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The Damped-Driven 2D Navier–Stokes System on Large Elongated Domains

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Abstract. The Navier–Stokes system with damping, which is motivated by Stommel–Charney model of ocean circulation, is considered in a large elongated periodic rectangular domain with area of the order α^{-1}, as $\alpha \to 0$. We obtain estimates for the dimension of the global attractor that are sharp as both $\alpha \to 0$ and $\nu \to 0$, where ν is the viscosity coefficient.

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1. Introduction

The two-dimensional Navier–Stokes system

$$\begin{gathered}\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta\, u - \nabla\, p + f, \\ \operatorname{div} u = 0, \qquad u(0) = u_0.\end{gathered} \tag{1.1}$$

remains in the last decades in the spotlight of the theory of infinite dimensional dissipative dynamical systems. Estimates for the degrees of freedom for this system are now traditionally interpreted in terms of the dimension of the attractor (see [1], [2], [10], [13], [21], [26], [27], [29]) and the number of determining finite dimensional projections (see [5], [6], [8], [13], [14], [15], [26] and the references therein).

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The system is considered in a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions $u_{|\partial\Omega} = 0$ or in a periodic rectangular domain $\Omega = [0, L/\alpha] \times [0, L]$, where $0 < \alpha \le 1$. We denote by $|\Omega|$ the area of Ω. The stationary data are combined in the dimensionless Grashof number G:

$$G = \frac{\|f\||\Omega|}{\nu^2}\,.$$

The best known estimates for the Hausdorff and fractal dimension of the global attractor are as follows. For the Dirichlet boundary conditions [29]

$$\dim_H \mathcal{A} \le \dim_F \mathcal{A} \le c\,G, \tag{1.2}$$

where $c = c(\Omega)$ is a dimensionless constant depending on the shape of Ω: $c(\lambda\Omega) = c(\Omega)$, for every $\lambda > 0$. It was shown in [3] that $c \le 1/(2\pi^{3/2})$.

For the periodic boundary conditions this estimate can be improved:

$$\dim_F \mathcal{A} \le c\,G^{2/3}(\ln(1+G))^{1/3}, \tag{1.3}$$

see [10], [11], [29], where c depends only on α, $c = c(\alpha)$ and $c(\alpha) \lesssim \alpha^{-1/3}$, as shown in [23].

The estimate (1.2) might be optimal, however, no lower bounds are known for the dimension of the global attractor for the Navier–Stokes system with Dirichlet boundary conditions.

The first lower bound for the periodic problem was obtained in [2] and was based on the analysis of instabilities of the classical Kolmogorov flows [24], [30]. The bifurcation parameter here is the area of Ω, that is, the parameter α. More precisely, setting $L = 2\pi$ and

$$f = f_{\mathrm{Kolm}} = \{f^1 = \lambda \sin x_2,\ f^2 = 0\}$$

in (1.1), the corresponding stationary solution $u_{\mathrm{Kolm}} = \nu^{-1} f_{\mathrm{Kolm}}$ is unstable for $\lambda \ge \lambda_0 \cdot \nu^2$, where $\lambda_0 = \lambda_0(\alpha)$ is bounded as $\alpha \to 0$, more precisely, $\lambda_0 \to \sqrt{2}$, as $\alpha \to 0$. Furthermore, the dimension of the unstable manifold $\dim M^{u_{\mathrm{Kolm}}} = 2\lfloor \frac{1}{\alpha} \rfloor$, where $\lfloor x \rfloor$ is the number of positive integers strictly less than x. Since $M^{u_{\mathrm{Kolm}}} \subset \mathcal{A}$ (see [2]), it follows that

$$\dim_F \mathcal{A} \ge \dim M^{u_{\mathrm{Kolm}}} = 2\left\lfloor \frac{1}{\alpha} \right\rfloor,$$

and the dimension of the global attractor grows like $1/\alpha$, as $\alpha \to 0$. The viscosity coefficient ν does not play a role here and the corresponding Grashof number $G_{\mathrm{Kolm}} \sim \alpha^{-3/2}$ is independent of ν.

It has later been shown in [31] that the rate of growth $1/\alpha$ of the dimension of the global attractor is sharp, and the following explicit estimate for the dimension of the attractor for the Kolmogorov flow problem was obtained in [17]

$$\dim_F \mathcal{A} \le \frac{1}{\alpha}\left(15 + \frac{6}{\pi}\,\widetilde{G}\right), \qquad \widetilde{G} = \frac{\alpha^{1/2} L^2 \|f\|}{\nu^2}\,. \tag{1.4}$$

We also observe that even if the upper and lower bounds are both of the order $1/\alpha$, their dependence on ν is essentially different: the lower bound is independent of ν, while the upper bound grows like ν^{-2}, as $\nu \to 0$.

For the periodic boundary conditions with $\alpha = 1$ (and $L = 2\pi$) it was shown in [22] that the estimate (1.3) is logarithmically sharp. Namely, for the family of right-hand sides

$$f = f_s = \{f_1 = \lambda s^3 \sin s x_2,\ f_2 = 0\}$$

the corresponding stationary solution $u_s = \nu^{-1}s^{-2}f_s$ is unstable for $\lambda \ge \Lambda_0 \cdot \nu^2$, and the dimension of the unstable manifold M^{u_s} is of the order s^2: $\dim M^{u_s} \ge c_1 s^2$. Since the corresponding Grashof number $G = G_s = \|f_s\||\Omega|/\nu^2 = c_2 s^3$, it follows that for the attractor $\mathcal{A} = \mathcal{A}_s$ corresponding to the Kolmogorov right-hand side f_s, one has

$$\dim_F \mathcal{A}_s \ge c_1 s^2 = c_3 G^{2/3}.$$

The aspect ratio here is fixed: $\alpha = 1$ and Λ_0, c_1, c_2, c_3 are absolute constants. We also observe that the viscosity ν in this lower bound again does not play a role and the bifurcation parameter is the parameter s, $s \to \infty$, characterizing the amplitude and spatial oscillations of the right-hand side f.

In this work we consider the system

$$\begin{gathered}\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta\, u - \mu u - \nabla\, p + f,\\ \operatorname{div} u = 0, \qquad u(0) = u_0,\end{gathered} \tag{1.5}$$

which is just (1.1) with additional damping term $-\mu u$ on the right-hand side. The term $-\mu u$, where μ is the Rayleigh friction coefficient or the Ekman pumping/dissipation constant, models the bottom friction in two-dimensional oceanic models or the Rayleigh friction in the planetary boundary layer for two-dimensional atmospheric models. This system has important applications is geophysical hydrodynamics [12], [25], and basically is the viscous Stommel–Charney barotropic ocean circulation model [7], [28]. The Coriolis term here is taken to be zero. It does not play a crucial role in our analysis since it is anti-symmetric. Its effect on the upper and lower bounds will considered elsewhere.

The system is considered in a periodic rectangular domain $\Omega = [0, L/\alpha] \times [0, L]$ with arbitrary (but fixed) $\mu > 0$ in the limit of the vanishing viscosity ($\nu \to 0^+$) and large domain ($\alpha \to 0^+$).

In the case of a square-shaped domain $\Omega = [0, L]^2$ ($\alpha = 1$) the following estimate for the Hausdorff and fractal dimensions of the attractor of the system (1.5) was obtained in [18]:

$$\dim_H \mathcal{A} \le \dim_F \mathcal{A} \le \left(\frac{6}{\pi^3}\right)^{1/2} \frac{L^2 \|\operatorname{rot} f\|_\infty}{\mu\nu}\,. \tag{1.6}$$

It was also shown that for the right-hand side

$$f = f_s = \{f_1 = \operatorname{const} \nu^2 s^3 \sin s x_2,\ f_2 = 0\},$$

where $L = 2\pi$, the following lower bound holds:

$$\dim_F \mathcal{A} \geq \text{const}\frac{L^2\|\operatorname{rot} f\|_\infty}{\mu\nu}\,,$$

which shows that the estimate (1.6) is sharp. Here the bifurcation parameter is $\nu \to 0^+$, while the integer s is no longer an independent parameter and is defined by the formula $s = (\mu/\nu)^{1/2} \to \infty$, as $\nu \to 0^+$.

Turning to the estimates of the determining modes and nodes [14], [15] (and other determining finite dimensional projections, see [5], [6]) we first recall that the best to date estimates for them (in the case when they are explicitly defined) for the periodic Navier–Stokes equations are as follows [13], [20]:

$$N_{\text{modes}}, N_{\text{modes}} \leq c(\alpha)G,$$

where it was shown in [19] that $c(\alpha) \sim \alpha^{-1/2}$, as $\alpha \to 0^+$, so that for a large elongated domain the numbers of the determining projections are proportional to the square of area of the domain: $N_{\text{modes}}, N_{\text{modes}} \sim \alpha^{-2}$.

On the other hand, for the damped Navier–Stokes system (1.5) the estimates for the numbers of the determining modes and nodes obtained in [19] are as follows:

$$N_{\text{modes}} \leq c_1\frac{|\Omega|\|\operatorname{rot} f\|_\infty}{\mu\nu}, \qquad N_{\text{nodes}} \leq c_2\frac{|\Omega|\|\operatorname{rot} f\|_\infty}{\mu\nu},$$

for some explicitly given absolute constants c_1 and c_2. These estimates are proportional to the area of the domain, which agrees with the physical intuition, and they suggest that a similar estimate (of the order α^{-1}) should hold for the Hausdorff and fractal dimension of the attractor, which as in the case $\alpha = 1$ is inversely proportional to ν, as $\nu \to 0^+$.

The main result of this work is the following estimate for the Hausdorff and fractal dimensions of the damped and driven Navier–Stokes system (1.5):

$$\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq 12\,\frac{L^2\|\operatorname{rot} f\|_\infty}{\alpha\mu\nu}\,,$$

where it is shown by means of the generalized Kolmogorov flows that this estimate is sharp as both $\nu \to 0^+$ and $\alpha \to 0^+$.

2. Dimension of the global attractor of the damped Navier–Stokes system in a large elongated domain

We consider the damped and driven Navier–Stokes system (1.5) in a periodic rectangular domain $\Omega = T^2_\alpha = [0, L/\alpha] \times [0, L]$, where without loss of generality we assume that $0 < \alpha \leq 1$. We also assume that f and u have mean values zero. We denote by $|\Omega|$ the area of Ω:

$$|\Omega| = \frac{L^2}{\alpha}\,.$$

Using the standard notation in the theory of the Navier–Stokes equations we denote by H the closure in $L_2(\Omega)^2$ of the set of trigonometric polynomials with divergence and mean value zero. The norm $\|\cdot\|$ and scalar product $(\cdot\,,\cdot)$ in H are those of $L_2(\Omega)$. The corresponding orthogonal Leray–Helmholtz projection is denoted by P, $P : L_2(\Omega)^2 \to H$. Applying P to the first equation in (1.5) we obtain the functional evolution equation

$$\partial_t u + B(u,u) + \nu A u = -\mu u + f, \qquad u(0) = u_0, \tag{2.1}$$

where $A = -P\Delta$ is the Stokes operator and $B(u,v) = P\big(\sum_{i=1}^2 u^i \partial_i v\big)$ is the nonlinear term. We also set

$$b(v,u,w) = (B(v,u),w) = \int_\Omega \sum\nolimits_{i,k=1}^2 v^k \partial_k u^i w^i \, dx. \tag{2.2}$$

Equation (2.1) has a unique solution $u(t)$ and the solution semigroup $S_t u_0 \to u(t)$ is well defined. The semigroup S_t has a global attractor $\mathcal{A}$ which is a compact strictly invariant set in H attracting under the action of S_t all bounded sets as $t \to \infty$. These facts are well known for the classical Navier–Stokes equations [2], [10], [21], [26], [27], [29]; the case $\mu > 0$ is similar. The solution semigroup S_t is uniformly differentiable in H with differential $L(t,u_0) : \xi \to U(t) \in H$, where $U(t)$ is the solution of the variational equation

$$\partial_t U = -\nu A U - \mu U \ - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t,u_0)U, \qquad U(0) = \xi. \tag{2.3}$$

Furthermore, the differential $L(t,u_0)$ depends continuously on the initial point $u_0 \in \mathcal{A}$ [2].

We estimate the Hausdorff and fractal dimension of the attractor $\mathcal{A}$ paying special attention to the dependence of the estimates on $\alpha \to 0^+$ and $\nu \to 0^+$.

Following [9], [10], [29], we estimate the numbers $q(m)$, that is, the sums of the first m global Lyapunov exponents:

$$q(m) \le \limsup_{t\to\infty} \sup_{u_0 \in \mathcal{A}} \sup_{\{v_j\}_{j=1}^m \in H\cap H^1} \frac{1}{t}\int_0^t \sum_{j=1}^m \big(\mathcal{L}(\tau,u_0)v_j, v_j\big) d\tau, \tag{2.4}$$

where $\{v_j\}_{j=1}^m \in H \cap H^1$ is an arbitrary L_2-orthonormal system of dimension m:

$$\int_\Omega v_i(x)\cdot v_j(x)\, dx = \delta_{ij}.$$

We first estimate the H^1-norm of the solutions on the attractor. Taking the scalar product of (2.1) with Au, using the identity $(B(u,u),Au) = 0$ (see, for example, [10], [29]) and integrating by parts we obtain

$$\begin{aligned}&\partial_t \|\operatorname{rot} u\|^2 + 2\nu\|Au\|^2 + 2\mu\|\operatorname{rot} u\|^2 \\ &\quad= 2(f, Au) = 2(\operatorname{rot} f, \operatorname{rot} u) \le \varepsilon\|\operatorname{rot} u\|^2 + \varepsilon^{-1}\|\operatorname{rot} f\|^2.\end{aligned}$$

Using the Poincaré inequality $\lambda_1\|\operatorname{rot} u\|^2 \le \|A\,u\|^2$, where λ_1 is the first eigenvalue of A, and setting $\varepsilon = \mu + \nu\lambda_1^{-1}$ we obtain

$$\partial_t\|\operatorname{rot} u\|^2 + (\mu + 2\nu\lambda_1^{-1})\|\operatorname{rot} u\|^2 \le (\mu + 2\nu\lambda_1^{-1})^{-1}\|\operatorname{rot} f\|^2,$$

which gives, in view of the Gronwall inequality, that on the attractor $u(t)\in\mathcal{A}$ the following estimate holds:

$$\|\operatorname{rot} u(t)\|^2 \le \frac{\|\operatorname{rot} f\|^2}{(\mu+2\nu\lambda_1^{-1})^2} < \frac{\|\operatorname{rot} f\|^2}{\mu^2}\,. \tag{2.5}$$

We now estimate the m-trace of the operator $\mathcal{L}$ in (2.4). Integrating by parts and using the identity $(B(u(t),v_j),v_j)=0$ (see [10], [29]) and the orthonormality of the v_js, we obtain

$$\sum_{j=1}^m \big(\mathcal{L}(t,u_0)v_j,v_j\big) = -\nu\sum_{j=1}^m \|\operatorname{rot} v_j\|^2 - \mu m - \sum_{j=1}^m b(v_j,u(t),v_j). \tag{2.6}$$

We now introduce the orthogonal projections M and N (see [31])

$$\mathrm{M}w(x_1)=\frac{1}{L}\int_0^L w(x_1,x_2)\,dx_2, \qquad \mathrm{N}=\mathrm{Id}-\mathrm{M}. \tag{2.7}$$

The spectral characterization of these projectors and the associated anisotropic Lieb–Thirring inequalities will be given in §3 (see [17]). For the moment we observe that if $\operatorname{div} w=0$, then $\operatorname{div}\mathrm{N}w=0$ and $\operatorname{div}\mathrm{M}w=0$. We represent u and v_j in the form $u=\mathrm{M}u+\mathrm{N}u$, $v_j=\mathrm{M}v_j+\mathrm{N}v_j$. Since $\mathrm{M}u$ and $\mathrm{M}v_j$ depend only on x_1, $\mathrm{M}u^1=\mathrm{M}v_j^1=0$ and $\int_0^L \mathrm{N}u(x_1,x_2)dx_2=0$, it follows that

$$b(\mathrm{M}v_j,u,\mathrm{M}v_j)=0,\quad b(\mathrm{M}v_j,\mathrm{M}u,\mathrm{N}v_j)=0,\quad b(\mathrm{N}v_j,\mathrm{M}u,\mathrm{M}v_j)=0.$$

For example, to see that the third equality holds we have

$$\begin{aligned} b(\mathrm{N}v,\mathrm{M}u,\mathrm{M}v) &= \int_\Omega \mathrm{N}v^1\partial_1\mathrm{M}u^2\mathrm{M}v^2dx \\ &= \int_0^{L/\alpha}\partial_1\mathrm{M}u^2(x_1)\mathrm{M}v^2(x_1)\int_0^L \mathrm{N}v^1(x_1,x_2)dx_2dx_1=0. \end{aligned}$$

Therefore, for the trilinear form b, defined in (2.2), the following equality holds

$$b(v_j,u,v_j)=b(\mathrm{N}v_j,u,\mathrm{N}v_j)+b(\mathrm{M}v_j,\mathrm{N}u,\mathrm{N}v_j)+b(\mathrm{N}v_j,\mathrm{N}u,\mathrm{M}v_j).$$

In view of Proposition 2.1 below

$$\sum_{j=1}^m b(v_j,u,v_j)\le 2^{-1/2}\|\operatorname{rot} u\|\|\rho_{\mathrm{N}v}\| + 2^{1/2}\|\operatorname{rot}\mathrm{N}u\|\|\rho_{\mathrm{M}v}\|^{1/2}\|\rho_{\mathrm{N}v}\|^{1/2}, \tag{2.8}$$

where

$$\rho_{\mathrm{M}v}(x)=\sum_{j=1}^m|\mathrm{M}v_j(x)|^2, \qquad \rho_{\mathrm{N}v}(x)=\sum_{j=1}^m|\mathrm{N}v_j(x)|^2. \tag{2.9}$$

For the first term using inequality (3.6) in the Theorem 3.1 below we find

$$2^{-1/2}\|\operatorname{rot} u\|\|\rho_{\mathrm{N}v}\| \le \frac{c_{\mathrm{N}}}{2\nu}\|\operatorname{rot} u\|^2+\frac{\nu}{4c_{\mathrm{N}}}\|\rho_{\mathrm{N}v}\|^2 \le \frac{c_{\mathrm{N}}}{2\nu}\|\operatorname{rot} u\|^2+\frac{\nu}{4}\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2. \tag{2.10}$$

Similarly, for the second term

$$
\begin{aligned}
&2^{1/2}\|\operatorname{rot}\mathrm{N}u\|\|\rho_{\mathrm{M}v}\|^{1/2}\|\rho_{\mathrm{N}v}\|^{1/2} \le 2^{1/2}\|\operatorname{rot}u\|\|\rho_{\mathrm{M}v}\|^{1/2}\|\rho_{\mathrm{N}v}\|^{1/2}\\
&\le 2^{1/2}\|\operatorname{rot}u\|c_{\mathrm{N}}^{1/4}\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2\right)^{1/4}\frac{c_{\mathrm{M}}^{1/4}}{L^{1/4}}m^{1/8}\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2\right)^{1/8}\\
&\le \frac{\sqrt{c_{\mathrm{M}}c_{\mathrm{N}}}}{2\nu}\|\operatorname{rot}u\|^2+\nu\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2\right)^{1/2}\frac{m^{1/4}}{L^{1/2}}\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2\right)^{1/4}\\
&\le \frac{\sqrt{c_{\mathrm{M}}c_{\mathrm{N}}}}{2\nu}\|\operatorname{rot}u\|^2+\frac{\nu}{4}\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2+\frac{\nu m^{1/2}}{L}\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2\right)^{1/2}\\
&\le \frac{\sqrt{c_{\mathrm{M}}c_{\mathrm{N}}}}{2\nu}\|\operatorname{rot}u\|^2+\frac{\nu}{4}\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2+\frac{\nu}{2}\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2+\frac{\nu m}{2L^2}.
\end{aligned}
\tag{2.11}
$$

It follows from (2.8)–(2.11) that

$$
\sum_{j=1}^m b(v_j,u,v_j)\le\frac{(c_{\mathrm{N}}+\sqrt{c_{\mathrm{M}}c_{\mathrm{N}}})}{2\nu}\|\operatorname{rot}u\|^2+\frac{\nu}{2}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2+\frac{\nu m}{2L^2}. \tag{2.12}
$$

Hence returning to (2.6) and setting $c_{\mathrm{N}}=c_{\mathrm{M}}=6$ (see (3.6), (3.7)) we obtain

$$
\sum_{j=1}^m\big(\mathcal{L}(t,u_0)v_j,v_j\big)\le-\frac{\nu}{2}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2+\frac{6}{\nu}\,\|\operatorname{rot}u\|^2+\left(\frac{\nu}{2L^2}-\mu\right)m. \tag{2.13}
$$

We now estimate the first term. By the orthogonality $\mathrm{M}H\perp\mathrm{N}H$ and by the orthonormality of $\{v_j\}_{j=1}^m$ we have for $\rho(x)=\sum_{j=1}^m|v_j(x)|^2$ and $\rho_{\mathrm{M}v}(x)$ and $\rho_{\mathrm{N}v}(x)$ defined in (2.9)

$$
m=\int_\Omega\rho(x)dx=\int_\Omega\sum_{j=1}^m|\mathrm{M}v_j(x)+\mathrm{N}v_j(x)|^2dx=\int_\Omega\rho_{\mathrm{M}v}(x)dx+\int_\Omega\rho_{\mathrm{N}v}(x)dx.
$$

Hence

$$
m^2\le2\left(\int_\Omega\rho_{\mathrm{M}v}(x)\,dx\right)^2+2\left(\int_\Omega\rho_{\mathrm{N}v}(x)\,dx\right)^2\le\frac{2L^2}{\alpha}(\|\rho_{\mathrm{N}v}\|^2+\|\rho_{\mathrm{M}v}\|^2),
$$

and by (3.6) and (3.7)

$$
\begin{aligned}
\frac{\alpha m^2}{2L^2}&\le c_{\mathrm{N}}\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2+\frac{c_{\mathrm{M}}}{L}m^{1/2}\left(\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2\right)^{1/2}\\
&\le c_{\mathrm{N}}\sum_{j=1}^m\|\operatorname{rot}\mathrm{N}v_j\|^2+c_{\mathrm{N}}\sum_{j=1}^m\|\operatorname{rot}\mathrm{M}v_j\|^2+\frac{c_{\mathrm{M}}^2m}{4c_{\mathrm{N}}L^2}\\
&=c_{\mathrm{N}}\sum_{j=1}^m\|\operatorname{rot}v_j\|^2+\frac{c_{\mathrm{M}}^2m}{4c_{\mathrm{N}}L^2}\,.
\end{aligned}
$$

Hence setting again $c_{\mathrm{N}} = c_{\mathrm{M}} = 6$ we find

$$\sum_{j=1}^{m} \| \operatorname{rot} v_j \|^2 \geq \frac{\alpha m^2}{2 c_{\mathrm{N}} L^2} - \frac{c_{\mathrm{M}}^2}{c_{\mathrm{N}}^2} \frac{m}{4L^2} = \frac{\alpha m^2}{12 L^2} - \frac{m}{4L^2}\,. \tag{2.14}$$

Using in (2.13) the estimate for the solutions on the attractor (2.5) and (2.14) we obtain

$$q(m) \leq -\frac{\nu}{24} \frac{m^2}{|\Omega|} + \left(\frac{5\nu}{8L^2} - \mu \right) m + \frac{6}{\nu} \frac{\| \operatorname{rot} f \|^2}{\mu^2}\,. \tag{2.15}$$

By the elementary estimate $m_* < a + c$ for the positive root m_* of the quadratic equation $m^2 - am - c^2 = 0$ we obtain that the Hausdorff dimension (see [29], [9]) and the fractal dimension (see [3]) of $\mathcal{A}$ satisfy $\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq m_*$, that is,

$$\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq \frac{1}{\alpha} (15 - 24 L^2 \mu/\nu)_+ + 12\, \frac{|\Omega|^{1/2} \| \operatorname{rot} f \|}{\mu\nu}\,, \tag{2.16}$$

where $x_+ = \max\{x, 0\}$.

Thus, we have proved the following theorem.

Theorem 2.1. *The Hausdorff and fractal dimensions of the global attractor $\mathcal{A}$ of the damped and driven Navier–Stokes system* (1.5) *satisfies the estimate*

$$\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq \frac{15}{\alpha} + 12\, \frac{|\Omega|^{1/2} \| \operatorname{rot} f \|}{\mu\nu}\,, \tag{2.17}$$

In particular, if ν is sufficiently small such that $\nu \leq (8/5) L^2 \mu$, then

$$\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq 12\, \frac{|\Omega|^{1/2} \| \operatorname{rot} f \|}{\mu\nu}\,. \tag{2.18}$$

Remark 2.1. Since $\| \operatorname{rot} f \| \leq |\Omega|^{1/2} \| \operatorname{rot} f \|_\infty$, we can write the estimate (2.18) in the form

$$\dim_H \mathcal{A} \leq \dim_F \mathcal{A} \leq 12\, \frac{|\Omega| \| \operatorname{rot} f \|_\infty}{\mu\nu} = 12 \frac{L^2 \| \operatorname{rot} f \|_\infty}{\alpha\mu\nu} \tag{2.19}$$

depending linearly on the area of the domain Ω.

Proposition 2.1 (see [17]). *If* $\operatorname{div} u(x) = 0$, *then:*

$$\begin{aligned} \left| \sum\nolimits_{k,i=1}^{2} v^k(x) \partial_k u^i(x) v^i(x) \right| &\leq \frac{1}{\sqrt{2}} |\nabla u(x)| |v(x)|^2, \\ \left| \sum\nolimits_{k,i=1}^{2} \left(v^k(x) \partial_k u^i(x) w^i(x) + w^k(x) \partial_k u^i(x) v^i(x) \right) \right| &\leq \sqrt{2} |\nabla u(x)| |v(x)| |w(x)|, \end{aligned} \tag{2.20}$$

where v and w are arbitrary vector functions and $|\nabla u| = \left(\sum_{k,i=1}^{2} (\partial_k u^i)^2 \right)^{1/2}$.

Proof. Setting $D = (\nabla u + \nabla u^*)/2$, where

$$\nabla u = \begin{pmatrix} \partial_1 u^1 & \partial_1 u^2 \\ \partial_2 u^1 & \partial_2 u^2 \end{pmatrix}, \quad \nabla u^* = \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix},$$

we have

$$\left|\sum\nolimits_{k,i=1}^{2} v^k \partial_k u^i v^i\right| = |\nabla u\, v \cdot v| = |Dv \cdot v| \le \lambda |v|^2.$$

Here $\lambda = \|D\|$ is the maximum (in absolute value) eigenvalue of D. Since $\operatorname{div} u = 0$, the trace of the 2×2 matrix D is zero and the eigenvalues of D are $\lambda > 0$ and $-\lambda$, where

$$\lambda^2 = (\partial_1 u^1)^2 + \frac{1}{4}(\partial_1 u^2 + \partial_2 u^1)^2 \le \frac{1}{2}\,|\nabla u|^2,$$

which proves the first inequality in (2.20). The left-hand side of the second inequality in (2.20) is equal to $2|Dv\cdot w| \le 2\lambda|v||w|$, hence the proof is complete. □

Lower bound. In this section we derive a lower bound for the global dimension of the attractor which is sharp both for $\nu \to 0^+$ and $\alpha \to 0^+$, while $\mu > 0$ is arbitrary and fixed.

We go over to the scalar vorticity equation. We introduce the stream function ψ, so that $u = \nabla^\perp \psi = \{-\partial_2\psi, \partial_1\psi\}$. Substituting this into (1.5) and applying the operator rot we obtain for $\varphi = \Delta\,\psi$ the equation

$$\partial_t\varphi - \nu\Delta\varphi + \mu\varphi + J(\Delta^{-1}\varphi, \varphi) = F = \operatorname{rot} f, \tag{2.21}$$

where the Jacobian

$$J(a,b) = \nabla^\perp a \cdot \nabla b = \partial_1 a \partial_2 b - \partial_2 a \partial_1 b. \tag{2.22}$$

Since the global attractor is the maximal strictly invariant compact set, it contains all the stationary solutions and their unstable manifolds along which the solutions tend to the stationary points as $t \to -\infty$ [2].

We use the following well-known family of Kolmogorov flows as stationary solutions [2], [18] [22], [24], [30].

We set $L = 2\pi$ and consider (1.5) in the periodic domain $[0, 2\pi/\alpha] \times [0, 2\pi]$. As in [18], for a large integer parameter s we consider a family of right-hand sides f:

$$f = f_s = \begin{cases} f_1 = \frac{1}{\sqrt{2\pi}}\nu^2\lambda s^2 \sin s x_2, \\ f_2 = 0, \end{cases} \tag{2.23}$$

where $\lambda = \lambda(s)$ ia a parameter to be defined later. Then

$$\operatorname{rot} f_s = F_s = -\frac{1}{\sqrt{2\pi}}\nu^2\lambda s^3 \cos s x_2, \tag{2.24}$$

so that

$$\|\operatorname{rot} f_s\| = \frac{\nu^2\lambda s^3}{\sqrt{\alpha}}, \qquad \|\operatorname{rot} f_s\|_\infty = \frac{1}{\sqrt{2\pi}}\nu^2\lambda s^3. \tag{2.25}$$

We consider the stationary solution (2.21) with right-hand side (2.24):

$$-\nu\Delta\varphi+\mu\varphi+J(\Delta^{-1}\varphi,\varphi)=-\frac{1}{\sqrt{2}\pi}\nu^2\lambda s^3\cos sx_2 \tag{2.26}$$

and look for its solution in the form

$$\varphi=\varphi_s=-\frac{1}{\sqrt{2}\pi}\nu\lambda sK\cos sx_2,\quad K=K(s,\mu,\nu). \tag{2.27}$$

Since φ_s depends only on x_2, it follows that $J(\Delta^{-1}\varphi_s,\varphi_s)\equiv 0$ and it is straight forward to see that for

$$K(s,\mu,\nu)=\frac{s^2}{s^2+\mu/\nu}\,, \tag{2.28}$$

φ_s defined in (2.27) is a solution of (2.26).

We consider the spectral problem for the equation linearized about the stationary solution φ_s

$$\mathcal{L}_{\varphi_s}\varphi=J(\Delta^{-1}\varphi_s,\varphi)+J(\Delta^{-1}\varphi,\varphi_s)-\nu\Delta\varphi+\mu\varphi=-\sigma\varphi. \tag{2.29}$$

The dimension of the unstable subspace with $\operatorname{Re}\sigma>0$ is a lower bound for the dimension of the global attractor $\mathcal{A}$.

Substituting the Fourier representation of φ

$$\varphi=\frac{\sqrt{\alpha}}{\sqrt{2}\pi}\sum_{k\in\mathbb{Z}^2_+}(a_k\cos k'x+b_k\sin k'x),\quad k\in\mathbb{Z}^2_+=\{k=(k_1,k_2),\ k_1\ge 0,\ |k|>0\},$$

$$k'=\{k_1\alpha,\ k_2\},\qquad k'x=k_1\alpha x_1+k_2x_2$$

into (2.29) and taking into the account the equality $J(a,b)=-J(b,a)$ we obtain

$$\begin{aligned}&\frac{\lambda Ks}{\sqrt{2}\pi}\sum_{k\in\mathbb{Z}^2_+}\left(\frac{1}{s^2}-\frac{1}{k'^2}\right)J(\cos sx_2,a_k\cos k'x+b_k\sin k'x)\\&\quad+\sum_{k\in\mathbb{Z}^2_+}(k'^2+\hat\sigma+\mu/\nu)(a_k\cos k'x+b_k\sin k'x)=0,\end{aligned} \tag{2.30}$$

where $\hat\sigma=\sigma/\nu$. We further act as in [18], where the case $\alpha=1$ has been considered, and use the formulas

$$\begin{aligned}J(\cos sx_2,\cos(\alpha k_1x_1+k_2x_2))&=-\alpha k_1s\,\sin sx_2\sin(\alpha k_1x_1+k_2x_2)\\&=\frac{\alpha k_1s}{2}\big(\cos(\alpha k_1x_1+(k_2+s)x_2)-\cos(\alpha k_1x_1+(k_2-s)x_2)\big),\\J(\cos sx_2,\sin(\alpha k_1x_1+k_2x_2))&=\alpha k_1s\sin sx_2\cos(\alpha k_1x_1+k_2x_2)\\&=\frac{\alpha k_1s}{2}\big(\sin(\alpha k_1x_1+(k_2+s)x_2)-\sin(\alpha k_1x_1+(k_2-s)x_2)\big).\end{aligned}$$

Substituting these formulas into (2.30) and setting equal the coefficients of $\cos k'x$ we find the equation for the coefficients $a_{k_1\,k_2}$ (the equation for $b_{k_1\,k_2}$ is exactly the same):

$$
\begin{aligned}
&-\frac{\lambda K\alpha k_1}{2\sqrt{2}\pi}\left(\frac{\alpha^2k_1^2+(k_2+s)^2-s^2}{\alpha^2k_1^2+(k_2+s)^2}\right)a_{k_1\,k_2+s}\\
&+\frac{\lambda K\alpha k_1}{2\sqrt{2}\pi}\left(\frac{\alpha^2k_1^2+(k_2-s)^2-s^2}{\alpha^2k_1^2+(k_2-s)^2}\right)a_{k_1\,k_2-s}\\
&\qquad\qquad+(\alpha^2k_1^2+k_2^2+\hat\sigma+\mu/\nu)a_{k_1\,k_2}=0.
\end{aligned}
\tag{2.31}
$$

We set

$$
a_{k_1\,k_2}\left(\frac{\alpha^2k_1^2+k_2^2-s^2}{\alpha^2k_1^2+k_2^2}\right)=:c_{k_1\,k_2}
$$

and

$$
\begin{gathered}
k_1=t,\quad k_2=sn+r,\quad \text{and}\quad c_{t\,sn+r}=e_n,\\
t=1,2,\dots,\quad r\in\mathbb{Z},\quad r_{\min}<r<r_{\max},
\end{gathered}
$$

where $r_{\min}, r_{\max}$ satisfy $r_{\max}-r_{\min}<s$ and will be indicated later. As a result, for each t and r we obtain the following recurrence relation:

$$
d_ne_n+e_{n-1}-e_{n+1}=0,\qquad n=0,\pm1,\pm2,\dots,
\tag{2.32}
$$

where

$$
d_n=\frac{2\sqrt{2}\pi(t'^2+(sn+r)^2)\,(t'^2+(sn+r)^2+\widetilde\sigma)}{(t'^2+(sn+r)^2-s^2)\Lambda t'},\quad \widetilde\sigma=\hat\sigma+\mu/\nu=\sigma/\nu+\mu/\nu,
\tag{2.33}
$$

where $t'=\alpha t$ and where we set

$$
\Lambda=\lambda K=\lambda(s)K(s,\mu,\nu)=\lambda(s)\cdot\frac{s^2}{s^2+\mu/\nu}\,.
\tag{2.34}
$$

We observe that up to the change $t'\to t$ the recurrence relation (2.32), (2.33) coincides with the corresponding relation in [18] with $\alpha=1$.

We look for non-trivial solutions $\{e_n\}$ of the recurrence relations (2.32), (2.33) tending to zero as $n\to\pm\infty$. Each such solution with

$$
\operatorname{Re}\widetilde\sigma>\frac{\mu}{\nu}
\tag{2.35}
$$

provides an unstable eigenmode φ of the spectral problem (2.29) with eigenvalue σ, $\operatorname{Re}\sigma>0$.

Lemma 2.1. *Let a pair of integers t, r satisfy the following conditions*

$$
\begin{gathered}
t'^2+r^2<s^2/3,\quad t'^2+(-s+r)^2>s^2,\quad t'^2+(s+r)^2>s^2,\quad t'\ge\delta s,\\
t'=\alpha t,\qquad r_{\min}<r<r_{\max},\quad r_{\min}=-\frac{s}{6},\ r_{\max}=\frac{s}{6},\quad 0<\delta<1/\sqrt{3},
\end{gathered}
\tag{2.36}
$$

where a sufficiently large integer $s>0$ is fixed. Then for any $\Lambda>0$ there exists a unique real $\widetilde\sigma=\widetilde\sigma(\Lambda)$ for which the recurrence relation (2.32), (2.33) *has a non-trivial solution, and, in addition, $\widetilde\sigma(\Lambda)$ is monotonely increasing as $\Lambda\to\infty$ and* *satisfies the two-sided inequality*

$$c_1(t', r, s)\Lambda \le \widetilde{\sigma}(\Lambda) \le c_2(t', r, s)\Lambda. \tag{2.37}$$

A unique $\Lambda_{\mu/\nu} = \Lambda_{\mu/\nu}(s)$ *solving the equation*

$$\widetilde{\sigma}(\Lambda_{\mu/\nu}) = \mu/\nu$$

satisfies the estimate

$$c_1(\delta)s < \Lambda_{\mu/\nu}(s) < c_2(\delta)\,\frac{s^2 + \mu/\nu}{s}\,. \tag{2.38}$$

The lemma is proved in [18] for $\alpha = 1$. The proof in [18] works without any changes in the case $\alpha < 1$ by a formal change $t \to t'$ (see also [23]).

We denote by $A(\delta)$ the region in (t', r)-plane defined by conditions (2.36). Clearly, $|A(\delta)| = a(\delta)\cdot s^2$, where $|A(\delta)|$ in the area of $A(\delta)$ and $a(\delta)$ is monotonely decreasing and $a(1/\sqrt{3}) = 0$. We denote by $d(s)$ the number of points (t, r) with integer coordinates such that the corresponding point $(t', r) \in A(\delta)$. Obviously,

$$d(s) := \#\{(t, r):\ (t', r) \in \mathbb{Z}^2 \cap A(\delta)\} \backsimeq \frac{|A(\delta)|}{\alpha} = a(\delta)\cdot\frac{s^2}{\alpha} \quad \text{as} \quad s \to \infty, \tag{2.39}$$

Next, taking into account that the analysis of the recurrence relation for $b_{k_1\,k_2}$ is exactly the same, we see from Lemma 2.1 that for each (t, r) such that $(t', r) \in \mathbb{Z}^2 \cap A(\delta)$ and the parameter Λ (see (2.34)) chosen as follows (see (2.38))

$$\Lambda = \Lambda_{\mu/\nu} = c_2(\delta)\,\frac{s^2 + \mu/\nu}{s} \tag{2.40}$$

there exists a unique real eigenvalue $\widetilde{\sigma} > \mu/\nu$ of multiplicity two. Therefore there exits a positive eigenvalue $\sigma > 0$ of the original spectral problem (2.29) of multiplicity two. Hence the dimension of the unstable manifold near the stationary solution φ_s is at least $2d(s)$. As a result, we find that

$$\dim \mathcal{A} \ge 2d(s) \backsimeq 2a(\delta)\cdot\frac{s^2}{\alpha}\,. \tag{2.41}$$

The integer parameter s has so far been arbitrary. We now set

$$s^2 = \frac{\mu}{\nu}\,.$$

(Strictly speaking we have to require that μ/ν is a complete square. But we already have the relation "$\backsimeq$" in (2.41).) Then

$$\dim \mathcal{A} \gtrsim 2a(\delta)\frac{1}{\alpha}\frac{\mu}{\nu}\,. \tag{2.42}$$

We now recall the definition of Λ (2.34):

$$\Lambda = \lambda(s)\cdot\frac{s^2}{s^2 + \mu/\nu}\,.$$

Setting $s^2 = \mu/\nu$ here and in (2.40) we obtain the equation for λ, from which

$$\lambda(s) = \lambda((\mu/\nu)^{1/2}) = 4c_2(\delta) \cdot s = 4c_2(\delta) \cdot \sqrt{\frac{\mu}{\nu}}. \tag{2.43}$$

We calculate

$$G_1 = \frac{|\Omega|^{1/2} \| \operatorname{rot} f\|}{\nu\mu} \qquad \text{and} \qquad G_2 = \frac{|\Omega| \| \operatorname{rot} f\|_\infty}{\nu\mu},$$

for $f = f_s$ and $s = (\mu/\nu)^{1/2}$. By (2.25) and (2.43) we obtain

$$\| \operatorname{rot} f_s\| = \frac{\nu^2 \lambda(s) s^3}{\sqrt{\alpha}} = 4c_2(\delta)\,\frac{\mu^2}{\sqrt{\alpha}}, \quad \| \operatorname{rot} f_s\|_\infty = \frac{1}{\sqrt{2}\pi} \nu^2 \lambda(s) s^3 = \frac{2\sqrt{2}}{\pi} c_2(\delta)\mu^2. \tag{2.44}$$

Hence

$$G_1 = 8\pi c_2(\delta)\,\frac{\mu}{\alpha\nu} \qquad \text{and} \qquad G_2 = 8\pi\sqrt{2} c_2(\delta)\,\frac{\mu}{\alpha\nu}\,. \tag{2.45}$$

Expressing the estimate (2.42) in terms of G_1 and G_2 (2.45) and optimizing with respect to $\delta \in (0, 1/\sqrt{3})$ we obtain

$$\begin{aligned} \dim \mathcal{A} &\gtrsim \mathrm{const}_1 G_1, \\ \dim \mathcal{A} &\gtrsim \mathrm{const}_2 G_2, \end{aligned} \tag{2.46}$$

where $\mathrm{const}_1 = (4\pi)^{-1} \max_{0<\delta<1/\sqrt{3}} a(\delta) c_2(\delta)^{-1}$ and $\mathrm{const}_2 = \mathrm{const}_1/\sqrt{2}$ are absolute constants.

Combining the results so obtained with Theorem 2.1 we have the following theorem.

Theorem 2.2. *The dimension of the global attractor $\mathcal{A}$ of the equation* (1.5) *with Kolmogorov right-hand side* (2.23) *satisfies the following two-sided estimate, which is sharp as both $\nu \to 0^+$ and $\alpha \to 0^+$:*

$$\begin{aligned} \mathrm{const}_1 \frac{|\Omega|^{1/2} \| \operatorname{rot} f\|}{\nu\mu} &\lesssim \dim_H \mathcal{A} \le \dim_F \mathcal{A} \le 12 \frac{|\Omega|^{1/2} \| \operatorname{rot} f\|}{\nu\mu}\,, \\ \mathrm{const}_2 \frac{|\Omega| \| \operatorname{rot} f\|_\infty}{\nu\mu} &\lesssim \dim_H \mathcal{A} \le \dim_F \mathcal{A} \le 12 \frac{|\Omega| \| \operatorname{rot} f\|_\infty}{\nu\mu}\,, \end{aligned} \tag{2.47}$$

where $|\Omega| = L^2/\alpha$.

3. Two-dimensional anisotropic Lieb–Thirring inequalities

We consider in this section the Lieb–Thirring inequalities on the two-dimensional torus $\Omega = T^2_\alpha = (0, L/\alpha) \times (0, L)$, $0 < \alpha \le 1$, elongated in the direction of x_1. Without loss of generality we set $L = 2\pi$.

To give the spectral characterization of the orthogonal projections M and N used in §2 (see (2.7)) we consider the spectrum $\sigma = \{\lambda_j\}_{j=1}^\infty$ of the problem

$$-\Delta\, w_j = \lambda_j w_j \tag{3.1}$$

in $H = L_2(T^2_\alpha) \cap \{\int f dx = 0\}$:

$$\sigma = \{\alpha^2 k_1^2 + k_2^2,\ k = (k_1, k_2) \in \mathbb{Z}^2_0\}, \quad \mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus 0. \tag{3.2}$$

We represent σ in the form

$$\sigma = \sigma_{\mathrm{N}} \cup \sigma_{\mathrm{M}},$$

where

$$\begin{aligned} &\sigma_{\mathrm{N}} = \{\alpha^2 k_1^2 + k_2^2,\ k_2 \neq 0,\ k = (k_1, k_2) \in \mathbb{Z}^2_0\}, \\ &\sigma_{\mathrm{M}} = \{\alpha^2 k_1^2,\ k_1 \in \mathbb{Z}_0\},\ \mathbb{Z}_0 = \mathbb{Z} \setminus 0. \end{aligned} \tag{3.3}$$

The corresponding spectral projections are denoted by N and M. The the Hilbert space H is the orthogonal sum of the two invariant (with respect to Δ) subspaces:

$$H = \mathrm{N}H \oplus \mathrm{M}H.$$

We observe that functions in $\mathrm{M}H$ depend only on x_1. Furthermore, the action of M amounts to taking the mean value over the 'short' coordinate x_2 [31]:

$$\mathrm{M}\varphi(x_1) = \frac{1}{L}\int_0^L \varphi(x_1, x_2) dx_2, \quad \mathrm{N} = \mathrm{Id} - \mathrm{M}.$$

Consider an L_2-orthonormal family of functions $\{f_j\}_{j=1}^m$:

$$(f_i, f_j) = \int f_i f_j dx = \delta_{ij}.$$

Then the corresponding families $\{\varphi_j\}_{j=1}^m$ and $\{\psi_j\}_{j=1}^m$, $f_j = \varphi_j + \psi_j$, where $\varphi = \mathrm{M}f$, $\psi = \mathrm{N}f$, are suborthonormal in L_2 in the sense of the following definition [16].

Definition 3.1. A family $\{\phi_j\}_{j=1}^m$ is called suborthonormal if for any $\xi \in \mathbb{R}^m$

$$\sum_{i,j=1}^m \xi_i \xi_j (\phi_i, \phi_j) \le \sum_{j=1}^m \xi_j^2. \tag{3.4}$$

In fact, for example, for $\{\varphi_j\}_{j=1}^m = \{\mathrm{M}f_j\}_{j=1}^m$ by the orthonormality of $\{f_j\}_{j=1}^m$ and orthogonality $(\varphi_i, \psi_j) = 0$ we have

$$\begin{aligned} |\xi|^2 = \sum_{i,j=1}^m \xi_i \xi_j (f_i, f_j) &= \sum_{i,j=1}^m \xi_i \xi_j (\varphi_i, \varphi_j) + \sum_{i,j=1}^m \xi_i \xi_j (\psi_i, \psi_j) \\ &= \sum_{i,j=1}^m \xi_i \xi_j (\varphi_i, \varphi_j) + \left\|\sum\nolimits_{j=1}^m \xi_j \psi_j\right\|^2, \end{aligned}$$

which gives that $\{\varphi_j\}_{j=1}^m$ satisfies (3.4).

The following theorem (see [17]) collects the Lieb–Thirring inequalities used in §2. It is important for us that for function in $\mathrm{N}H$ the constants in the corresponding inequalities are bounded by absolute constants as $\alpha \to 0$.

Theorem 3.1. *Let the families of scalar functions* $\{\varphi_j\}_{j=1}^m$ *and vector functions* $\{v_j\}_{j=1}^m$ *be suborthonormal in* $L_2(T^2_\alpha)$*, have mean value zero and* $\operatorname{div} v_j = 0$*. Then for*

$$\rho_v(x) = \sum_{j=1}^m |v_j(x)|^2, \quad \rho_\varphi(x) = \sum_{j=1}^m \varphi_j(x)^2.$$

the following inequalities hold:

$$\int_\Omega \rho_\varphi(x)^2 dx \le \frac{c_{\mathrm{LT}}}{\alpha} \sum_{j=1}^m \|\nabla\varphi_j\|^2, \quad \int_\Omega \rho_v(x)^2 dx \le \frac{c_{\mathrm{LT}}}{\alpha} \sum_{j=1}^m \|\operatorname{rot} v_j\|^2, \quad c_{\mathrm{LT}} \le \frac{6}{\pi}\,. \tag{3.5}$$

If, in addition, $v_j = \mathrm{N}v_j$, $\varphi_j = \mathrm{N}\varphi_j$*, then for* $\rho_{\mathrm{N}v}(x) = \sum_{j=1}^m |\mathrm{N}v_j(x)|^2$ *and* $\rho_{\mathrm{N}\varphi}(x) = \sum_{j=1}^m \mathrm{N}\varphi_j(x)^2$ *the following inequalities hold:*

$$\int_\Omega \rho_{\mathrm{N}\varphi}(x)^2 dx \le c_{\mathrm{N}} \sum_{j=1}^m \|\nabla \mathrm{N}v_j\|^2, \quad \int_\Omega \rho_{\mathrm{N}v}(x)^2 dx \le c_{\mathrm{N}} \sum_{j=1}^m \|\operatorname{rot} \mathrm{N}v_j\|^2, \quad c_{\mathrm{N}} \le \frac{12}{\pi}\,. \tag{3.6}$$

Finally, if $v_j = \mathrm{M}v_j$, $\operatorname{div} v_j = 0$*, then* $\rho_{\mathrm{M}v}(x) = \sum_{j=1}^m |\mathrm{M}v_j(x)|^2$ *satisfies*

$$\int_\Omega \rho_{\mathrm{M}v}(x)^2 dx \le \frac{c_{\mathrm{M}}}{L} \sum_{j=1}^m \|\operatorname{rot} \mathrm{M}v_j\|, \quad c_{\mathrm{M}} \le 6. \tag{3.7}$$

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