KELDYSH INSTITUTE OF APPLIED MATHEMATICS
RUSSIAN ACADEMY OF SCIENCES

Yu. B. Radvogin

A CANONICAL FORM OF THE MULTI-COMPONENT FILTRATION SYSTEM. HYPERBOLICITY AND STABILITY

Moscow, 2001 .

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Yu.B. Radvogin. *A canonical form of the multi-component filtration system. Hyperbolicity and stability.*

Abstract. The propagation of small discontinuities (ε-breaks) is studied as applied to the multidimensional multi-component filtration problem. The characteristic properties of the problem are investigated. The canonical form of the governing system is presented and the concept of characteristic is specified in connection with the problem under consideration. This form consists of a "hyperbolic" subsystem and a "parabolic" equation as well. Stability of the flow is discussed.

[1] This work is supported by RFBR, Grant No. 00–01–00387.

1 Introduction

The presented work is devoted to the investigation of the multi-component filtration system which does not belong to any standard class of differential equations (hyperbolic, parabolic etc). Nevertheless, some properties of the solutions to this problem make it possible to consider in some sense the governing system as hyperbolic. In fact, it was shown in [1] that the velocities of weak discontinuities of solutions are finite and they are the eigenvalues of an eigenvalue problem with a linear restiction. For the case $N = 2$ — two-component model — this result means that there exists a unique value of a weak discontinuity velocity. It has also been found in [2] that the velocity of the $\varepsilon-$jumps (small amplitude breaks) is the same as that of the propagation of weak discontinuities. These facts make it possible to determine the notion of characteristic. Using the above results one can "split" the governing system into the "advection" and "parabolic" equations, see [2]. Such a form of the system can be called canonical. Note that exactly this form can be used for the creation of high resolution numerical algorithms because it is impossible to capture discontionuities without using characteristic properties of the problem.

The main aim or this work is the extension of the results for the two-component problem to the multicomponent case and the discussion of some questions connected with well-posedness of the corresponding Cauchy problem.

We consider the filtration of an M-phase fluid consisting of N components. Let the pressure P be the same for all phases. Then the governing system can be written in the form of N conservation laws:

$$\frac{\partial}{\partial t}\left(m\sum_{\alpha=1}^{M} c_{i\alpha}n_\alpha s_\alpha\right) + Div\left(\sum_{\alpha=1}^{M} c_{i\alpha}n_\alpha \mathbf{u}_\alpha\right) = 0. \tag{1}$$

Here, m is the porosity, n_α is the molar density of the αth phase, $c_{i\alpha}$ is the molar concentration of the ith component in αth phase, s_α is the phase saturation (the volume concentration of the αth phase in the mixture), and $\mathbf{u}_\alpha$ is a filtration velocity of the αth phase.

According to the Darcy law we have

$$\mathbf{u}_\alpha = -\frac{K}{\mu_\alpha}k_{r\alpha}(s_\alpha)\nabla P, \tag{2}$$

where K is the total permeability, $k_{r\alpha}$ and μ_α are the relative permeability and viscosity of the αth phase, respectively.

The system (1)–(2) has to be supplemented with the two conditions:

$$\sum_{i=1}^{N} c_{i\alpha} = 1, \qquad \sum_{\alpha=1}^{M} s_\alpha = 1. \tag{3}$$

Denote by c_i the molar concentration of the ith component in the whole mixture. Let n be the molar density of the mixture. Then we have

$$nc_i = \sum_{\alpha=1}^{M} c_{i\alpha}n_\alpha s_\alpha. \tag{4}$$

Since all the velocities $\mathbf{u}_i$ are proportional to the common vector ∇P, it follows that by neglecting the compressibility of the porous media (e. g. assuming $m = \text{Const}$) we can rewrite (1) as follows:

$$\frac{\partial nc_i}{\partial t} + Div(\varphi_i \mathbf{Q}) = 0, \tag{5}$$

where

$$\mathbf{Q} = -\frac{K}{m}\nabla P, \tag{6}$$

and

$$\varphi_i = \sum_{\alpha=1}^{M} c_{i\alpha}n_\alpha \frac{k_{r\alpha}(s_\alpha)}{\mu_\alpha}. \tag{7}$$

The normalization condition is

$$\sum_{i=1}^{N} c_i = 1. \tag{8}$$

Under the thermodynamic equilibrium assumption $s_\alpha = s_\alpha(c_1, ..., c_N, P)$. Hence one can simply consider n and φ_i as known functions of $c_1, ...c_N$ and P. Such a formulation makes it possible to study some general properies of solutions.

2 Characteristic directions and characteristic relations

For simplicity we shall consider in detail only the one-dimensional case. The corresponding results for multidimensional problem will be given in § 1.

We start with the governing system of the form:

$$\frac{\partial nc_i}{\partial t} + \frac{\partial \varphi_i Q}{\partial x} = 0, \quad i = 1, ..., N, \tag{9}$$

$$\sum_{i=1}^{N} c_i = 1. \tag{10}$$

Here, φ_i, and n depend on $c_1, ..., c_N$, and P, and $Q = -K(x)\frac{\partial P}{\partial x}$.

Using (10) one can eliminate one of c_i for example c_N. Summing all the equations (9) and taking into account (10) we obtain

$$\frac{\partial n\vec{c}}{\partial t} + \frac{\partial \vec{\varphi} Q}{\partial x} = 0, \tag{11}$$

$$\frac{\partial n}{\partial t} + \frac{\partial \sigma Q}{\partial x} = 0. \tag{12}$$

where $\vec{c} = (c_1, ..., c_{N-1})^*$, $\vec{\varphi} = (\varphi_1, ..., \varphi_{N-1})^*$, $\vec{\varphi} = \vec{\varphi}(\vec{c}, P)$, $n = n(\vec{c}, P)$, $\sigma = \sum_{i=1}^{N} \varphi_i$. (Here and below the arrow denotes a vector of the $(N-1)-$dimensional concentration space and the astersk denotes a column vector.)

Introducing $\vec{\varphi} = \sigma\vec{\psi}$ we obtain the final form of the governing system:

$$\frac{\partial n\vec{c}}{\partial t} + \frac{\partial \vec{\psi}\sigma Q}{\partial x} = 0, \tag{13}$$

$$\frac{\partial n}{\partial t} + \frac{\partial \sigma Q}{\partial x} = 0. \tag{14}$$

Now we consider an $\varepsilon-$discontinuity line. The standard Hugoniot conditions are of the form:

$$\begin{aligned} &D\,[n\vec{c}] - [\vec{\psi}\sigma Q] = 0, \\ &D\,[n] - [\sigma Q] = 0. \end{aligned} \tag{15}$$

Here, the brackets denote the jumps of the variables and D denotes the velocity of the $\varepsilon-$jump (that is $D = dx/dt$, where $x = x(t)$ is the equation of the line). It should be emphasized that $[P]$=0 for any type of breaks.

Replacing all $[f]$ by df one can rewrite (15) in the form

$$D\,n\,d\vec{c} + D\,\vec{c}\,dn - \vec{\psi}\,d(\sigma Q) - \sigma Q\,d\vec{\psi} = 0, \tag{16}$$

$$D\,dn - d(\sigma Q) = 0. \tag{17}$$

It follows from (16)-(17) that

$$D\,n\,d\vec{c} - D(\vec{\psi} - \vec{c})dn - \sigma Q d\vec{\psi} = 0. \tag{18}$$

Taking into account $dP = 0$ we have

$$dn = \nabla_c n \cdot d\vec{c}, \quad d\vec{\psi} = \frac{\partial \vec{\psi}}{\partial \vec{c}}\,d\vec{c}.$$

Here, $\nabla_c = \left(\frac{\partial}{\partial c_1}, ..., \frac{\partial}{\partial c_{N-1}}\right)$.

It remains to introduce $\lambda:\ D = \lambda Q$. Finally, we obtain the matrix form of (18):

$$(B - \lambda A)d\vec{c} = 0, \tag{19}$$

where

$$B = \{b_{ij}\}, \qquad b_{ij} = \sigma \frac{\partial \psi_i}{\partial c_j}, \tag{20}$$

$$A = \{a_{ij}\}, \qquad a_{ij} = -(\psi_i - c_i) \frac{\partial n}{\partial c_i} + n\delta_{ij}. \tag{21}$$

(δ_{ij} is the Kronecker delta).

This equation means that:

(i) each real eigenvalue $\lambda^{(k)}$ of the matrix BA^{-1} defines the kth $\varepsilon-$discontinuity direction: $dx/dt = D^{(k)} = \lambda^{(k)} Q$;

(ii) the $\varepsilon-$jump $d\vec{c}$ (or $[\vec{c}]$) is a right-side annihilator of the matrix $B - \lambda A$ for each real $\lambda = \lambda^{(k)}$.

Note that these $D^{(k)}$ coincide with the velocities of weak discontinuities mentioned in [1]. Such a situation is typical for hyperbolic systems.

Below we consider only real eigenvalues. Note that if N is even, then at least one real eigenvalue exists.

By analogy with hyperbolic systems we call such directions "characteristi" dirctions. Following this analogy we construct the corresponding characteristic relations. In the two-component case this problem has been solved by means of the special way, see [2]. The key idea was based on the determination the function θ such that the product θQ is continous on the $\varepsilon-$jumps: $[\theta Q] = 0$. This function is a solution of some ordinary differencial equation. Using the function θ an "advection" equation was singled out from the governing system. In the multi-component case we will apply the same idea but now the implementation of it looks much more complicated because of the existance of several characteristic directions. As is already clear, now the required function θ has to be a vector: $\theta = \vec{\theta}$.

Let us choose some eigenvalue $\lambda^{(k)}$ and represent the scalar σ in the form of a scalar product of two vectors:

$$\sigma = \vec{\theta}^{(k)} \cdot \vec{\mu}, \tag{22}$$

where $\vec{\mu}$ is the same for all $\lambda^{(k)}$.

Let $\vec{\varphi}^{(k)}$ be the left-side annihilator of $B - \lambda^{(k)} A$:

$$\vec{\varphi}^{(k)}(B - \lambda^{(k)} A) = 0. \tag{23}$$

We return to (13) and exclude $\dfrac{\partial n}{\partial t}$ using (14). The result is

$$n \frac{\partial \vec{c}}{\partial t} + (\vec{\psi} - \vec{c}) \frac{\partial \sigma Q}{\partial x} + \sigma Q \frac{\partial \vec{\psi}}{\partial x} = 0. \tag{24}$$

Now we replace σ in the term $\dfrac{\partial \sigma Q}{\partial x}$ by $\vec{\theta}^{(k)} \cdot \vec{\mu}$, see (22).

$$n \frac{\partial \vec{c}}{\partial t} + (\vec{\psi} - \vec{c}) \left(\vec{\mu} \frac{\partial \vec{\theta}^{(k)} Q}{\partial x} + \vec{\theta}^{(k)} Q \frac{\partial \vec{\mu}}{\partial x} \right) + \sigma Q \frac{\partial \vec{\psi}}{\partial x} = 0. \tag{25}$$

But

$$\sigma \frac{\partial \vec{\psi}}{\partial x} = \sigma \frac{\partial \vec{\psi}}{\partial \vec{c}} \frac{\partial \vec{c}}{\partial x} + \sigma \frac{\partial \vec{\psi}}{\partial P} \frac{\partial P}{\partial x} = B \frac{\partial \vec{c}}{\partial x} + \sigma \frac{\partial \vec{\psi}}{\partial P} \frac{\partial P}{\partial x};$$

$$\frac{\partial \vec{\mu}}{\partial x} = \frac{\partial \vec{\mu}}{\partial \vec{c}} \frac{\partial \vec{c}}{\partial x} + \frac{\partial \vec{\mu}}{\partial P} \frac{\partial P}{\partial x}.$$

Hence (25) can be rewritten as follows:

$$n \frac{\partial \vec{c}}{\partial t} + Q(\vec{\psi} - \vec{c}) \left(\vec{\theta}^{(k)} \frac{\partial \vec{\mu}}{\partial \vec{c}} \frac{\partial \vec{c}}{\partial x} \right) + QB \frac{\partial \vec{c}}{\partial x} + \vec{f}^{(k)} = 0, \tag{26}$$

where

$$\vec{f}^{(k)} = Q \left((\vec{\psi} - \vec{c}) \left(\vec{\theta}^{(k)} \frac{\partial \vec{\mu}}{\partial P} \right) + \sigma \frac{\partial \vec{\psi}}{\partial P} \right) \frac{\partial P}{\partial x} + (\vec{\psi} - \vec{c}) \left(\vec{\mu} \frac{\partial \vec{\theta}^{(k)} Q}{\partial x} \right). \tag{27}$$

Since we intend to obtain the canonical form we add the term

$nQBA^{-1} \dfrac{\partial \vec{c}}{\partial x}$ to both sides of (26). Finally, we have

$$n \frac{\partial \vec{c}}{\partial t} + nQBA^{-1} \frac{\partial \vec{c}}{\partial x} = nQBA^{-1} \frac{\partial \vec{c}}{\partial x} - QB \frac{\partial \vec{c}}{\partial x} - Q(\vec{\psi} - \vec{c}) \left(\vec{\theta}^{(k)} \frac{\partial \vec{\mu}}{\partial \vec{c}} \frac{\partial \vec{c}}{\partial x} \right) - \vec{f}^{(k)}. \tag{28}$$

Now we multiply (28) by $\vec{\varphi}^{(k)}$ from the left. Because $\vec{\varphi}^{(k)}BA^{-1} = \lambda^{(k)}\vec{\varphi}^{(k)}A$ we obtain

$$n\vec{\varphi}^{(k)} \cdot \left(\frac{\partial \vec{c}}{\partial t} + \lambda^{(k)}Q\frac{\partial \vec{c}}{\partial x}\right) = g^{(k)} - \vec{\varphi}^{(k)} \cdot \vec{f}^{(k)}, \tag{29}$$

where

$$g^{(k)} = n\,\vec{\varphi}^{(k)}\lambda^{(k)}Q \cdot \frac{\partial \vec{c}}{\partial x} - Q\,\vec{\varphi}^{(k)}B\,\frac{\partial \vec{c}}{\partial x} - Q\left(\vec{\varphi}^{(k)} \cdot (\vec{\psi} - \vec{c})\right)\left(\vec{\theta}^{(k)}\frac{\partial \vec{\mu}}{\partial \vec{c}}\,\frac{\partial \vec{c}}{\partial x}\right). \tag{30}$$

But $\vec{\varphi}^{(k)}B = \lambda^{(k)}\vec{\varphi}^{(k)}A$. Hence,

$$g^{(k)} = Q\lambda^{(k)}\vec{\varphi}^{(k)}(nE - A)\frac{\partial \vec{c}}{\partial x} - Q\left(\vec{\varphi}^{(k)} \cdot (\vec{\psi} - \vec{c})\right)\left(\vec{\theta}^{(k)}\frac{\partial \vec{\mu}}{\partial \vec{c}}\,\frac{\partial \vec{c}}{\partial x}\right).$$

Here, E is the identity matrix.

We introduce

$$\xi^{(k)} = \vec{\varphi}^{(k)} \cdot (\vec{\psi} - \vec{c}), \tag{31}$$

$$\tilde{A} = nE - A; \quad \tilde{A} = \{\tilde{a}_{ij}\}, \quad \tilde{a}_{ij} = (\psi_i - c_i)\frac{\partial n}{\partial c_j}. \tag{32}$$

Then

$$g^{(k)} = Q\left(\lambda^{(k)}\vec{\varphi}^{(k)}\tilde{A} - \xi^{(k)}\vec{\theta}^{(k)}\frac{\partial \vec{\mu}}{\partial \vec{c}}\right) \cdot \frac{\partial \vec{c}}{\partial x}.$$

Let us find $\vec{\theta}^{(k)}$ and $\vec{\mu}$ so that $g^{(k)} = 0$.

It means that the identity

$$\lambda^{(k)}\vec{\varphi}^{(k)}\tilde{A} - \xi^{(k)}\vec{\theta}^{(k)}\frac{\partial \vec{\mu}}{\partial \vec{c}} \equiv 0 \tag{33}$$

must be valid under the condition (22). It follows from (33) that

$$\vec{\theta}^{(k)} = \frac{\lambda^{(k)}}{\xi^{(k)}}\,\vec{\varphi}^{(k)}\tilde{A}\left(\frac{\partial \vec{\mu}}{\partial \vec{c}}\right)^{-1}. \tag{34}$$

Substituting (34) into (22) we obtain the main equation in the form

$$\lambda^{(k)}\vec{\varphi}^{(k)}\tilde{A}\left(\frac{\partial \vec{\mu}}{\partial \vec{c}}\right)^{-1}\vec{\mu} = \sigma\xi^{(k)}, \tag{35}$$

or in the form

$$\vec{\varphi}^{(k)} \cdot \left(\lambda^{(k)}\tilde{A}\left(\frac{\partial \vec{\mu}}{\partial \vec{c}}\right)^{-1}\vec{\mu} - \sigma(\vec{\psi} - \vec{c})\right) = 0. \tag{36}$$

The equation (35) (or (36)) is a non-linear first-order PDE with respect to $\vec{\mu}(\vec{c}, P)$, that is $N-1$ functions $\mu_i(c_1, ...c_{N-1}, P)$, $i = 1, ..., N-1$. Note that this equation does not contain $\partial/\partial P$. Hence $\vec{\mu}$ contains P as a parameter.

Any solution of (35) or (36) gives us the required vector $\vec{\mu}$. The second multiplier $\vec{\theta}^{(k)}$ can easily be found from (34). Since $\vec{\mu}$ is independent of $\lambda^{(k)}$ and $\vec{\varphi}^{(k)}$, this vector is the same for all $\lambda^{(k)}$.

Taking into account that $g^{(k)} = 0$ for all (real) $\lambda^{(k)}$ we can write the kth characteristic relation (corresponding to $\lambda^{(k)}$, see (29)) in the form

$$n\vec{\varphi}^{(k)} \cdot \left(\frac{\partial \vec{c}}{\partial t} + \lambda^{(k)}Q\frac{\partial \vec{c}}{\partial x}\right) + \vec{\varphi}^{(k)} \cdot \vec{f}^{(k)} = 0. \tag{37}$$

The vector $\vec{f}^{(k)}$ contains the term $\vec{\mu} \cdot \frac{\partial}{\partial x}(\vec{\theta}^{(k)}Q)$. Let us clarify the meaning of the product $\vec{\theta}^{(k)}Q$. We consider the $\varepsilon-$jumps. Keeping in mind (22) we rewrite (17) in the form

$$\lambda^{(k)}Q\,dn - d((\vec{\theta}^{(k)} \cdot \vec{\mu})Q) = 0.$$

That is,

$$\lambda^{(k)}Q(\nabla_c n \cdot d\vec{c}) - \vec{\theta}^{(k)} \cdot Qd\vec{\mu} - \vec{\mu} \cdot d(\vec{\theta}^{(k)}Q) = 0.$$

or

$$\lambda^{(k)}Q(\nabla_c n \cdot d\vec{c}) - \vec{\theta}^{(k)}Q\frac{\partial \vec{\mu}}{\partial \vec{c}}d\vec{c} = \vec{\mu} \cdot d(\vec{\theta}^{(k)}Q). \tag{38}$$

But $\vec{\theta}^{(k)}\left(\dfrac{\partial\vec{\mu}}{\partial\vec{c}}d\vec{c}\right)=\left(\vec{\theta}^{(k)}\dfrac{\partial\vec{\mu}}{\partial\vec{c}}\right)d\vec{c}=\dfrac{\lambda^{(k)}}{\xi^{(k)}}\vec{\varphi}^{(k)}\tilde{A}$, see (34). Hence the left side of (38) goes over to

$$\lambda^{(k)}Q(\nabla_c n\cdot d\vec{c})-\vec{\theta}^{(k)}Q\frac{\partial\vec{\mu}}{\partial\vec{c}}d\vec{c}=\lambda^{(k)}Q(\nabla_c n\cdot d\vec{c})-\frac{\lambda^{(k)}}{\xi^{(k)}}\vec{\varphi}^{(k)}Q\,\tilde{A}d\vec{c}=$$

$$\frac{\lambda^{(k)}}{\xi^{(k)}}Q(\xi^{(k)}(\nabla_c n\cdot d\vec{c})-\vec{\varphi}^{(k)}\tilde{A}d\vec{c})=\frac{\lambda^{(k)}}{\xi^{(k)}}Q((\vec{\varphi}^{(k)}\cdot(\vec{\psi}-\vec{c}))(\nabla_c n\cdot d\vec{c})-\vec{\varphi}^{(k)}\tilde{A}\,d\vec{c})=0,$$

see (32). Therefore $\vec{\mu}\cdot d(\vec{\theta}^{(k)}Q)=0$.

This means that

$$\vec{\mu}\cdot[\vec{\theta}^{(k)}Q]\;=0 \tag{39}$$

for the $\varepsilon-$jumps. (Recall that the brackets $[*]$ denote breaks).

This relation is the natural generarization of the two-component case: $[\theta Q]=0$.

3 Canonical form

Let all $\lambda^{(k)}$ be real. (The situation arising when some $\lambda^{(k)}$ are complex will be discussed in § 5.) Then the equation (37) considered for all values of k can be treated as a system. Denote by S a matrix whose rows are the vectors $\vec{\varphi}^{(k)}$. Then the system (37) (k=1,...,N-1) can be written in the matrix form

$$n(S\frac{\partial\vec{c}}{\partial t}+\Lambda QS\frac{\partial\vec{c}}{\partial x})+\vec{F}=0. \tag{40}$$

Here, Λ is a diagonal matrix with elements $\lambda^{(k)}:\ \vec{F}=(\hat{f}_1,...,\hat{f}_{N-1})^*,\ \hat{f}_k=\vec{\varphi}^{(k)}\cdot\vec{f}^{(k)}$ ($\vec{f}^{(k)}$ is defined by (27)). It follows from the identity $S(B-\Lambda A)=0$ that $BA^{-1}=S^{-1}\Lambda S$. Muliplying (40) by S^{-1} from the left we obtain

$$n(\frac{\partial\vec{c}}{dt}+QBA^{-1}\frac{d\vec{c}}{dx})+S^{-1}\vec{F}=0. \tag{41}$$

The system (41) (or (40)) can be considered as a "hyperbolic" subsystem of the governing equations. It remains to construct the missing "parabolic" equation.

First, we rewrite the system (13)–(14) as follows:

$$\begin{gathered}n\,\frac{\partial\vec{c}}{\partial t}+\vec{c}\,\frac{\partial n}{\partial t}+\sigma\,\frac{\partial\vec{\psi}}{\partial x}+\vec{\psi}\,\frac{\partial\sigma Q}{\partial x}=0.\\ \frac{\partial n}{\partial t}+\frac{\partial\sigma Q}{\partial x}=0.\end{gathered} \tag{42}$$

or, in more detail,

$$\begin{gathered}n\,\frac{\partial\vec{c}}{\partial t}+\vec{c}\,(n_P\frac{\partial P}{\partial t}+\nabla_c n\cdot\frac{\partial\vec{c}}{\partial t})+\sigma Q\,\frac{\partial\vec{\psi}}{\partial x}+\vec{\psi}\,\frac{\partial\sigma Q}{\partial x}=0,\\ n_P\frac{\partial P}{\partial t}+\nabla_c n\cdot\frac{\partial\vec{c}}{\partial t}+\frac{\partial\sigma Q}{\partial x}=0.\end{gathered} \tag{43}$$

Now one can eliminate $\dfrac{\partial\vec{c}}{\partial t}$ from (43). As the result we have

$$nn_P\frac{\partial P}{\partial t}=\Big(\nabla_c n\cdot(\vec{\psi}-\vec{c})-n\Big)\,\frac{\partial\sigma Q}{\partial x}+\sigma Q(\nabla_c n\cdot\frac{\partial\vec{\psi}}{\partial x}). \tag{44}$$

Because $Q=-K(x)P_x,\ K>0$ the last equation can be treated as parabolic under the condition

$$\nabla_c n\cdot(\vec{\psi}-\vec{c})<n. \tag{45}$$

Another form of (45) is

$$\nabla_C n\cdot(\vec{\varphi}-\sigma\vec{c})<\sigma n. \tag{46}$$

(Note that the natural thermodynamic inequality $n_P>0$ is supposed to be valid.)

The resulting canonical form consists of the "hyperbolic" subsystem (41) and the "parabolic" equation (44). We now explain the role of this form.

(1). The equation (44) is parabolic with respect to P for a given concentration vector $\vec{c}$;

(2). The subsystem (41) is hyperbolic with respect to $\vec{c}$ (for the case of real eigenvalues) for a given pressure P. Note that the terms $\vec{\mu}\cdot\dfrac{\partial\vec{\theta}^{(k)}Q}{\partial x}$ do not change the velocities of weak discontinuities or $\varepsilon-$breaks which are defined by $\lambda^{(k)}$ and Q, see also (39).

For the two-component case (more exactly for the Buckley-Leverett model) it was shown in [2] that the $\varepsilon-$perturbations of the initial data propagate as the $\varepsilon-$perturbations of the solution only along characteristic directions. As for non-characteristic directions the moving perturbations are of ε^2-order. Apparently, this feature takes place for the multi-component case as well.

The above mentioned canonical form is very convenient for the creation of high resolution numerical algorithms besause precisely this form describes the propagation of perturbations in the right way.

4 Multidimensional case

We begin with an analysis of the $\varepsilon-$breaks. Let L be the surface of $\varepsilon-$discontinuity, $L:\ \Phi(t,X)=0,\ X\in R^3$. Denote by $\mathbf{n}$ the normal to $L:\ \mathbf{n}=(\Phi_t,\nabla\Phi)$. Then the Hugoniot conditions are

$$[nc_i]\Phi_t+[\varphi_i\mathbf{Q}]\cdot\nabla\Phi=0,\ i=1,...,N. \tag{47}$$

or of the form (compare with (15))

$$[nc_i]\Phi_t+[\sigma\psi_i\mathbf{Q}]\cdot\nabla\Phi=0,\quad (\text{here}\quad i=1,...,N-1.) \tag{48}$$

$$[n]\Phi_t+[\sigma\mathbf{Q}]\cdot\nabla\Phi=0. \tag{49}$$

Introducing $\lambda=\Phi_t/(\mathbf{Q}\cdot\nabla\Phi)$ we obtain the eigenvalue problem (19–21). By definiton we have

$$\Phi_t-\lambda(\mathbf{Q}\cdot\nabla\Phi)=0. \tag{50}$$

This equation means that the normal $\mathbf{n}$ is ortogonal to $\mathbf{l}=(1,-\lambda\mathbf{Q})$. Therefore the concentracions c_i (more exactly some invariants of c_i) propagate in the $\mathbf{Q}-$direction (or in $\nabla P-$direction which is the same.) The eigenvalues λ define velocities of this process.

Now we turn to the characteristic relations and canonical form. The corresponding formulas can be obtained in a formal way from the one-dimensional case. One only have to be careful when replacing the operator $\partial/\partial x$ by Div or $Grag$. For brievity we simply shall associate the multi-dimensional formulas with the corresponding one-dimensional prototypes.

Similary to the one-dimensional case we choose some real eigenvalue $\lambda^{(k)}$. Let us use the same representation $\sigma=\vec{\theta}^{(k)}\cdot\vec{\mu}$. Denote by $\dfrac{\partial\vec{c}}{\partial\mathbf{s}}$ the column vector with components $\dfrac{\partial c_i}{\partial\mathbf{s}}=\mathbf{Q}\cdot Grad\,c_i$ and by $Div\,(\vec{\theta}^{(k)}\mathbf{Q})$ a row vector with components $Div\,({\theta_i}^{(k)}\mathbf{Q})$. (The definition of $\dfrac{\partial\vec{\psi}}{\partial\mathbf{s}}$ is completely analogous.)

It can be seen that

$$(24)\longrightarrow\qquad n\frac{\partial\vec{c}}{\partial t}+(\vec{\psi}-\vec{c})Div\,(\sigma\mathbf{Q})+\sigma\frac{d\vec{\psi}}{\partial\mathbf{s}}=0. \tag{51}$$

Since

$$Div\,(\sigma\mathbf{Q})=\vec{\mu}\cdot Div\,(\vec{\theta}^{(k)}\mathbf{Q})+\vec{\theta}^{(k)}\cdot\frac{\partial\vec{\mu}}{\partial\vec{s}},$$

we obtain

$$(26)\longrightarrow\qquad n\frac{\partial\vec{c}}{\partial t}+(\vec{\psi}-\vec{c})(\vec{\theta}^{(k)}\frac{\partial\vec{\mu}}{\partial\vec{c}}\frac{\partial\vec{c}}{\partial\mathbf{s}})+B\frac{\partial\vec{c}}{\partial\mathbf{s}}+\vec{f}^{(k)}=0; \tag{52}$$

$$(27)\longrightarrow\qquad \vec{f}^{(k)}=\left((\vec{\psi}-\vec{c})\left(\vec{\theta}^{(k)}\cdot\frac{\partial\vec{\mu}}{\partial P}\right)+\sigma\frac{\partial\vec{\psi}}{\partial P}\right)\frac{\partial P}{\partial\mathbf{s}}+(\vec{\psi}-\vec{c})(\vec{\mu}\cdot Div\,(\vec{\theta}^{(k)}\,\mathbf{Q})). \tag{53}$$

(In (52) the matrix $B=\sigma\dfrac{\partial\vec{\psi}}{\partial\vec{c}}$, see (20)).

Formulas (52)–(53) are obtained from their prototypes by changing the term $Q\dfrac{\partial\vec{c}}{\partial x}$ by $\dfrac{\partial\vec{c}}{\partial\mathbf{s}}$ and $\dfrac{\partial}{\partial x}(\vec{\theta}^{(k)}Q)$ by $Div\,(\vec{\theta}^{(k)}\,\mathbf{Q})$.

Thus,

$$(29) \longrightarrow \quad n\vec{\varphi}^{(k)} \cdot \left(\frac{\partial \vec{c}}{\partial t} + \lambda^{(k)} Q \frac{\partial \vec{c}}{\partial \mathbf{s}} \right) = g^{(k)} - \vec{\varphi}^{(k)} \cdot \vec{f}^{(k)}, \tag{54}$$

where

$$(30) \longrightarrow \quad g^{(k)} = n\,\vec{\varphi}^{(k)} \lambda^{(k)} \cdot \frac{\partial \vec{c}}{\partial \mathbf{s}} - \vec{\varphi}^{(k)} B \frac{\partial \vec{c}}{\partial \mathbf{s}} - \left(\vec{\varphi}^{(k)} \cdot (\vec{\psi} - \vec{c}) \right) \left(\vec{\theta}^{(k)} \frac{\partial \vec{\mu}}{\partial \vec{c}} \frac{\partial \vec{c}}{\partial \mathbf{s}} \right). \tag{55}$$

It is easily seen that equations (35) and (36) are unchanged. Concerning the characteristic relations the multidimensional version has the form (compare with (37))

$$n\vec{\varphi}^{(k)} \cdot \left(\frac{\partial \vec{c}}{\partial t} + \lambda^{(k)} \frac{\partial \vec{c}}{\partial \mathbf{s}} \right) + \vec{\varphi}^{(k)} \cdot \vec{f}^{(k)} = 0. \tag{56}$$

As for the “hyperbolic”subsystem we now have the analogies of (40), (41):

$$n(S\frac{\partial \vec{c}}{\partial t} + \Lambda S \frac{\partial \vec{c}}{\partial \mathbf{s}}) + \vec{F} = 0, \tag{57}$$

$$n(\frac{\partial \vec{c}}{dt} + BA^{-1} \frac{d\vec{c}}{d\mathbf{s}}) + S^{-1}\vec{F} = 0, \tag{58}$$

where as before $\vec{F} = (\hat{f}_1, ..., \hat{f}_{N-1})^*$, $\hat{f}_k = \vec{\varphi}^{(k)} \cdot \vec{f}^{(k)}$ (but now $\vec{f}^{(k)}$ is defined in (53)).

It remains to represent the “parabolic” equation (see (44)):

$$nn_P \frac{\partial P}{\partial t} = \left(\nabla_c n \cdot (\vec{\psi} - \vec{c}) - n \right) Div\,(\sigma \mathbf{Q}) + \sigma(\nabla_c n \cdot \frac{\partial \vec{\psi}}{\partial \mathbf{s}}). \tag{59}$$

Obviously, the parabolicity condition (45) (or (46)) remains the same.

5 Hyperbolicity and stability

Return to the complex eigenvalue case. To illustrate the essence of the problem we consider the following system:

$$\frac{\partial \vec{U}}{\partial t} + A(x, t, \vec{U}) \frac{\partial \vec{U}}{\partial x} = 0. \tag{60}$$

Here, $\vec{U}$ is a $n-$vector, A is an $(n \times n)-$matrix.

The standard question is about the stability of the uniform stationary solution $\vec{U}_0$ or the well-posedness of the corresponding Cauchy problem (which is the same). Evidently, if A has complex eigenvalues (non-hyperbolicity), then $\vec{U} = \vec{U}_0$ is an unstable solution. Applying the frozen coefficient principle one can extend the stability condition to an arbitrary stationary solution.

A typical object for the application of this concept is the two-phase (or the multi-phase) flow system with the common pressure. (Recall that unlike the Darcy law used in the filtration problems this model is based on the momentum equation.) From the very beginning it has been found that the uniform one-dimensional stationary two-phase flow of compressible media is stable only if some misterious inequality is fulfilled, see for example [3]. Later it was shown that this condition arises in one-dimensional case only. If we take into account multi-dimensional perturbations, then the flow becomes unconditionally unstable [4].

It seems that the same situation must take place in the multi-component filtration problems. However, here the specific problem arises: the fact is that in the common case the procedure of singling out a subsystem in the form (60) uses by itself a reality of all eigenvalues of the matrix $C = BA^{-1}$. Because of that one needs to analize small perturbations of the mean flow for the original form of the governing equations. But we then now have a new problem: there is no nontrivial uniform stationary sulution of this problem. In fact, if $P = \text{Const}$, then $\nabla P = 0$, which leads to the degeneration of the governing system. Therefore it is necessary to use another stationary solution (not degenerated). However, there are the filtration problems, where the subsystem (60) can be created directly. The question is about the incompressible filtration problem. In this case the system (11)–(12) has the form:

$$\frac{\partial \vec{c}}{\partial t} + \frac{\partial \vec{\varphi} Q}{\partial x} = 0, \tag{61}$$

$$\frac{\partial \sigma Q}{\partial x} = 0. \tag{62}$$

We obtained the required subsystem:

$$\frac{\partial \vec{c}}{\partial t} + \sigma Q \frac{\partial \vec{\psi}}{\partial x} = \frac{\partial \vec{c}}{\partial t} + QB\frac{\partial \vec{c}}{\partial x} = 0. \tag{63}$$

As the mean flow we take $(\vec{c}_0, P_0)$: $\vec{c}_0 = \text{Const},\ P = P_0(x)$ is a linear function of x. Now it is clearly seen that the stability condition completely coincides with the hyperbolicity condition of the system (63). There are many problems where this condition is violated (at least in some domains of the mean flow parameters). But there also exist unconditionally stable flows. As an example of such a flow we consider the three-component filtration model in the form (61)-(62), where each φ_i depends only on c_i : $\varphi_i = \varphi_i(c_i)$.

Now we obtain the system

$$\begin{aligned} &\frac{\partial c_1}{\partial t} + \sigma Q \frac{\partial \psi_1}{\partial x} = 0, \\ &\frac{\partial c_2}{\partial t} + \sigma Q \frac{\partial \psi_2}{\partial x} = 0, \end{aligned} \tag{64}$$

where $\psi_i = \varphi_i/\sigma$. Taking into account that here

$$\sigma = \varphi_1(c_1) + \varphi_2(c_2) + \varphi_3(1 - c_1 - c_2)$$

we obtain

$$\begin{aligned} &\sigma b_{11} = \sigma \dot{\varphi}_1 - \varphi_1(\dot{\varphi}_1 - \dot{\varphi}_2) = \dot{\varphi}_1(\varphi_2 + \varphi_3) + \varphi_1\dot{\varphi}_3; \\ &\sigma b_{12} = -\varphi_1(\dot{\varphi}_2 - \dot{\varphi}_3); \\ &\sigma b_{21} = -\varphi_2(\dot{\varphi}_1 - \dot{\varphi}_3); \\ &\sigma b_{22} = \sigma\dot{\varphi}_2 - \varphi_2(\dot{\varphi}_2 - \dot{\varphi}_3) = \dot{\varphi}_2(\varphi_1 + \varphi_3) + \varphi_2\dot{\varphi}_3. \end{aligned} \tag{65}$$

Here, $\dot{\varphi}_i$ denotes differentiation with respect to c_i. It is easily seen that the eigenvalues of B are real if and only if

$$D = (b_{11} - b_{22})^2 + 4b_{12}b_{21} \geq 0. \tag{66}$$

We introduce $\delta_1 = \dot{\varphi}_1 - \dot{\varphi}_3, \quad \delta_2 = \dot{\varphi}_2 - \dot{\varphi}_3$. Then

$$b_{11} - b_{22} = \varphi_1(\dot{\varphi}_3 - \dot{\varphi}_2) + \varphi_2(\dot{\varphi}_1 - \dot{\varphi}_3) + \varphi_3(\dot{\varphi}_1 - \dot{\varphi}_2) = \delta_1(\varphi_2 + \varphi_3) - \delta_2(\varphi_1 + \varphi_3).$$

Hence,

$$D = \left(\delta_1(\varphi_2 + \varphi_3) - \delta_2(\varphi_1 + \varphi_3)\right)^2 + 4\delta_1\delta_2\varphi_1\varphi_2.$$

This is a quadratic form of $\delta_1\ \delta_2$. The discriminant $\triangle$ of this form is

$$\begin{aligned} \triangle = 4\left((2\varphi_1\varphi_2 - (\varphi_1 + \varphi_3)(\varphi_2 + \varphi_3))^2 - (\varphi_1 + \varphi_3)^2(\varphi_2 + \varphi_3)^2\right) = \\ -16\varphi_1\varphi_2\varphi_3\sigma \leq 0. \end{aligned}$$

Therefore $D \geq 0$ and the roots of the characteristic equation are real. Hence the system (64) is hyperbolic and the flow is stable.

The analyzis of the stability is the necessary procedure especially if numerical methods are used. The fact is that often the “hysical” and “numerical” instability are indistinguishable. The main question connected with the stability problem is what we have to do if the flow is unstable. Probably in this case the system has to be somehow regularized. In particular, this instability can be removed by taking into account the cappilar forces that leads to the two-pressure model (by analogy with the already mentioned two-phase problem).

It remains to note that the majority of one-dimensional stability conditions can formally be extended to the multi-dimensional case. It follows from the fact that the multi-component filtration problem is essentially one-dimensional (see § 4).

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