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The Impact of Grid Cell Geometry on the Least-Squares Gradient Reconstruction.

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Abstract

We consider the problem of the least-squares approximation on two-dimensional unstructured grids with "bad" cells. We discuss how the accuracy of the least-squares approximation depends on the cell geometry. We analyze a simple geometry and demonstrate that introducing weight coefficients into the problem may help to essentially improve the accuracy of the least-squares approximation. Based on the results of our analysis, a heuristic choice of the weights in a general least-squares procedure is suggested. Our approach is illustrated by numerical tests.

Introduction. [1]

We discuss the issue of a least-squares approximation on two-dimensional unstructured grids with "bad" cell geometry. The issues related to approximation and interpolation are covered in many books (e.g. see [2]). Nevertheless, to our best knowledge, the approximation over grid cells, which are almost degenerate, is not widely addressed in the literature. A general discussion of the topic can be found in [1], [5]. The impact of the cell geometry on the quality of approximation has been studied in [3, 4, 7].

Generally, a least-squares reconstruction is difficult for analysis, since a data set of an arbitrary dimension is used for the approximation. Thus we first discuss a simple geometry and demonstrate that data measuring at remote points which lie beyond an actual domain of interest, may considerably worsen the approximation. As a result of our analysis, we elaborate the weighting coefficients which correct the impact of "bad" geometry (i.e. remote points) in a general least-squares procedure. Our approach is illustrated by numerical tests.

1. Problem statement.

Consider a data set $\mathbf{U} = (U_1, U_2, ..., U_N)$ where the data U_i represent a continuous function $U(x, y)$ at points $P_i = (x_i, y_i)$, $i = 1, ..., N$. We have to fit the data $\mathbf{U}$ to the function

$$u(x, y) = \sum_{k=1}^{M} u_k \phi_k(x, y), \quad M \leq N, \tag{1}$$

where $(u_1, u_2, ..., u_M)$ are fitting parameters, and $\phi_k(x, y)$, $k = 1, ..., M$ are basis functions. We define the merit function F^2 as follows (e.g. see [6])

$$F^2 = \sum_{i=1}^{N} \left[\frac{U_i - \sum_{k=1}^{M} u_k \phi_k(P_i)}{\sigma_i} \right]^2. \tag{2}$$

Below we loosely refer to parameter σ_i as the weight of the i-th data point.

A least-squares approach considers the vector $\mathbf{u} = (u_1, u_2, ..., u_M)$ as the best fit to a given data set, if $\mathbf{u}$ minimizes the function (2). Thus, the parameters u_k can be found from the M conditions

$$\frac{\partial F^2}{\partial u_k} = 0, \quad k = 1, ..., M,$$

which are called the normal equations of the least-squares problem. Taking into account the definition (2), we obtain the normal equations in the following form

$$\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left[U_i - \sum_{j=1}^{M} u_j \phi_j(P_i) \right] \phi_k(P_i) = 0, \ \ k = 1, ..., M.$$

We introduce the weighted data **b** and the design matrix $[A]$ as follows

$$b_i = U_i/\sigma_i, \ \ \ A_{ij} = \phi_j(P_i)/\sigma_i, \ \ \ i = 1, ..., N, \ j = 1, ..., M.$$

The normal equations can be written as $[\alpha] \, \mathbf{u} = [\beta]$, where the matrix $[\alpha] = [A]^T [A]$, and the right-hand side $\beta = [A]^T \mathbf{b}$. They are to be solved for the vector of parameters $\mathbf{u} = (u_1, ..., u_M)$,

$$\mathbf{u} = [\alpha]^{-1}[\beta]. \tag{3}$$

Below we consider the approximation (1) with linear basis functions,

$$u(x, y) = u_0 + u_1(x - x_0) + u_2(y - y_0), \tag{4}$$

where the origin $P_0 = (x_0, y_0)$ is chosen to serve the needs of the problem under consideration. Our main purpose is to understand how the fitting parameters **u** in (4) depend on the geometry $\{P\}$. The problem can be illustrated by the following example. Consider the simplest geometry, $P_1 = (-\Delta x, 0)$, $P_2 = (0, \Delta y)$, $P_3 = (\Delta x, 0)$, and $P_4 = (0, -\Delta y)$. We reconstruct $U(x, y)$ at $P_0 = (0, 0)$, assuming $\sigma_i = 1, \forall i = 1, ..., 4$. The matrix $[\alpha]^{-1}$ is diagonal,

$$[\alpha]^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2\Delta x^2} & 0 \\ 0 & 0 & \frac{1}{2\Delta y^2} \end{bmatrix},$$

and the reconstruction (4) is

$$u(x, y) = \frac{U_1 + U_2 + U_3 + U_4}{4} + \frac{(U_3 - U_1)}{2\Delta x} x + \frac{(U_2 - U_4)}{2\Delta y} y.$$

One can see that the least-squares procedure provides a second order approximation to the function $U(x, y)$ and gradient $(\partial U(x, y)/\partial x, \partial U(x, y)/\partial y)_{|P_0}$. However, this result should be essentially attributed to the geometry of the problem. Now let us have an arbitrary geometry, $P_i = (\Delta x_i, \Delta y_i)$, $i = 1, ..., 4$. The entries of the matrix $[\alpha]^{-1}$ are not "decoupled" anymore, as each of them

depends now on each of deviations $(\Delta x_i, \Delta y_i)$. Hence, each fitting parameter u_k can be presented as

$$u_k = \sum_{l=1}^{N} C_{kl}(\{P\})U_l,$$

where coefficients $C_{kl}(\{P\})$ are defined by the geometry $\{P\}$.

We are going to study how far the geometric coefficients $C_{kl}(\{P\})$ are responsible for the quality of the least-squares approximation. In particular, we are interested in the gradient reconstruction at a given point P_0, in which case $u_0 = U(x_0, y_0)$ and $\mathbf{u} = (u_1, u_2)$. The matrix $[\alpha]^{-1}$ used for the gradient reconstruction takes the form

$$[\alpha]^{-1} = \begin{bmatrix} \dfrac{\sum_{i=1}^{N}(y_i - y_0)^2/\sigma_i^2}{\Delta} & \dfrac{-\sum_{i=1}^{N}(x_i - x_0)(y_i - y_0)/\sigma_i^2}{\Delta} \\ \dfrac{-\sum_{i=1}^{N}(x_i - x_0)(y_i - y_0)/\sigma_i^2}{\Delta} & \dfrac{\sum_{i=1}^{N}(x_i - x_0)^2/\sigma_i^2}{\Delta} \end{bmatrix}, \quad (5)$$

where

$$\Delta = \sum_{i=1}^{N}(x_i - x_0)^2/\sigma_i^2 \sum_{i=1}^{N}(y_i - y_0)^2/\sigma_i^2 - \left(\sum_{i=1}^{N}(x_i - x_0)(y_i - y_0)/\sigma_i^2\right)^2.$$

The gradient error is

$$e_\nabla(P_0) = ||\nabla U(x, y) - \nabla u(x, y)||_{|P_0} =$$

$$\sqrt{\left((\partial U(x, y)/\partial x)_{|P_0} - u_1\right)^2 + \left((\partial U(x, y)/\partial y)_{|P_0} - u_2\right)^2}$$

where ∇ is a formal notation for the gradient vector, $\nabla = (\partial/\partial x, \partial/\partial y)$.

We begin our consideration with the following simple configuration. Let $P_1 = (-H, h_1)$, $P_2 = (0, h_0)$, and $P_3 = (H, h_1)$, where $H \gg h_0$ (see configuration I in fig.1). We define the function $U(x, y)$ as

$$U(x, y) = ax^2 + y, \quad (6)$$

where the parameter $a = -0.001$, and reconstruct the gradient (u_1, u_2) at the origin $P_0 = (0, 0)$. The analytic gradient is $\partial U/\partial x = 2ax$, $\partial U/\partial y = 1$, so that $\nabla U(P_0) = (0, 1)$.

An unweighted least-squares approach (i.e. $\sigma_i = 1$, $\forall i = 1, ..., N$) gives us the matrix $[\alpha]^{-1}$ and the right-hand side β as follows
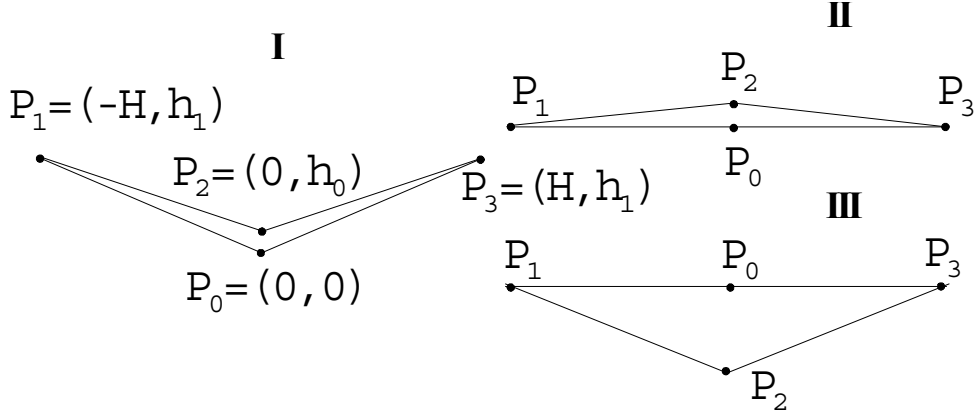
Figure 1: *The geometry for the least-squares approximation. Configuration I has outliers in the data set. Configurations II, III have no outliers.*

$$[\alpha]^{-1} = \begin{bmatrix} \dfrac{1}{2H^2} & 0 \\ 0 & \dfrac{1}{2h_1^2 + h_0^2} \end{bmatrix}, \quad \beta = \begin{bmatrix} (U_3 - U_1)H \\ U_2 h_0 + (U_3 + U_1)h_1 \end{bmatrix},$$

where $U_i = U(P_i)$. According to (3), the gradient (u_1, u_2) is reconstructed as

$$u_1 = \frac{U_3 - U_1}{2H}, \quad u_2 = U_2 \frac{h_0}{h_0^2 + 2h_1^2} + (U_1 + U_3)\frac{h_1}{h_0^2 + 2h_1^2}. \tag{7}$$

For the given geometry, a second order approximation to the gradient would be

$$u_1 = \frac{U_3 - U_1}{2H}, \quad , u_2 = \frac{U_2}{h_0}. \tag{8}$$

It can be seen from (7) that the least-squares procedure results in a correct reconstruction of the x-component of the gradient, while the value u_2 depends on both h_0 and h_1. Let $H = 100$, $h_1 = 10$, and $h_0 = 1$. The relations (6), (7) give us $u_1 = 0.0$, $u_2 = 4.97512 \cdot 10^{-3}$, so that the error in the gradient norm is $e_\nabla = 9.95025 \cdot 10^{-1}$.

Let us estimate which value h_1 would be appropriate for the accurate reconstruction (8) of the gradient. Substituting u_2 into (7) yields

$$\frac{1}{h_0^2 + 2h_1^2}[U_2 h_0 + (U_1 + U_3)h_1] - \frac{U_2}{h_0} = 0.$$

Taking into account that $U_1 = U_3 = aH^2 + h_1$ and $U_2 = h_0$, we find that $h_1 = 0$. This trivial solution means that our domain of interest, where the

data $\mathbf{U}$ are to be defined, has only two characteristic lengths, one of which is h_0 and another one is H.

Suppose that we carry out a physical experiment and measure by mistake our function $\mathbf{U}$ at some points which lie beyond the characteristic domain. Such measurements (called outliers in the statistics), included into the data set, may seriously affect the results of the least-squares procedure used to treat the experimental data. For the configuration I, the length $h_1 \neq 0$ indicates that the geometry of the domain associated with the function $U(x, y)$ has not been correctly defined. Thus, we may suggest outliers in our problem, i.e. the data at the points which lie beyond the actual characteristic domain.

What are outliers in our data set? An evident answer is to recognize the remote points P_1 and P_3 as the outliers. The configuration II shown in the figure provides $h_1 = 0$ and gives us the gradient (8) at the point P_0. Another, less evident conclusion, is to admit that it is not correct to measure the data at the point P_0. An appropriate choice would be to put the point P_0 in the vicinity of the midpoint of the edge $P_1 - P_3$ (see configuration III in the figure).

2. The choice of weights in the least-squares procedure.

Now let us look how we can improve the gradient estimate (7) by using weights in the least-squares procedure. Since the configuration is symmetric relative to the y-axis, the equal weights $\sigma_1 = \sigma_3 \equiv \sigma$ are assumed for the data at the points P_1 and P_3. We denote the weight of U_2 as δ.

For the weighted least-squares approximation, the matrix $[\alpha]^{-1}$ and the right-hand side β are given by

$$[\alpha]^{-1} = \begin{bmatrix} \dfrac{\sigma^2}{2H^2} & 0 \\ 0 & \dfrac{\sigma^2\delta^2}{2h_1^2\delta^2 + \sigma^2 h_0^2.} \end{bmatrix}, \quad \beta = \begin{bmatrix} (U_3 - U_1)\dfrac{H}{\sigma^2} \\ (U_3 + U_1)\dfrac{h_1}{\sigma^2} + U_2\dfrac{h_0}{\delta^2} \end{bmatrix}. \quad (9)$$

The gradient is

$$u_1 = (U_3 - U_1)\frac{1}{2H},$$

$$u_2 = \frac{\sigma^2\, h_0^2}{\sigma^2\, h_0^2 + \delta^2\, 2h_1^2}\frac{U_2}{h_0} + \frac{\delta^2\, h_1}{\sigma^2\, h_0^2 + \delta^2\, 2h_1^2}(U_1 + U_3) =$$

$$A_w(\sigma, \delta)\frac{U_2}{h_0} + B_w(\sigma, \delta)(U_1 + U_3),$$

where

$$A_w(\sigma, \delta) = \frac{\sigma^2 h_0^2}{\sigma^2 h_0^2 + \delta^2 2h_1^2}, \quad B_w(\sigma, \delta) = \frac{\delta^2 h_1}{\sigma^2 h_0^2 + \delta^2 2h_1^2}.$$

It can be seen from the expression above that the values $A_w(\sigma, \delta) = 1$, $B_w(\sigma, \delta) = 0$ are required to get the consistent approximation (8). Solving these equations, we obtain $\delta = 0$, $\sigma \in \mathsf{R}$, which solution is irrelevant to the least-squares problem.

For practical purposes, we need to evaluate σ and δ in magnitude in order to obtain asymptotic estimates $A_w(\sigma, \delta) \rightarrow 1$, $B_w(\sigma, \delta) \rightarrow 0$. To get such estimates we use the following approach. Let us consider the unweighted least-squares procedure in which new basis functions are exploited instead of the basis $\phi_1 = x$, $\phi_2 = y$. The new basis functions are

$$\tilde{\phi}_1 = f(x, y)\phi_1, \quad \tilde{\phi}_2 = g(x, y)\phi_2,$$

where the functions $f(x, y)$ and $g(x, y)$ must take the required values $1/\sigma$ and $1/\delta$ at points P_i. This approach is equivalent to the weighting procedure in which the data vector $U(P_i)$ remains unweighted. The matrix $[\alpha]^{-1}$ and the right-hand side β are now given by

$$[\alpha]^{-1} = \begin{bmatrix} \dfrac{\sigma^2}{2H^2} & 0 \\ 0 & \dfrac{\sigma^2\delta^2}{2h_1^2\delta^2 + \sigma^2 h_0^2.} \end{bmatrix}, \quad \beta = \begin{bmatrix} (U_3 - U_1)\dfrac{H}{\sigma} \\ (U_3 + U_1)\dfrac{h_1}{\sigma} + U_2\dfrac{h_0}{\delta} \end{bmatrix}. \quad (10)$$

The fitting parameters are

$$u_1 = (U_3 - U_1)\frac{\sigma}{2H}, \quad (11)$$

and

$$u_2 = \frac{\sigma^2\,\delta^2}{\delta^2\,2h_1^2 + \sigma^2 h_0^2}\left[U_2\frac{h_0}{\delta} + (U_1 + U_3)\frac{h_1}{\sigma}\right]. \quad (12)$$

Again, we seek for the weights (σ, δ) which provide the second order approximation (8) to the gradient. Comparing (11) and (8), we get $\sigma = 1$. Substituting (8) and $\sigma = 1$ into (12) yields

$$U_2 h_0\delta + 2(U_1 + U_3)h_1\delta^2 = 2h_1^2\delta^2 + h_0^2.$$

Solving the above equation for δ, we require that the discriminant $D = U_2^2 h_0^2 + 8h_0^2 h_1(U_1 + U_3 - h_1) > 0$. Taking into account the explicit form of the function (6), we obtain the following estimate

$$h_1 < h_1^{max} = 125\frac{h_0^2}{H^2}, \quad (13)$$

i.e. $h_1 < 1/80$ in case that $h_0 = 1$, $H = 100$. Only if the points P_1 and P_3 lie in the narrow band between 0 and h_1^{max}, the weights will help to get the accurate gradient.

Since our geometry does not meet the requirement (13), a formal conclusion should be that the accurate reconstruction of both gradient components is impossible. However, we essentially used our knowledge of the function $U(x, y)$ in our analysis. The symmetry of the function $U(x, y)$ allows to get the required value $u_1 = 0$ along the x-axis. Thus, below we analyze the y-component of the gradient, assuming an arbitrary weight in the expression (11).

Let us rewrite (12) as

$$u_2 = A(\sigma, \delta) \frac{U_2}{h_0} + B(\sigma, \delta)(U_1 + U_3),$$

where

$$A(\sigma, \delta) = \frac{\sigma^2 \delta h_0^2}{\sigma^2 h_0^2 + \delta^2 2h_1^2}, \quad B(\sigma, \delta) = \frac{\sigma \delta^2 h_1}{\sigma^2 h_0^2 + \delta^2 2h_1^2}.$$

First we define the parameter σ. The equation $A(\sigma, \delta) = 1$ yields

$$\sigma^* = \frac{h_1}{h_0} \sqrt{\frac{2\delta^2}{\delta - 1}}. \tag{14}$$

Substituting σ^* into the equation $B(\sigma^*, \delta^*) = 0$, we obtain

$$\frac{\sqrt{2} h_1 \delta \sqrt{\delta - 1}}{2h_0^2 + 2h_0 h_1(\delta - 1)} = 0.$$

The two roots of the equation are $\delta_1^* = 0$ and $\delta_2^* = 1$. The choice of $\delta_1^* = 0$ is irrelevant to the least-squares problem. The value $\delta_2^* = 1$ gives us $\sigma^* = \infty$ with a consistent asymptotic estimate $u_2 \to 1$, as $\sigma \to \infty$. Hence, for practical purposes one should take $\delta = 1$ and choose the weights (14) for the data at the remote points P_1 and P_3 as big as possible to obtain the gradient reconstruction u_2 with the desired accuracy.

To evaluate the order of magnitude for σ, we analyze the matrix (5) in the presence of outliers. Let points $P_1, P_2, ..., P_{N-1}$ be (x_i, y_i), where $x_i \sim l$, $y_i \sim l$, and the point P_N be a remote point, $x_N \sim L$, $y_N \sim L$, where we assume $L \gg Nl$. Consider the diagonal entries of the matrix $[\alpha]^{-1}$ for the unweighted least-squares approximation,

$$\alpha_{11}^{-1} = \frac{\sum\limits_{i=1}^{N} y_i^2}{\sum\limits_{i=1}^{N} x_i^2 \sum\limits_{i=1}^{N} y_i^2 - \left(\sum\limits_{i=1}^{N} x_i y_i\right)^2}, \tag{15}$$

where the origin $(x_0, y_0) = (0, 0)$.

Each sum in (15) can be rearranged as

$$\alpha_{11}^{-1} \sim \frac{\sum\limits_{i=1}^{N-1} y_i^2 + L^2}{(\sum\limits_{i=1}^{N-1} x_i^2 + L^2)(\sum\limits_{i=1}^{N-1} y_i^2 + L^2) - (\sum\limits_{i=1}^{N-1} x_i y_i + L^2)^2} =$$

$$\frac{\sum\limits_{i=1}^{N-1} y_i^2 + L^2}{\sum\limits_{i=1}^{N-1} x_i^2 \sum\limits_{i=1}^{N-1} y_i^2 + \sum\limits_{i=1}^{N-1} x_i^2 L^2 + \sum\limits_{i=1}^{N-1} y_i^2 L^2 + L^4 - (\sum\limits_{i=1}^{N-1} x_i y_i)^2 - 2\sum\limits_{i=1}^{N-1} x_i y_i L^2 - L^4},$$

or, taking into account $\sum\limits_{i=1}^{N-1} x_i^2 \sim \sum\limits_{i=1}^{N-1} y_i^2 \sim (N-1)l^2$,

$$\alpha_{11}^{-1} \sim \frac{L^2}{\sum\limits_{i=1}^{N-1} x_i^2 L^2 + \sum\limits_{i=1}^{N-1} y_i^2 L^2 - 2\sum\limits_{i=1}^{N-1} x_i y_i L^2} = \frac{1}{\sum\limits_{i=1}^{N-1} (x_i - y_i)^2}.$$

A similar estimate holds for α_{22}^{-1}.

Since we assume $x_i \sim l$, $y_i \sim l$, their difference can be very small, $x_i - y_i \sim \epsilon \to 0$. (Note the singularity in the estimate above which arise in degenerated case that all of the points P_i are placed at the same straight line.) The entries of $[\alpha]^{-1}$ will grow with grid refinement and affect the gradient reconstruction (3), unless the outlier is suppressed by a weight σ_L, $L/\sigma_L \to 0$, as $L \to \infty$.

For the configuration I, we have $L \sim \sqrt{H^2 + h_1^2} \approx H$. Thus, we choose $\sigma = H^2$ in which case the weighted least-squares procedure yields $u_2 = 0.999998$, $e_\nabla = 2 \cdot 10^{-6}$ for values $h_0 = 1$, $h_1 = 10$, $H = 100$.

3. A general weighting procedure. Numerical results.

The analysis made in the previous section concerns a particular configuration and function $U(x, y)$. Nevertheless, it allows us to elaborate heuristic weights in the least-squares procedure. A general recommendation is that once outliers have been detected in the problem, those data must have small values $\rho_i = (1/\sigma_i)$. Based on the analysis above we suggest that the weights σ_i can be defined as

$$\sigma_i = r_i^2 = (x_i - x_0)^2 + (y_i - y_0)^2,$$

where (x_0, y_0) is the point where the gradient is reconstructed. Formally, the

weights can be scaled as

$$\tilde{\sigma}_i = \sigma_i/r_{min}^2, \quad r_{min} = \min_{i=1,\dots,N}\{r_i\},$$

so that the point closest to the origin P_0 has $\tilde{\sigma}_{min} = 1$.

Below we consider numerical tests which illustrate the choice of the weight coefficients. We define the geometry for our test cases as follows. Let points $P_1 = (-H, 0)$, $P_2 = (0.02, h)$, $P_3 = (H, -0.01)$, and $P_4 = (-0.07, -h)$, where h and H are the controlling parameters for the configuration. Let $H = 1$, $h = 1$, so the points P_i lie very close to a unit circle. We are going to imitate a refinement procedure by halving h at each "refinement step", the parameter H being fixed. We reconstruct the gradient (u_1, u_2) at $P_0 = (0, 0)$ and compare the error $e_\nabla(h)$ for the unweighted and weighted least-squares approximation. We are also interested in the diagonal elements of the matrix $[\alpha]^{-1}$, which, in our opinion, may help to detect the outliers.

We begin our consideration with a function

$$U(x) = \frac{1}{2}((x - A)^2 + (y - A)^2), \tag{16}$$

where the parameter $A = 2$. The gradient is $\partial U/\partial x = x - A$, $\partial U/\partial y = y - A$, $\nabla U(P_0) = (-2, -2)$.

First, we reconstruct the gradient by using an unweighted least-squares approach. The error $e_\nabla(h)$ and the diagonal entries α_{ii}^{-1}, $i = 1, 2$ are shown in Table 1.

Table 1.

The unweighted least-squares procedure for the function (16).

h	$e_\nabla(h)$	α_{11}^{-1}	α_{22}^{-1}
1.0000	0.0152469	0.4993633	0.5005867
0.5000	0.01270758	0.4992012	2.001396
0.2500	0.04471311	0.498952	7.996793
0.1250	0.1690752	0.4987531	31.89793
0.06250	0.652037	0.4995403	126.5818
0.03125	2.506628	0.5056627	493.8101
0.01562	9.159516	0.5333749	1817.866
0.007813	28.16445	0.6214184	5610.583
0.003907	59.85515	0.7776305	11945.89
0.001953	84.41483	0.9052282	16863.26
0.0009766	94.74845	0.9624739	18936.42
0.0004883	98.10343	0.9827952	19611.53

It can be seen from the table that the error in the gradient grows, as h is getting smaller. The error in the gradient norm is mostly due to the y-component of the gradient. Actually, at the last step of the refinement we have $e_x = |\partial U/\partial x_{|P_0} - u_1| = 0.48743$ and $e_y = |\partial U/\partial y_{|P_0} - u_2| = 98.103$.

Let us evaluate the characteristic lengths l_x and l_y from the relation

$$\frac{U(l_x, 0) - U(0, 0)}{l_x} \sim 1, \quad \frac{U(0, l_y) - U(0, 0)}{l_y} \sim 1.$$

For "homogeneous" function (16), the characteristic domain is $l_x = l_y$. Our "anisotropic" refinement procedure makes the points P_1 and P_3 lie outside the characteristic domain. The presence of the outliers in the problem is evidenced by different orders of magnitude in α_{11}^{-1} and α_{22}^{-1}, as one may expect $\alpha_{11}^{-1} \sim \alpha_{22}^{-1}$, if the characteristic domain is chosen adequate to the function (16). Thus, the weighting procedure is required to eliminate the outliers.

Table 2.

The weighted least-squares procedure for the function (16).

h	$e_\nabla(h)$	α_{11}^{-1}	α_{22}^{-1}
1.0000	0.01489822	0.4994162	0.5030751
0.5000	0.05869409	0.4927092	0.1304421
0.2500	0.1923858	0.4118147	0.0430285
0.1250	0.2295866	0.1353229	0.01894057
0.0625	0.1119552	0.024711	0.006007634
0.03125	0.06687997	0.009721229	0.002569299
0.01562	0.06012273	0.007275371	0.003624347
0.007812	0.07669123	0.00675025	0.01162409
0.003906	0.127259	0.006623483	0.04456423
0.001953	0.2398984	0.006590117	0.1765079
0.0009766	0.4704899	0.006577596	0.703892
0.0004883	0.9289341	0.00656544	2.809573

The results of the weighted least-squares approximation are shown in Table 2. Comparing the unweighted and weighted gradient reconstruction, one can see that weighting reduces the gradient error. However, the weighting procedure is efficient only in the presence of outliers. At initial steps of the refinement, the error is smaller for the unweighted least-squares approach. Also, it can be seen from Table 2 that at last steps of the refinement the geometry distortion is so strong that the error begins to grow again.

Table 3.

The unweighted least-squares procedure for the function (17).

h	$e_\nabla(h)$	α_{11}^{-1}	α_{22}^{-1}
1.0000	2.153319	0.4993633	0.5005867
0.5000	2.153135	0.4992012	2.001396
0.2500	2.18501	0.498952	7.996793
0.1250	2.429651	0.4987531	31.89793
0.0625	4.095294	0.4995403	126.5818
0.03125	11.7667	0.5056627	493.8101
0.01562	38.88393	0.5333749	1817.866
0.007812	114.0223	0.6214184	5610.583
0.003906	236.5301	0.7776305	11945.89
0.001953	329.5013	0.9052282	16863.26
0.0009766	367.5444	0.9624739	18936.42
0.0004883	379.3713	0.9827952	19611.53

Table 4.

The weighted least-squares procedure for the function (17).

h	$e_\nabla(h)$	α_{11}^{-1}	α_{22}^{-1}
1.0000	2.153138	0.4994162	0.5030751
0.5000	2.171339	0.4927092	0.1304421
0.2500	2.021146	0.4118147	0.0430285
0.1250	1.099674	0.1353229	0.01894057
0.0625	0.6035814	0.024711	0.006007634
0.03125	0.5099836	0.009721229	0.002569299
0.0156	0.4908921	0.007275371	0.003624347
0.007812	0.4869887	0.00675025	0.01162409
0.003906	0.4880201	0.006623483	0.04456423
0.001953	0.4951019	0.006590117	0.1765079
0.0009766	0.5184101	0.006577596	0.703892
0.0004883	0.5765298	0.00656544	2.809573

Another solution to Laplace's equation is

$$U(x,y) = A \log \left(\sqrt{R_1(x,y)/R_2(x,y)} \right), \qquad (17)$$

where $R_1(x,y) = (x-a)^2 + y^2$, $R_2(x,y) = (x-4a)^2 + y^2$. In our numerical calculations, $a = 1.0$, $A = 1.0$. The gradient is $\partial U/\partial x = A((x-a)/R_1(x,y) - (x-4a)/R_2(x,y))$, $\partial U/\partial y = A(y/R_1(x,y) - y/R_2(x,y))$, $\nabla U(P_0) = (-0.25, 0)$.

The results for the unweighted and weighted least-squares approach are shown in Table 3 and Table 4, respectively. The results are similar to those for the function (16). Again, the outliers in the problem are evidenced by different orders of magnitude in the diagonal elements of $[\alpha]^{-1}$.

Our next test is to consider the function

$$U(x, y) = x + \exp(Ay), \tag{18}$$

where the parameter $A = 5$. The gradient is $\partial U/\partial x = 1$, $\partial U/\partial y = A \exp(Ay)$, $\nabla U(P_0) = (1, 5)$.

Table 5.
The least-squares procedure (LS) for the function (18). The gradient error for the unweighted and weighted LS approach.

h	$e_\nabla(h)$ (unweighted LS)	$e_\nabla(h)$ (weighted LS)
1.0000	69.28902	69.63832
0.5000	7.106995	7.835687
0.2500	1.408095	5.731806
0.1250	0.3305086	5.29683
0.0625	0.07890232	1.911436
0.03125	0.01458822	0.6712025
0.01562	0.009495008	0.2829979
0.007812	0.03703674	0.133406
0.003906	0.07644436	0.06547207
0.0019531	0.1057604	0.03250343
0.0009766	0.1176681	0.0161402
0.0004883	0.1213422	0.007900425

The gradient error for the unweighted and weighted least-squares approach is shown in Table 5. For the function (18), one may expect the characteristic length $l_y \ll l_x$, so that the data at the points P_2 and P_4 will be outliers at the first steps of the refinement. It can be seen from the table that the weighting procedure does not recognize the outliers for the function (18), as a uniform initial geometry provides almost equal weights for the points of the configuration. On the other hand, the outliers, which exist on the initial grid, can be eliminated by means of the refinement. The unweighted least-squares procedure is appropriate for the problem, as the gradient error reduces with the grid refinement. Only that the grid parameter $h \ll l_y$ (i.e. at last steps of the refinement), the unweighted approach fails to reconstruct the gradient, so that weighting is of use.

Conclusions.

- We have obtained preliminary results concerning the problem of the least-squares approximation on bad grids. It has been shown that if data used for the approximation are measured at the points which lie beyond an actual characteristic domain, the least-squares procedure will give us a reconstruction with poor accuracy.

- We have suggested a heuristic choice of the weights in the least-squares approximation. It has been demonstrated that introducing weight coefficients into the problem may help to eliminate outliers and improve the accuracy of the least-squares procedure. The issue of weighting requires a further thorough study and has to be considered together with the problem of the detection of outliers.

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