$G = \psi_0 \xi,\ H = h_0 \text{ for } \xi \to +\infty, \tag{3.15}$$

$$G = dG/d\xi = 0 \text{ for } \xi = 0. \tag{3.16}$$

The problem (3.14)–(3.16) has an invariant manifold $(G'H)' = 0$ on which it is reduced to one equation

$$\Delta \overset{\text{def}}{=} 2(\xi H^n H') - 2\xi H^n H'^2 + (\xi + c_2)H' = 0,$$

where c_2 is an arbitrary constant, with the boundary conditions

$$\begin{aligned} H &\to 1 && \text{for} \quad \xi \to +\infty, \\ H &\to +\infty && \text{for} \quad \xi \to +0. \end{aligned}$$

An analysis of solutions to the latter problem by methods of PG revealed that for $n \in (0, 1)$ it has solutions of the form

$$H \sim c_3 |\log \xi|^{1/n},\ \ \xi \to 0,$$

where c_3 is an arbitrary constant.

Thus, for $n \in (0, 1)$, in the boundary layer $r^2/x < \text{const}$, as $x \to +\infty$ and $\xi = r^2/x \to 0$, we obtained the asymptotic form of the flow

$$\psi \sim c_1 r^2 |\log \xi|^{-1/n},\ \ \rho \sim c_2 |\log \xi|^{-1/n},\ \ h \sim c_3 |\log \xi|^{1/n},$$

i.e. near the needle, the density tends to zero, and the temperature increases to infinity as the distance to the point of the needle tends to $+\infty$.

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Иллюстрации

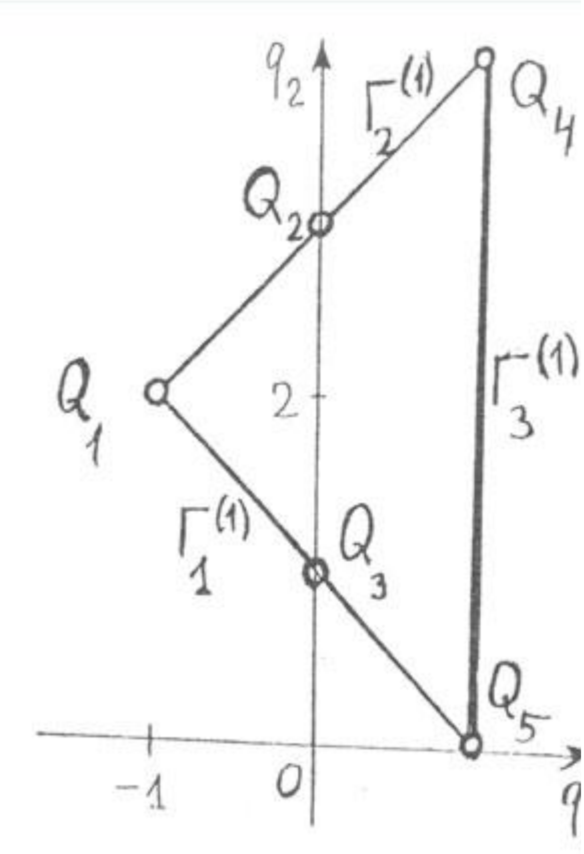


Fig. 1

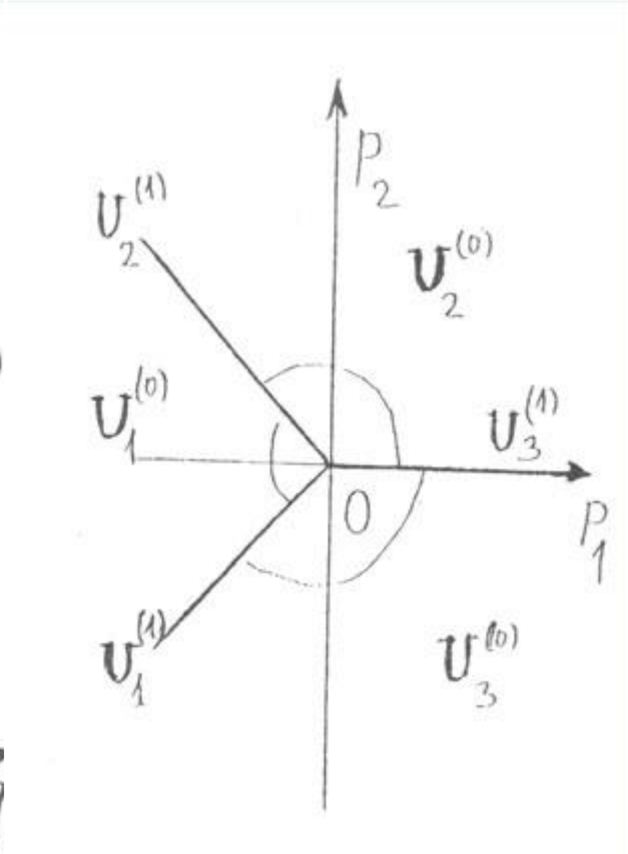


Fig. 2

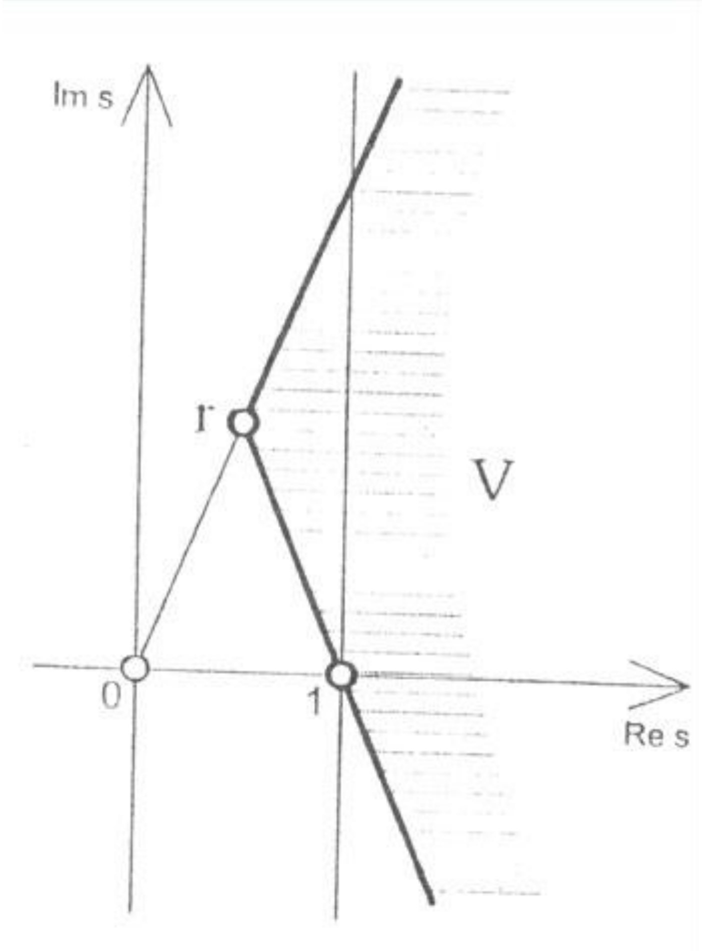


Fig. 3

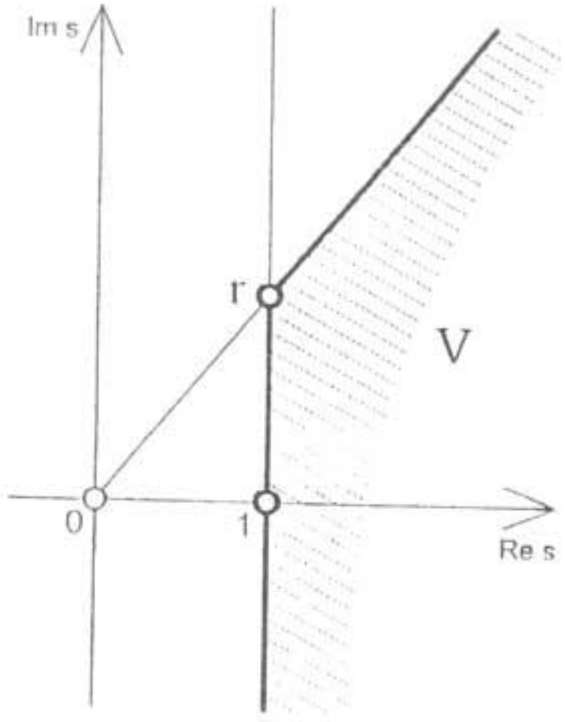


Fig. 4

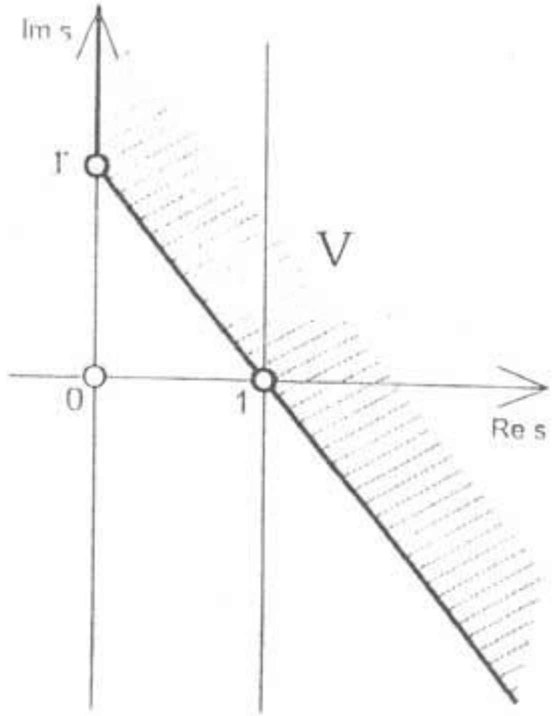


Fig. 5

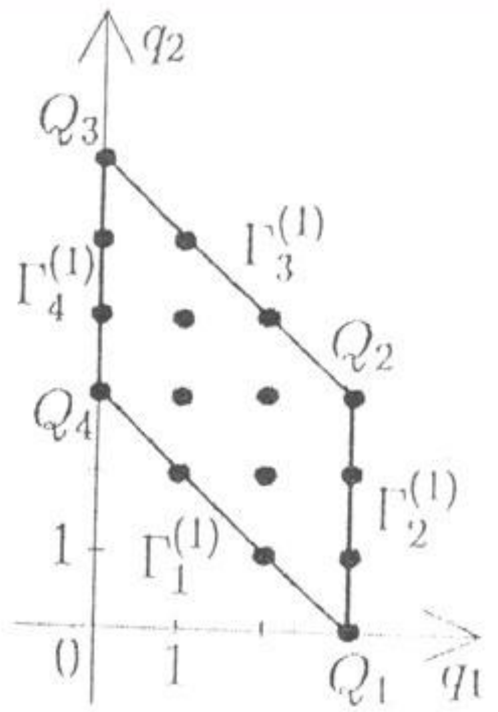


Fig. 6

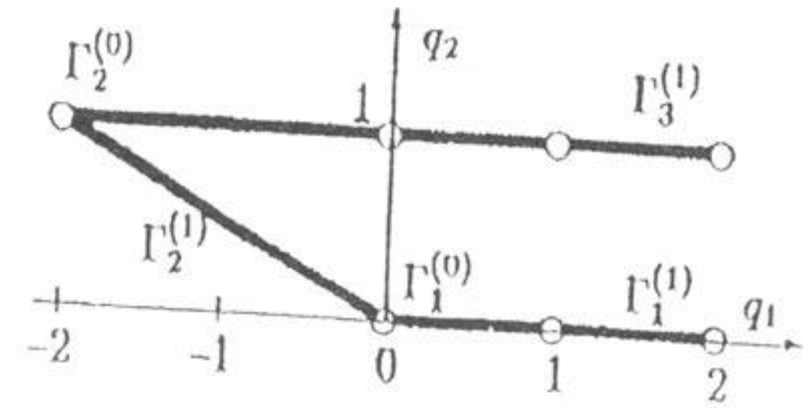


Fig. 7

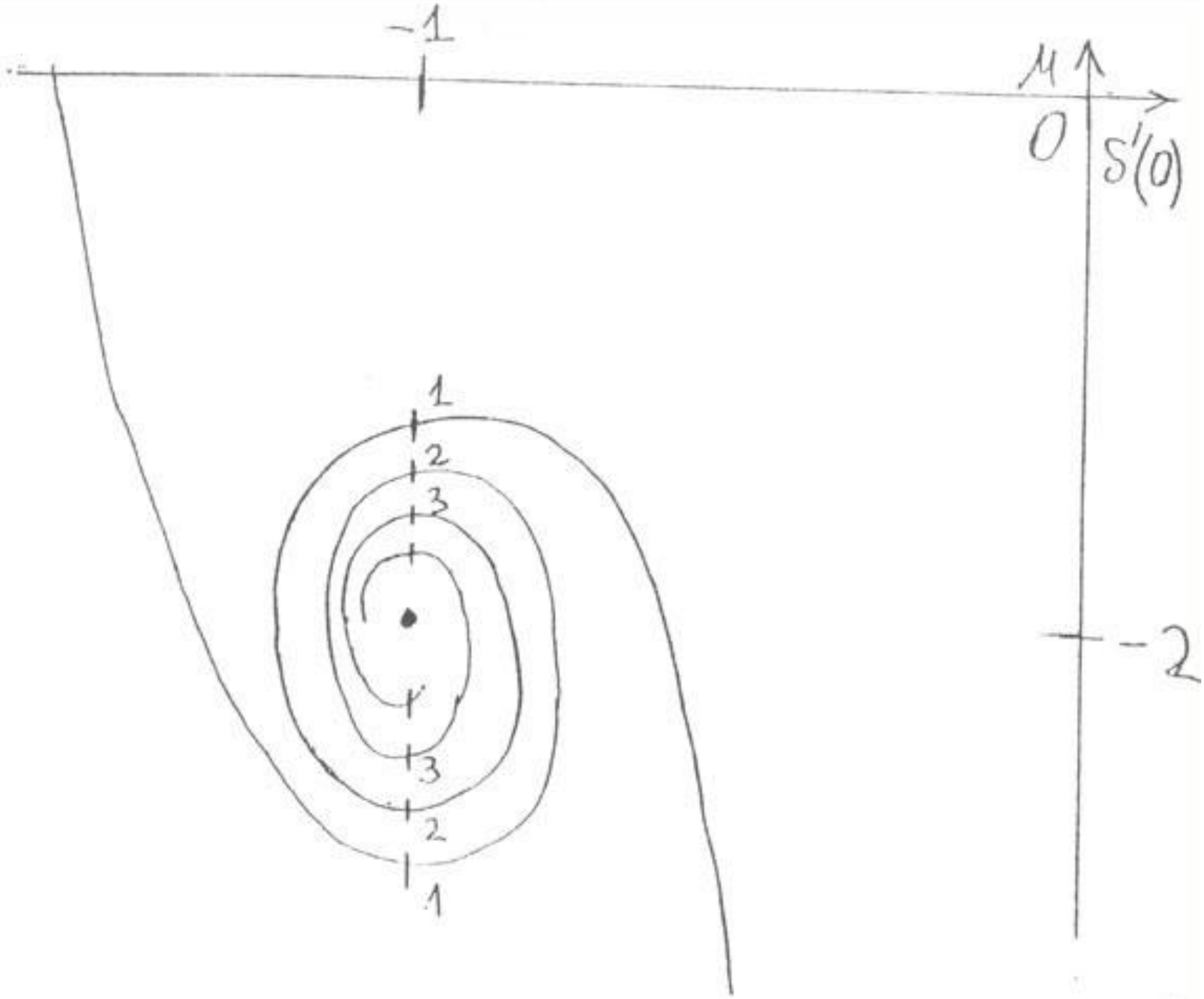
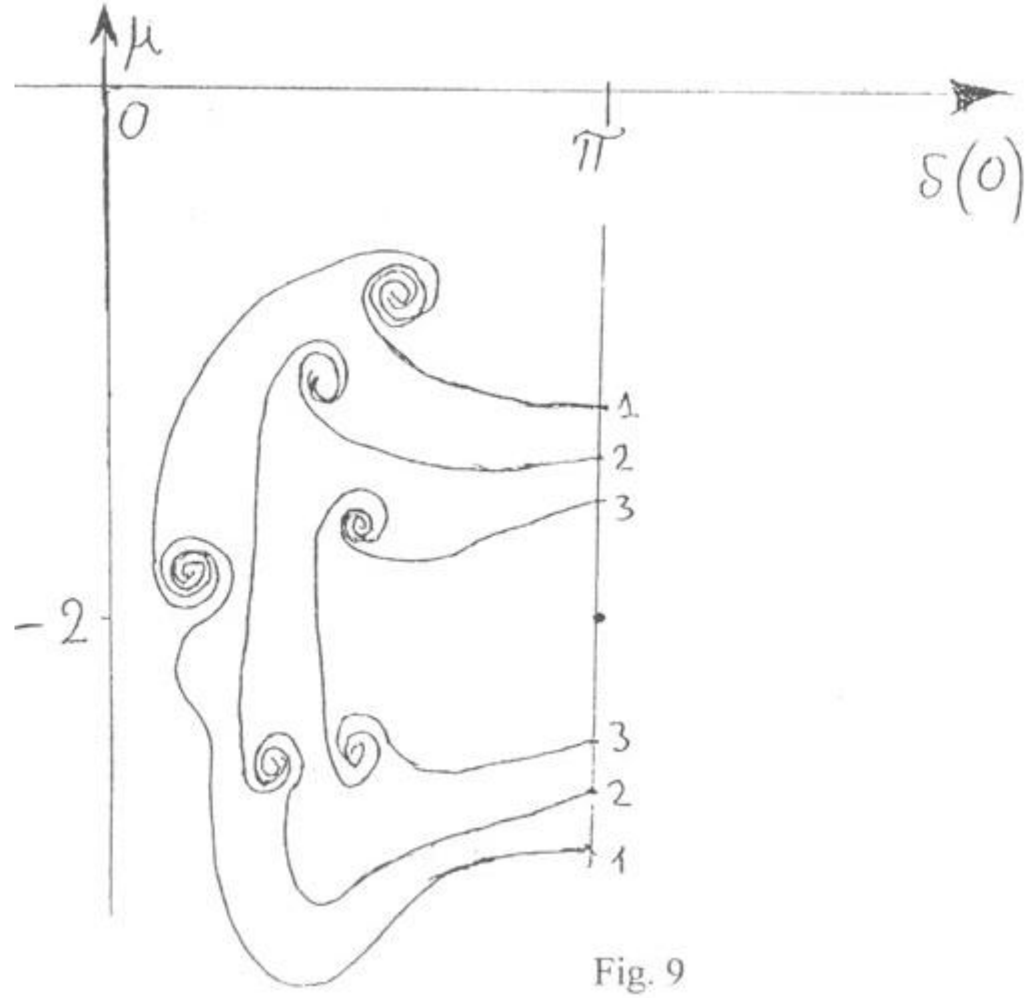
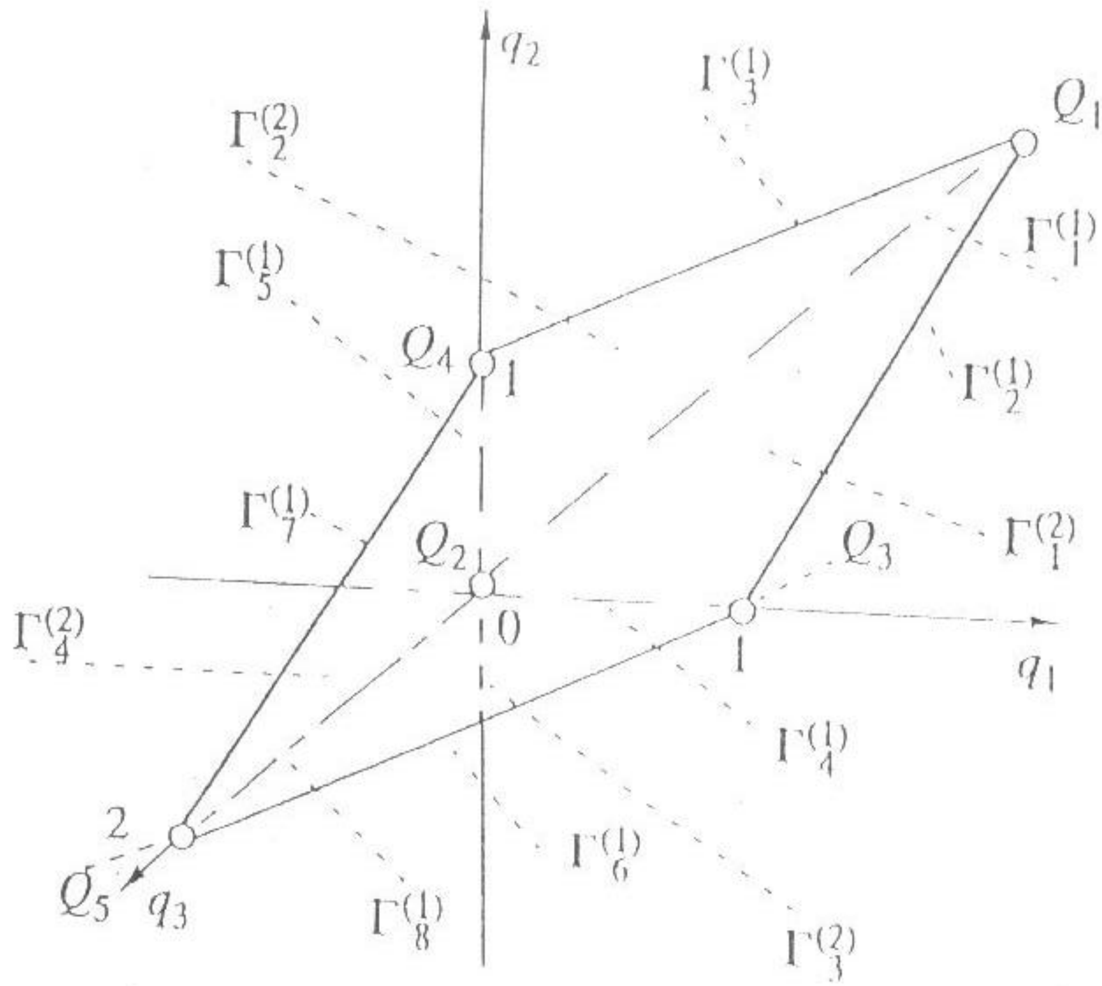


Fig. 8



μ
0
π
δ(0)
−2
1
2
3
3
2
1

Fig. 9



q_2
$\Gamma_3^{(1)}$
Q_1
$\Gamma_2^{(2)}$
$\Gamma_1^{(1)}$
$\Gamma_5^{(1)}$
Q_4
1
$\Gamma_2^{(1)}$
$\Gamma_7^{(1)}$
Q_2
Q_3
$\Gamma_1^{(2)}$
$\Gamma_4^{(2)}$
0
1
q_1
$\Gamma_4^{(1)}$
2
$\Gamma_6^{(1)}$
Q_5
q_3
$\Gamma_8^{(1)}$
$\Gamma_3^{(2)}$

Fig. 10

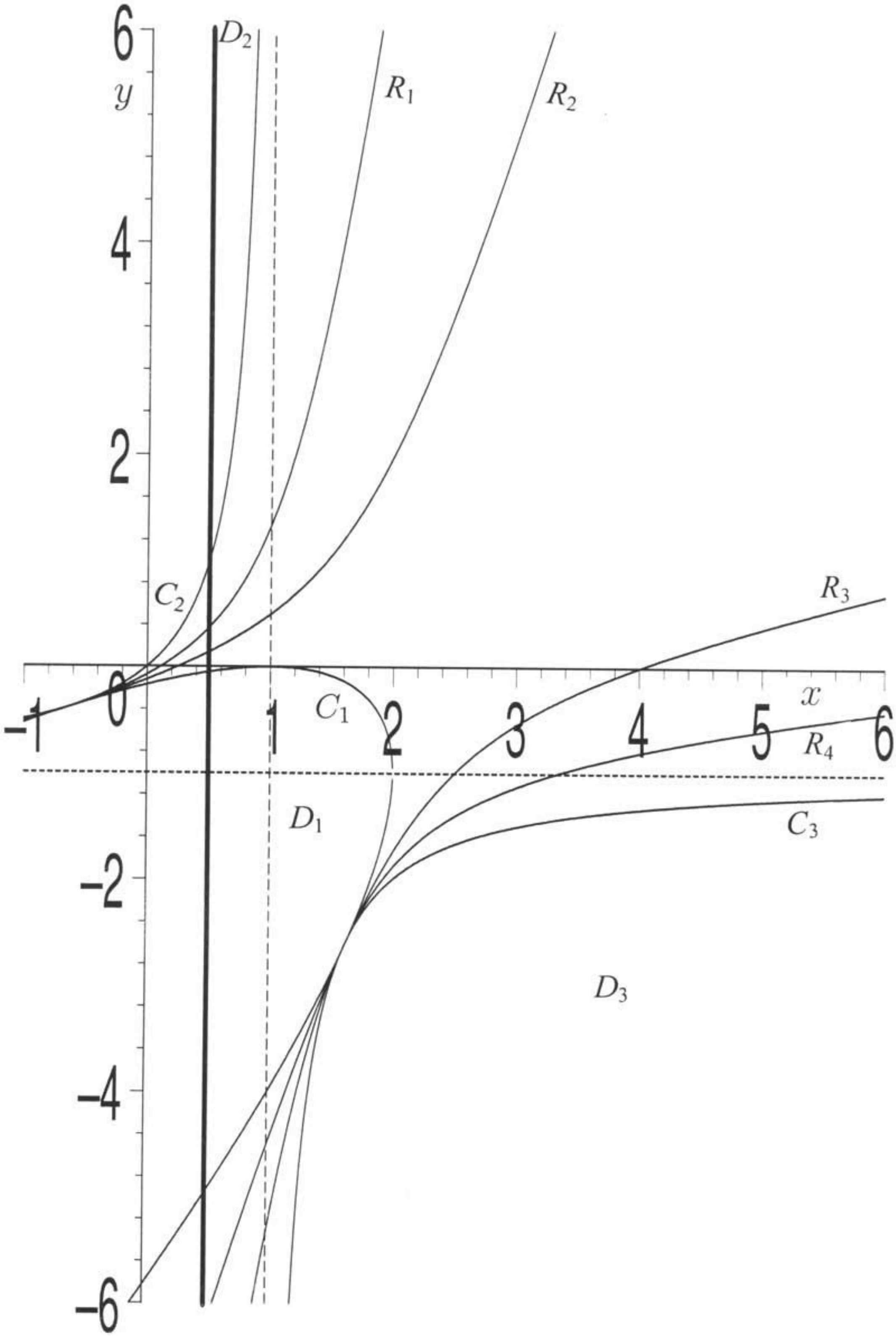


Fig. 11

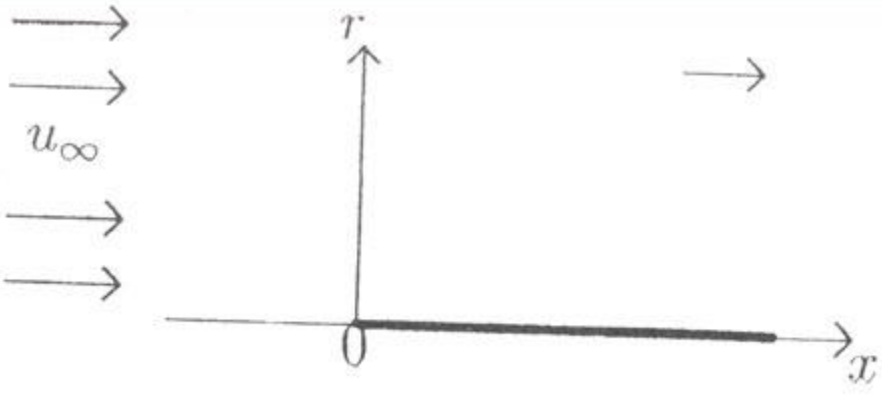


Fig. 12

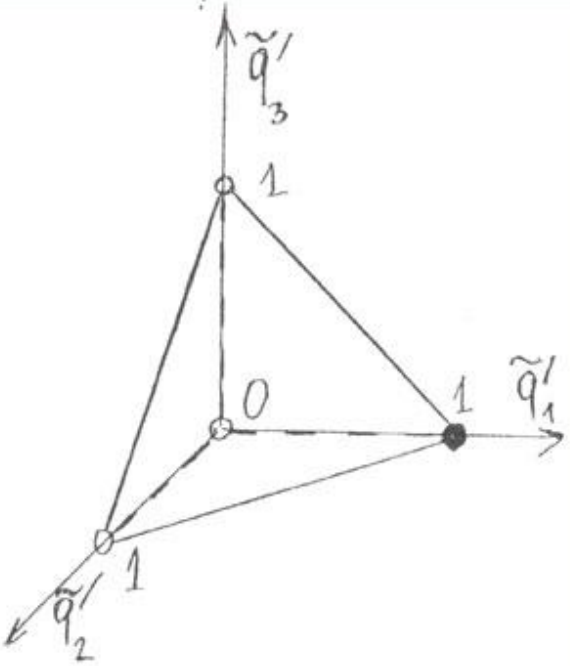


Fig. 13

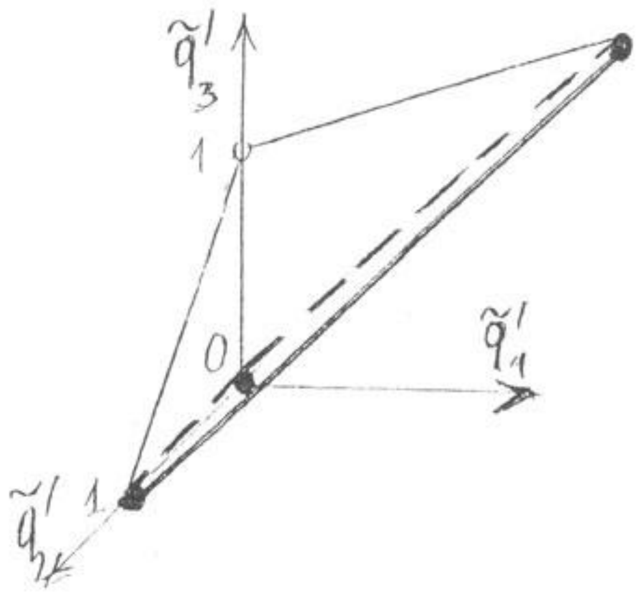


Fig. 14

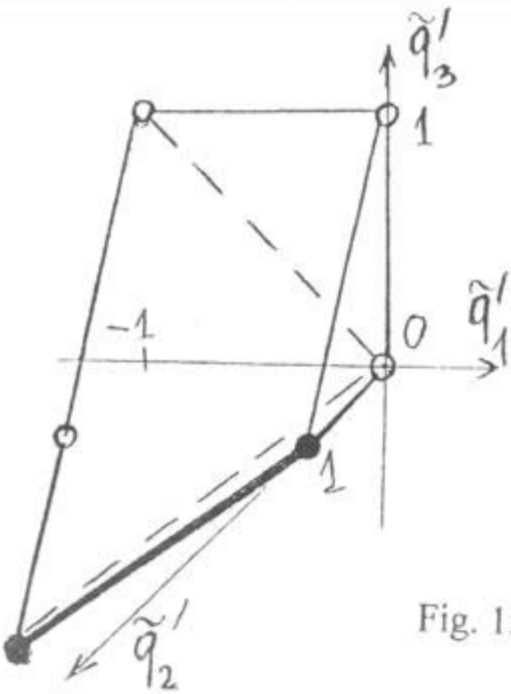


Fig. 15