Equivalence of Delsarte's bounds for codes and designs in symmetric association schemes, and some applications*

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Abstract

In order to obtain bounds on the sizes of codes and designs in association schemes Delsarte introduced two extremum problems for the systems of p-numbers and q-numbers. We prove that the Delsarte bound for codes obtained with the help of either of these systems is equivalent to that for designs obtained by using the other system. In particular, this means that the universal Delsarte's bound of 1973 for designs is equivalent to the sphere packing bound for codes. Furthermore, the universal bound for codes obtained by the author in 1978 gives rise to a new universal bound for designs, in particular, for block designs. This bound improves upon the known bounds when the strength of the design is sufficiently large. Moreover, a relationship between bounds for orthogonal arrays and block designs is obtained which gives new lower bounds on the size of orthogonal arrays with the help of those for block designs.

Keywords: codes, designs, association schemes, Delsarte's extremum problems, bounds, duality, orthogonal arrays, block designs

1 Introduction

In the Delsarte theory [4] of (symmetric) association schemes extremum problems for obtaining bounds on the size of codes and designs are of essential interest. In particular, for P- and Q-polynomial association schemes X,

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upper bounds on the maximum size $A(X,d)$ of a code $C \subseteq X$ with the minimum distance $d(C) \geq d$ and lower bounds on the minimum size $B(X,d)$ of a code $C \subseteq X$ with the minimum dual distance $d'(C) \geq d$ (or a $(d-1)$-design in X) were considered. In order to obtain bounds on $A(X,d)$ and $B(X,d)$ Delsarte used a linear programming problem (a problem $\mathfrak{A}$) for every of the systems Q and P of orthogonal polynomials. If we denote by $A_Q(X,d)$ and $A_P(X,d)$ the objective functions of the optimal solutions of the problem $\mathfrak{A}$ for the systems Q and P, respectively, the Delsarte bounds can be written in the following form:

$$A(X,d) \leq A_Q(X,d) \quad \text{and } B(X,d) \geq \frac{|X|}{A_P(X,d)}. \tag{1}$$

Furthermore, for the systems Q and P he considered another extremum problem (a problem $\mathfrak{B}$) which is not the dual linear program to the first one. This gave rise to the two more bounds

$$A(X,d) \leq \frac{|X|}{B_P(X,d)} \quad \text{and } B(X,d) \geq B_Q(X,d) \tag{2}$$

where $B_Q(X,d)$ and $B_P(X,d)$ denote the objective functions of the optimal solutions of the problem $\mathfrak{B}$ for the systems Q and P, respectively.

One of the main results of this paper is the equality

$$A_P(X,d)B_Q(X,d) = B_P(X,d)A_Q(X,d) = |X|. \tag{3}$$

This equality implies the equivalence of the Delsarte bounds (1) and (2) and has some other interesting applications. The proof of (3) is based on a notion of U-duality of polynomials (U is either Q or P). By the definition, the polynomial $D_U(h)$, which is U-dual to a polynomial h, is expansible over the system U with coefficients being proportional to certain values of h. We show that certain values of $D_U(h)$ are proportional to the coefficients of the expansion of h over the system $\overline{U}$ ($\overline{U} = P$ if $U = Q$, and $\overline{U} = Q$ if $U = P$). This is also used to describe the annihilators and the dual annihilators of codes for which the bounds (1) and (2) are attained.

It should be noted that in this paper a result similar to (3) is proved in the general form: for T-designs and T-codes in an arbitrary association scheme X. However, all applications of this result are given for P- and Q-polynomial association schemes X with n classes under the additional restriction that the eigenvalues $p_1(j)$ and $q_1(j)$ are decreasing functions in j. In particular, we prove that $\overline{U}$-dual polynomial to the polynomial $h^{U,d}(x) = \prod_{i=d}^{n} \frac{x-u_1(i)}{u_1(0)-u_1(i)}$ is proportional to $h^{\overline{U},n-d+2}(x)$. This implies that the coefficients of the expansions of these polynomials over systems Q and P are nonnegative (this was proved in [3] for the Hamming space) and yields a pair of bounds on the size of a code $C \subseteq X$ with given values $d(C)$ and $d'(C)$. We call this pair of bounds *total* because they imply the inequality $d(C)+d'(C) \leq n+2$ which can be attained only together with each of these bounds. The known Singleton and Johnson bounds for codes (see [15]) are special cases of the packing bound of the total pair, and MDS-codes and Steiner systems, for which this bound is attained, belong to the class of codes C with the property $d(C)+d'(C)=n+2$.

The two more pairs of bounds on the size of a code $C \subseteq X$ with given values $d(C)$ and $d'(C)$ are connected with the optimal solutions of the problems $\mathfrak{A}$ and $\mathfrak{B}$ for the system Q of orthogonal polynomial under some additional restrictions. In particular, Dunkl [9] noted that the Delsarte bound [4] on the size of $(d-1)$-designs (for odd d) is in fact based on the use of the polynomial h which gives the optimal solution of the problem $\mathfrak{B}$ for the system Q in the class of polynomials of degree less that d (this solution belongs to Schoenberg and Szegö [19]). Dunkl [9] also extended this bound to the case of even d. The Rao bound [17] for the Hamming space H_2^n and the Ray-Chaudhuri-Wilson bound [18] for the Johnson space J_w^n are special cases of this bound on the size of designs. The use of the polynomial $D_P(h)$ gives rise to the sphere packing bound for codes $C \subseteq X$ (in the case of even $d(C)$ as well) which is a generalization of the Hamming bound.

The third pair of bounds is obtained with the help of the polynomial g proposed in [10] as a solution of the problem $\mathfrak{A}$ for the system Q. It turned out (see [20], [12]-[14]) that this polynomial also gives the optimal solution of the problem $\mathfrak{A}$ for the system Q in the class of polynomials of a restricted degree. The use of the polynomial $D_P(g)$ gives rise to a new bound for designs.

It is worth pointing out that there exists a relationship between the objective functions of the optimal solutions of the "restricted" problems $\mathfrak{A}$ and $\mathfrak{B}$ for the same system of orthogonal polynomials (Theorem 6). This allows us to simplify computing the bounds. Notice also, that the second pair of bounds on the size of a code $C \subseteq X$ with given values $d(C)$ and $d'(C)$ yields good results for sufficiently small $d(C)$ and $d'(C)$ and the third pair does for sufficiently large $d(C)$ and $d'(C)$.

One more application of (3) is connected with the problem of finding for designs an analogue of the following Bassalygo-Elias inequality for codes:

$$A(H_2^n, 2d) \leq \frac{2^n}{\binom{n}{w}} A(J_w^n, d). \tag{4}$$

Rodemich (see [8]) proved that $A_Q(H_2^n, 2d) \leq \frac{2^n}{\binom{n}{w}} A_Q(J_w^n, d)$, i.e., (4) remains valid for linear programming bounds for the systems Q. Using this result and (3) we have

$$B(H_2^n, 2d) \geq B_Q(H_2^n, 2d) \geq B_P(J_w^n, d) = \frac{\binom{n}{w}}{A_Q(J_w^n, d)}. \tag{5}$$

This gives new bounds for orthogonal arrays.

For the Hamming space, the equality (3) and the three pairs of bounds on the size of a code C with given values $d(C)$ and $d'(C)$ were published in [13] with detailed proofs. However, it should be noted that the inequality (5), based on the general form of (3), gives rise to a better asymptotic bound for binary orthogonal arrays as compared with that in [13].

2 Symmetric Association Schemes

We use some known facts of Delsarte's theory of association schemes (see [4], [1], [2]). A (symmetric) association scheme $X = \{X, d(x,y)\}$ (with n classes) can be considered as a finite set X with a given function $d(x,y)$ which is defined for any $x, y \in X$, takes all values from $N_n = \{0, 1, \ldots, n\}$ and has the following properties : $d(x,y) = 0$ if and only if $x = y$; $d(x,y) = d(y,x)$ for any $x, y \in X$; for any $x, y \in X$ and any $i, j \in N_n$, the number of points z such that $d(x,z) = i, d(z,y) = j$ depends on $d(x,y)$ only (this number is called an *intersection number* and denoted by $p^k_{i,j}$, where $k = d(x,y)$).

For association schemes (AS), $p^k_{i,j} = p^k_{j,i}$ and $v_i = p^0_{i,i}$ is the number of points $y \in X$ such that $d(x,y) = i$ for a fixed $x \in X$. In general, the function $d(x,y)$ does not satisfy the triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$. However, for many significant examples of association schemes the function $d(x,y)$ has this property and is a metric. In this case the association scheme is called *metric*.

By definition of an AS, the adjacency matrices $A_i, i \in N_n$, of order $|X|$ (defined by $(A_i)_{x,y} = 1$ if $d(x,y) = i$, and $(A_i)_{x,y} = 0$ otherwise) are linearly independent and generate an $(n+1)$-dimensional (over $\mathbb{R}$) commutative algebra $\mathfrak{C}$ of symmetric matrices, which is called the Bose-Mesner algebra. We consider the $|X|$-dimensional vector space $V = \{\mathbf{u}(x) : X \to \mathbb{R}\}$ of real functions on X with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{|X|} \sum_{x \in X} \mathbf{u}(x)\mathbf{v}(x).$$

For an AS with n classes (as for the $(n+1)$-dimensional commutative algebra of symmetric matrices) there exists a decomposition $V = V_0 + \cdots + V_n$ of V into a direct sum of pairwise orthogonal subspaces V_i, $i \in N_n$, where each V_i is a maximal common eigenspace of $A_0, A_1, \ldots, A_n$. We can assume that V_0 consists of constants only since the eigenvector of all ones can belong only to a one-dimensional common eigenspace.

Let $r_i = \dim V_i$, $i \in N_n$, $r_0 = 1$, and $\{\mathbf{v}_{i,j}(x), j = 1, \ldots, r_i\}$ be any orthonormal basis of V_i. The matrices

$$E_i(x,y) = \frac{1}{|X|} \sum_{j=1}^{r_i} \mathbf{v}_{i,j}(x)\mathbf{v}_{i,j}(y), \quad i = 0, 1, \ldots, n, \tag{6}$$

do not depend on the choice of the bases of V_i and form the basis of irreducible idempotents of $\mathfrak{C}$.

It follows that there exist two systems of real numbers $p_i(j)$ and $q_i(j)$, $i, j \in N_n$, such that

$$A_i = \sum_{i=0}^{n} p_i(j) E_j,$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{n} q_i(j) A_j. \tag{7}$$

Note that (7) can be rewrite in the form

$$|X| E_i(x, y) = q_i(d(x, y)). \tag{8}$$

The column space of E_j is an eigenspace of each A_i, and the corresponding eigenvalue $p_i(j)$ has the multiplicity $r_j = \text{rank } E_j = \ tr\ E_j$. We use the following important properties of these p-numbers $p_i(j)$ and q-numbers $q_i(j)$:

$$p_0(j) = 1,\ q_0(j) = 1,\ p_i(0) = v_i,\ q_i(0) = r_i, \tag{9}$$

$$r_i p_j(i) = v_j q_i(j), \tag{10}$$

$$\sum_{d=0}^{n} p_i(d) p_j(d) r_d = v_i |X| \delta_{i,j}, \tag{11}$$

$$\sum_{d=0}^{n} q_i(d) q_j(d) v_d = r_i |X| \delta_{i,j}, \tag{12}$$

$$p_i(d) p_j(d) = \sum_{k=0}^{n} p_{i,j}^k p_k(d) \tag{13}$$

where $p_{i,j}^k$ are the intersection numbers.

Any non-empty subset $C \subseteq X$ is called a *code* (in the association scheme $\{X, d(x, y)\}$). We also use the term a *design* for a set or multiset C in X which contains every $x \in \dot{X}$ with a multiplicity $m(x)$. A design C consisting of distinct elements ($m(x) = 1$ for any $x \in C$) is called *simple*. Thus, a code is the same as a simple design. We denote by $|C|$ the number of elements of a code or design C taking into account identical elements in the case of designs, i.e.,

$$|C| = \sum_{x \in C} m(x). \tag{14}$$

For any code (design) $C \subseteq X$, we define the *inner distribution*

$$a_i = a_i(C) = \frac{1}{|C|} \left| \{(x, y) : x, y \in C, d(x, y) = i\} \right|, \quad i \in N_n, \tag{15}$$

and the *dual distribution*

$$a'_i = a'_i(C) = \sum_{d=0}^{n} a_d q_i\left(d\right) = \frac{1}{|C|} \sum_{x,y \in C} q_i\left(d(x,y)\right), \quad i \in N_n. \tag{16}$$

Note that

$$a_0(C) = \frac{1}{|C|} \sum_{x \in C} m^2(x), \;\; a'_0(C) = \sum_{x \in C} m(x) = |C|, \tag{17}$$

$$a_i = \frac{1}{|X|} \sum_{d=0}^{n} a'_d p_i(d). \tag{18}$$

From (6), (8), and (16) it follows the *Delsarte inequalities*

$$|C| a'_i(C) = \sum_{x,y \in C} \sum_{j=1}^{r_i} \mathbf{v}_{i,j}(x) \mathbf{v}_{i,j}(y) = \sum_{j=1}^{r_i} \left(\sum_{x \in C} \mathbf{v}_{i,j}(x) \right)^2 \geq 0, \tag{19}$$

which are foundation for obtaining bounds on the optimal sizes of codes and designs. Since any nonnegative definite matrix of the form $f(d(x,y)), x, y \in X$, has nonnegative coefficients in the expansion over basis (6), (8) also implies that for any $i, j \in N_n$ there exist $q_{i,j}^k$ such that

$$q_i(d) q_j(d) = \sum_{k=0}^{n} q_{i,j}^k q_k(d) \text{ and } q_{i,j}^k \geq 0. \tag{20}$$

For any $T \subseteq N_n^1 = \{1, ..., n\}$, a code C is called a T-*code* if $a_i(C) = 0$ when $i \in T$, and a design C is called a T-*design* if $a'_i(C) = 0$ when $i \in T$. For an association scheme $\{X, d(x,y)\}$ there exist the two problems of the important interest: find the maximum number $A(X,T)$ of elements of a T-code and the minimum number $B(X,T)$ of distinct elements of a T-design. One of the main results of this paper consists in the fact that these problems are equivalent under the condition that we use the Delsarte extremum problems described below.

3 Delsarte's extremum problems for codes and designs

We shall use the letter u instead of either p or q, and use $\overline{u}$ for the other one. Then (9)-(12) can be rewritten as follows:

$$\sum_{d=0}^{n} u_i(d) u_j(d) \overline{u}_d(0) = u_i(0) |X| \delta_{i,j}, \tag{21}$$

$$u_0(i) = 1, \;\; u_i(0) \overline{u}_j(i) = \overline{u}_j(0) u_i(j). \tag{22}$$

Consider the $(n+1)-$dimensional vector space $F(N_n)$ of real functions defined on the set N_n. In particular, p_i and q_i are elements of $F(N_n)$ which equal $p_i(j)$ and $q_i(j)$ when $j \in N_n$. Note that functions p_i and q_i, $i \in N_n$, form the two bases in $F(N_n)$, since otherwise A_i and E_i, $i \in N_n$, would not be bases in $\mathfrak{C}$. This means that any function $f \in F(N_n)$ has the unique u-expansion (u is either p or q)

$$f = \sum_{j=0}^{n} f_j(u) u_j, \tag{23}$$

with the real coefficients

$$f_i(u) = \frac{1}{u_i(0)|X|} \sum_{d=0}^{n} f(d) u_i(d) \overline{u}_d(0). \tag{24}$$

Put

$$\Omega_u(f) = \frac{f(0)}{f_0(u)} \text{ if } f_0(u) \neq 0. \tag{25}$$

For any $T \subseteq N_n^1$, we say that $f \in F(N_n)$ has the property $\mathfrak{A}_u(T)$ if

$$f_0(u) > 0, \;\; f_i(u) \geq 0 \text{ for } i \in N_n^1, \tag{26}$$

$$f(0) > 0, \; f(i) \leq 0 \text{ for } i \in N_n^1 \backslash T, \tag{27}$$

and has the property $\mathfrak{B}_u(T)$ if

$$f_0(u) > 0, \; f_i(u) \leq 0 \text{ for } i \in N_n^1 \backslash T, \tag{28}$$

$$f(0) > 0, \;\; f(i) \geq 0 \text{ for } i \in N_n^1. \tag{29}$$

Let

$$A_u(T) = A_u(X,T) = \min \Omega_u(f), \tag{30}$$

where the minimum is taken over all functions $f \in F(N_n)$ with the property $\mathfrak{A}_u(T)$, and

$$B_u(T) = B_u(X,T) = \max \Omega_u(f), \tag{31}$$

where the maximum is taken over all functions $f \in F(N_n)$ with the property $\mathfrak{B}_u(T)$. It should be noted that the both problems are linear programming ones, because without loss of generality one can assume that $f_0(u) = 1$ and then $\Omega_u(f) = f(0) = \sum_{j=0}^{n} f_j(u) u_j(0)$.

For any function $f \in F(N_n)$, from (15), (16), (18), and (23) it follows that

$$\sum_{i=0}^{n} a_i(C) f(i) = \sum_{j=0}^{n} a'_j(C) f_j(q)$$

and

$$\sum_{i=0}^{n} a_i(C) f_i(p) = \frac{1}{|X|} \sum_{j=0}^{n} a'_j(C) f(j).$$

These identities, the definitions of a T-code and T-design, and properties $\mathfrak{A}_u(T)$ and $\mathfrak{B}_u(T)$ imply the following statement [4].

Theorem 1 (Delsarte's extremum problems) *If C is a T-code and $f \in F(N_n)$ has the property $\mathfrak{A}_q(T)$, then*

$$|C| \leq \Omega_q(f); \tag{32}$$

if C is a simple T-design and $f \in F(N_n)$ has the property $\mathfrak{B}_q(T)$, then

$$|C| \geq \Omega_q(f). \tag{33}$$

Equality in (32) and (33) takes place if and only if

$$a_j(C) f(j) = 0, \quad j \in N_n^1, \tag{34}$$

and

$$a'_j(C) f_j(q) = 0, \quad j \in N_n^1. \tag{35}$$

If C is a T-code and $f \in F(N_n)$ has the property $\mathfrak{B}_p(T)$, then

$$|C| \Omega_p(f) \leq |X|; \tag{36}$$

if C is a simple T-design and $f \in F(N_n)$ has the property $\mathfrak{A}_p(T)$, then

$$|C| \Omega_p(f) \geq |X|. \tag{37}$$

Equality in (36) and (37) takes place if and only if

$$a'_j(C) f(j) = 0, \quad j \in N_n^1, \tag{38}$$

and

$$a_j(C) f_j(p) = 0, \quad j \in N_n^1. \tag{39}$$

Remark 1 *Note that for an arbitrary (not necessarily simple) design C, which contains M distinct elements x with multiplicities $m(x)$, these bounds for T-designs are valid with $|C|$ replaced by M. It is true (see (17)), since*

$$\frac{a'_0(C)}{a_0(C)} = \frac{\left(\sum_{x \in C} m(x)\right)^2}{\sum_{x \in C} m^2(x)} \leq M.$$

Thus, there exist the following bounds for the maximum number $A(X,T)$ of elements of a T-code and the minimum number $B(X,T)$ of distinct elements of a T-design:

$$A(X,T) \leq \min\left(A_q(X,T), \frac{|X|}{B_p(X,T)}\right) \tag{40}$$

and

$$B(X,T) \geq \max\left(B_q(X,T), \frac{|X|}{A_p(X,T)}\right). \tag{41}$$

In the next section we show that for any $T \in N_n$, $A_p(X,T)B_q(X,T) = B_p(X,T)A_q(X,T) = |X|$ and hence, the optimal bounds for codes and designs in an AS can be obtained with the help of a solution for either of the problems for different systems.

4 The equivalence of Delsarte's bounds for codes and designs

We introduce a notion of u-duality (u is either p or q) on $F(N_n)$. For any $f \in F(N_n)$ the function

$$D_u(f) = |X|^{-\frac{1}{2}} \sum_{i=0}^{n} f(i) u_i \tag{42}$$

is called *u-dual* to f. Thus, if $g = D_u(f)$ then $g_i(u)$ is proportional to $f(i)$. Now we verify that $g(j)$ is proportional to the coefficient $f_j(\overline{u})$.

Theorem 2 *For any $f \in F(N_n)$,*

$$D_{\overline{u}}(D_u(f)) = f, \tag{43}$$

$$\Omega_{\overline{u}}(f)\Omega_u(D_u(f)) = |X|. \tag{44}$$

If f has the property $\mathfrak{A}_{\overline{u}}(T)$ or $\mathfrak{B}_{\overline{u}}(T)$, then the u-dual function $D_u(f)$, respectively, has the property $\mathfrak{B}_u(T)$ or $\mathfrak{A}_u(T)$.

Proof. From (21) and (22) it follows that for any $j \in N_n$

$$\sum_{d=0}^{n}\sum_{i=0}^{n} f(i)u_i(d)\overline{u}_d(j) = |X| f(j).$$

In accord to the definition (42), this gives (43) and shows that if $g = D_u(f)$ then

$$g(j) = |X|^{\frac{1}{2}} f_j(\overline{u}) \text{ for any } j \in N_n \ .$$

Therefore, by (25) and (42),

$$\Omega_u(D_u(f)) = |X|\frac{f_0(\overline{u})}{f(0)}$$

and (44) holds. The remaining statements are obtained if one compares (26)-(27) with (28)-(29). $\square$

In particular, Theorem 2 implies the equivalence of the extremum problems of Delsarte which are used for obtaining bounds on the sizes of T-codes and T-designs in symmetric association schemes (see (40)-(41)). This equivalence means that if $f \in F(N_n)$ yields an optimal solution of the problem $\mathfrak{A}$ or $\mathfrak{B}$ for the system $\overline{u}$ (u is either p or q), the u-dual function $D_u(f)$ yields an optimal solution of the problem $\mathfrak{B}$ or $\mathfrak{A}$, respectively, for the system u. This can be expressed in the form of the following statement.

Theorem 3 *For any symmetric association scheme X and any $T \subseteq N_n^1$,*

$$A_p(X,T)B_q(X,T) = B_p(X,T)A_q(X,T) = |X|.$$

5 Dual polynomials for polynomial graphs

Denote $F_n[t]$ the set of polynomials over $\mathbb{R}$ of degree at most n in a real t. An association scheme $X = \{X, d(x,y)\}$ is called *P-polynomial* if there exist polynomials $P_i \in F_n[t]$ of degree $i, i = 0, 1, ..., n$, such that

$$p_i(j) = P_i(p_1(j)), \quad j \in N_n, \tag{45}$$

and *Q-polynomial* if there exist polynomials $Q_i \in F_n[t]$ of degree $i, i = 0, 1, ..., n$, such that

$$q_i(j) = Q_i(q_1(j)), \quad j \in N_n. \tag{46}$$

Delsarte [4] proved that the first property of an AS with n classes is equivalent to the fact that $d(x,y)$ is the path metric on the graph of diameter n with the set of vertices $V = X$ and the set of edges $E = \{(x,y) : d(x,y) = 1\}$. We call an AS X a *polynomial graph* if X is P-polynomial and Q-polynomial. We also shall use the letter U instead of either P or Q, and use $\overline{U}$ for the other one. Then the orthogonality conditions for the system $\{U_i, i = 0, 1, ..., n\}$ take the following form:

$$\sum_{d=0}^{n} U_i(u_1(d))U_j(u_1(d))\overline{u}_d(0) = u_i(0)|X|\delta_{i,j} \tag{47}$$

(see (21)). In the case of polynomial graphs for any U there exists one-to-one mapping between $F(N_n)$ and $F_n[t]$ for which any function $f \in F(N_n)$ is mapped to the polynomial $h = h^{U,f} \in F_n[t]$ defined as follows:

$$h=\sum_{j=0}^{n}h_j(U)U_j$$

where $h_j(U)=f_j(u)$ (see (23)). Note that $f(i)=h(u_1(i))$ for any $i\in N_n$. This mapping allows us to express the linear programming bounds (30) and (31) in terms of the corresponding extremum problems for the systems $P=\{P_i, i=0,1...,n\}$ and $Q=\{Q_i, i=0,1...,n\}$ of orthogonal polynomials.

For any polynomial $h=\sum_{j=0}^{n}h_j(U)U_j$ put

$$\Omega_U(h)=\frac{h(u_1(0))}{h_0(U)} \text{ if } h_0(U)\neq 0. \tag{48}$$

For any $T\subseteq N_n^1$, we say that a polynomial $h\in F_n[t]$ has the property $\mathfrak{A}_U(T)$, if

$$h_0(U)>0,\ h_i(U)\geq 0 \text{ for } i\in N_n^1, \tag{49}$$

$$h(u_1(0))>0,\ h(u_1(i))\leq 0 \text{ for } i\in N_n^1\backslash T, \tag{50}$$

and has the property $\mathfrak{B}_U(T)$, if

$$h_0(U)>0,\ h_i(U)\leq 0 \text{ for } i\in N_n^1\backslash T, \tag{51}$$

$$h(u_1(0))>0,\ h(u_1(i))\geq 0 \text{ for } i\in N_n^1. \tag{52}$$

Denote

$$A_U(T)=A_U(X,T)=\min\Omega_U(h), \tag{53}$$

where the minimum is taken over all polynomials $h\in F_n[t]$ with property $\mathfrak{A}_U(T)$, and

$$B_U(T)=B_U(X,T)=\max\Omega_U(h), \tag{54}$$

where the maximum is taken over all polynomials $h\in F_n[t]$ with property $\mathfrak{B}_U(T)$. Note that $f\in F(N_n)$ has the property $\mathfrak{A}_u(T)$ ($\mathfrak{B}_u(T)$) if and only if the corresponding polynomial $h=h^{U,f}\in F_n[t]$ has the property $\mathfrak{A}_U(T)$ ($\mathfrak{B}_U(T)$) and that

$$\Omega_u(f)=\Omega_U(h). \tag{55}$$

Therefore,

$$A_U(X,T)=A_u(X,T),\quad B_U(X,T)=B_u(X,T)$$

and hence, by Theorems 1 and 3, for any polynomial graph X and any $T\subseteq N_n^1$,

$$A_P(X,T)B_Q(X,T)=B_P(X,T)A_Q(X,T)=|X|, \tag{56}$$

and

$$A(X,T)\leq A_Q(X,T)=\frac{|X|}{B_Q(X,T)}, \tag{57}$$

$$B(X,T) \geq B_Q(X,T) = \frac{|X|}{A_P(X,T)}. \tag{58}$$

One more way to prove (56) consists in extending the u-duality on $F(N_n)$ to the corresponding U-duality on $F_n[t]$. For any $h \in F_n[t]$ the polynomial

$$D_U(h) = |X|^{-\frac{1}{2}} \sum_{i=0}^{n} h(\overline{u}_1(i)) U_i \tag{59}$$

is called *U-dual* to h. Thus, if $g = D_U(h)$ then $g_i(U) = |X|^{-\frac{1}{2}}\ h(\overline{u}_1(i))$. Using the arguments of the proof of Theorem 2 one can show that

$$g(u_1(i)) = |X|^{\frac{1}{2}} h_i(\overline{U}) \tag{60}$$

and verify the following analogue of this theorem is true.

Theorem 4 *For any polynomial graph X and any $h \in F_n(t)$,*

$$D_{\overline{U}}(D_U(h)) = h, \tag{61}$$

$$\Omega_{\overline{U}}(h)\Omega_U(D_U(h)) = |X|. \tag{62}$$

If h has the property $\mathfrak{A}_{\overline{U}}(T)$ or $\mathfrak{B}_{\overline{U}}(T)$, then the U-dual polynomial $D_U(h)$, respectively, has the property $\mathfrak{B}_U(T)$ or $\mathfrak{A}_U(T)$.

The U-dual polynomials are also useful to formulate the necessary and sufficient conditions under whose the bounds of Theorem 1 are attained for a polynomial graph X. For a code $C \subseteq X$, a polynomial $f \in F_n(t)$ is called an *annihilator* if $a_j(C)f(q_1(j)) = 0$ for all $j \in N_n^1$, and *dual annihilator* if $a'_j(C)f(p_1(j)) = 0$ for all $j \in N_n^1$. Using (60) we can rephrase the conditions (34)-(35) and (38)-(39) as follows: the polynomial f is an annihilator and $D_P(f)$ is a dual annihilator for a code C or, respectively, the polynomial f is a dual annihilator and $D_Q(f)$ is an annihilator for a code C.

As an example, for any system U (U is either P or Q) and any $d \in N_{n+1}^1$, consider the polynomial

$$h^{U,d}(t) = \prod_{i=d}^{n} \frac{t - u_1(i)}{u_1(0) - u_1(i)}. \tag{63}$$

of degree $n - d + 1$ (we assume that $h^{U,n+1}(t) = 1$) and put

$$M_U(d) = M_U(X,d) = \sum_{j=0}^{d-1} h^{U,d}(u_1(j))\overline{u}_j(0). \tag{64}$$

Since $u_0(0) = 1$, from (47) and (48) it follows that

$$\Omega_U(h^{U,d}) = \frac{|X|}{M_U(X,d)}. \tag{65}$$

Lemma 1 *For any $d \in N^1_{n+1}$,*

$$D_{\overline{U}}(h^{U,d}) = |X|^{-\frac{1}{2}} M_U(d) h^{\overline{U},n-d+2}$$

Proof. According to (59), the $\overline{U}$-dual to $h^{U,d}$ polynomial is equal to $g = |X|^{-\frac{1}{2}} \sum_{i=0}^{n} h^{U,d}(u_1(i))\overline{U}_i$. Note that g takes the value $|X|^{-\frac{1}{2}} M_U(d)$ at $t = \overline{u}_1(0)$ and has exactly $d-1$ zeros $\overline{u}_1(l), l = n-d+2, ..., n$, since by (23) and (45)-(47),

$$|X|^{\frac{1}{2}} g(\overline{u}_1(l)) u_l(0) = \sum_{i=0}^{n} h^{U,d}(u_1(i)) \overline{U}_i(\overline{u}_1(l)) u_l(0)$$

$$= \sum_{i=0}^{n} h^{U,d}(u_1(i)) U_l(u_1(i)) \overline{u}_i(0) = 0$$

for $l = n-d+2, ..., n$. The polynomial g is uniquely defined by these properties. This completes the proof because the polynomial $|X|^{-\frac{1}{2}} M_U(d) h^{\overline{U},n-d+2}$ has the same properties (see (63)). $\square$

6 The total pair of bounds for codes and designs

From now on we consider polynomial graphs X of diameter n for which $q_1(j)$ and $p_1(j)$ are *decreasing functions* in $j \in N_n$. It is known (see [1], [2], [12]) that this is true for all known infinite families of polynomial graphs except for the Hermitian forms. Moreover, we discuss bounds for T-codes and T-designs only for the case $T = \{1, ..., d-1\}$ where $2 \leq d \leq n$. For this case we write $A(X, d)$, $\mathfrak{A}_U(d)$, $A_U(X, d)$, $B(X, d)$, $\mathfrak{B}_U(d)$, and $B_U(X, d)$ instead of $A(X, T)$, $\mathfrak{A}_U(T)$, $A_U(X, T)$, $B(X, T)$, $\mathfrak{B}_U(T)$, and $B_U(X, T)$, respectively. The *minimal distance* $d(C)$ of a code $C \subseteq X$ is the maximum number d such that C is a $\{1, ..., d-1\}$-code. The *dual distance* $d'(C)$ of a design $C \subseteq X$ is the maximum number d such that C is a $\{1, ..., d-1\}$-design. The value $t(C) = d'(C) - 1$ is also called the (maximum) *strength* of a design $C \subseteq X$. Thus, $A(X, d)$ is the maximum size of a code $C \subseteq X$ for which $d(C) \geq d$, and $B(X, d)$ is the minimum number of distinct elements of a design $C \subseteq X$ for which $d'(C) \geq d$ or $t(C) \geq d-1$.

We call *total* the following pair of bounds because they imply the inequality $d(C) + d'(C) \leq n+2$ which can be attained only together with each of these bounds.

Theorem 5 (Total pair of bounds) *In a polynomial graph X of diameter n, for any code C,*

$$\frac{|X|}{M_Q(n - d'(C) + 2)} = M_P(d'(C)) \leq |C| \leq \frac{|X|}{M_Q(d(C))} = M_P(n - d(C) + 2).$$

Each of these bounds is attained if and only if $d(C)+d'(C)=n+2$. In this case $h^{Q,d(C)}$ is an annihilator and $h^{P,d'(C)}$ is a dual annihilator for C.

Proof. Using Theorem 4, Lemma 1, and (65) we have

$$M_U(d)M_{\overline{U}}(n-d+2)=|X|$$

and

$$h^{U,d}=\frac{M_U(d)}{|X|}\sum_{j=0}^{n}h^{\overline{U},n-d+2}(\overline{u}_1(j))U_j.$$

Therefore, by the assumption that $u_1(j)$ (u is either p or q) is a decreasing function in $j\in N_n$, we have that

$$h^{U,d}(u_1(j))>0 \text{ for } j=0,1,...,d-1, \tag{66}$$

$$h_j^{U,d}(U)>0 \text{ for } j=0,1,...,n-d+1, \tag{67}$$

and the polynomial $h^{U,d}$ has the properties $\mathfrak{A}_U(d)$ and $\mathfrak{B}_U(n-d+2)$ (and hence $h^{\overline{U},n-d+2}$ has the properties $\mathfrak{A}_{\overline{U}}(n-d+2)$ and $\mathfrak{B}_{\overline{U}}(d)$). If we use Theorem 1, (55), (65) for the polynomial $h^{U,d}$ with $U=Q$ and $d=d(C)$ (which has the property $\mathfrak{A}_Q(d)$), we get the upper bound and the fact that this bound is attained if and only if $h^{Q,d}$ is an annihilator and $h^{P,n-d+2}$ is a dual annihilator for the code C. Note that (35) and (67) imply the inequality $d'(C)\geq n-d(C)+2$ in the case when this bound is attained. Analogously, the use of $h^{U,n-d+2}$ with $U=Q$ and $d=d'(C)$ (which has the property $\mathfrak{B}_Q(d)$) gives the desirable lower bound and the fact that this bound is attained if and only if $h^{Q,n-d+2}$ is an annihilator and $h^{P,d}$ is a dual annihilator for the code C. Note that (34) and (66) imply the inequality $d(C)\geq n-d'(C)+2$ in the case when this lower bound is attained. The bounds which have been proved imply the inequality $d'(C)+d(C)\leq n+2$ because $M_U(d)$ is an increasing function in $d\in N_n$. This completes the proof. $\square$

Now we compute $M_U(d)=M_U(X,d)$ for some families of polynomial graphs X of the diameter n and check that q_i and p_i are decreasing functions on N_n. For any integers n and r, $n\geq 2, r\geq 2$, we denote by H_r^n the *Hamming space* consisting of all vectors $x=(x_1,...,x_n)$ over $\{0,1,...,r-1\}$ with the distance $d(x,y)$ being equal to the number of coordinate places where x and y differ. For any integers n, r ($r\geq n\geq 2$), and q which is a degree of a prime, we denote by $H(n,r,q)$ the *space of bilinear forms* consisting of matrices x of order $r\times n$ over the field $GF(q)$ with the distance $d(x,y)=\text{rank}(x-y)$ (The author hope that readers can distinguish these numbers q from q-numbers.) Delsarte [4], [5], [7] proved that H_r^n and $H(n,r,q)$ are self-dual (i.e., $p_i=q_i$) polynomial graphs of diameter n and found their parameters. To describe simultaneously these parameters it is useful to assume that q might be equal to one, put $H(n,r,1)=H_r^n$, and use the following notation, according to which for any $d=0,1,\dots$, $\begin{bmatrix} d \\ i \end{bmatrix}=1$ if $i=0$ and

$$\begin{bmatrix} d \\ i \end{bmatrix} = \begin{cases} \binom{d}{i} = \frac{d(d-1)\cdots(d-i+1)}{i!} & \text{if } q=1, i=1,2,\dots \\ \frac{(q^d-1)\cdots(q^{d-i+1}-1)}{(q^i-1)\cdots(q-1)} & \text{if } q\neq 1, i=1,2,\dots \end{cases}$$

If also put

$$q^{(r)} = \begin{cases} r & \text{if } q=1, \\ q^r & \text{if } q\neq 1, \end{cases}$$

these parameters for $X=H(n,r,q)$ can be described as follows:

$$q_1(d) = \begin{bmatrix} n-d \\ 1 \end{bmatrix} q^{(r)} - \begin{bmatrix} n \\ 1 \end{bmatrix}, \quad d\in N_n,$$

$$v_i = r_i = q_i(0) = \begin{bmatrix} n \\ i \end{bmatrix} (q^{(r)}-1)(q^{(r)}-q)\cdots(q^{(r)}-q^{i-1}), \quad i\in N_n,$$

$$q_i(d) = Q_i(q_1(d)) = \sum_{j=0}^{i} (-1)^{i-j} \begin{bmatrix} n-j \\ i-j \end{bmatrix} \begin{bmatrix} n-d \\ j \end{bmatrix} q^{\binom{i-j}{2}} \left(q^{(r)}\right)^j,$$

$$|X| = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (q^{(r)}-1)(q^{(r)}-q)\cdots(q^{(r)}-q^{i-1}) = \left(q^{(r)}\right)^n.$$

Therefore, using (64) we have

$$M_Q(H(n,r,q),d) = \sum_{j=0}^{d-1} v_j \prod_{i=d}^{n} \frac{\begin{bmatrix} n-j \\ 1 \end{bmatrix} - \begin{bmatrix} n-i \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-i \\ 1 \end{bmatrix}}$$

$$= \sum_{j=0}^{d-1} \begin{bmatrix} d-1 \\ j \end{bmatrix} (q^{(r)}-1)(q^{(r)}-q)\cdots(q^{(r)}-q^{j-1}) = \left(q^{(r)}\right)^{d-1}.$$

For any integers v and n, $v\geq 2n\geq 2$, we denote by J_n^v the *Johnson space* consisting of all n-subsets of the v-set N_v^1 where the distance $d(x,y)$ between two elements $x,y\in J_n^v$ is defined by $n-|x\bigcap y|$. For any integers v, n, $v\geq 2n\geq 2$, and q (which is a degree of a prime), we denote by $J(v,n,q)$ the *Grassmann space* consisting of all n-dimensional linear subspaces of the space $(GF(q))^v$ where the distance $d(x,y)$ between two elements $x,y\in J(v,n,q)$ is defined by $n-\dim x\bigcap y$. As in the case of the Hamming space we put $J(v,n,1)=J_n^v$. Delsarte [4]-[6] proved that $J(v,n,q)$ is a polynomial graph of diameter n and found the following its parameters:

$$q_1(d) = \begin{bmatrix} v \\ 1 \end{bmatrix} \left(1 - \frac{\begin{bmatrix} v \\ 1 \end{bmatrix}\begin{bmatrix} d \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ 1 \end{bmatrix}\begin{bmatrix} v-n \\ 1 \end{bmatrix}} q^{-d} \right), \quad d\in N_n,$$

$$p_1(d) = \begin{bmatrix} n \\ 1 \end{bmatrix} \begin{bmatrix} v-n \\ 1 \end{bmatrix} q - \begin{bmatrix} d \\ 1 \end{bmatrix} \begin{bmatrix} v+1-d \\ 1 \end{bmatrix}, \quad d \in N_n,$$

$$r_i = q_i(0) = \begin{bmatrix} v \\ i \end{bmatrix} - \begin{bmatrix} v \\ i-1 \end{bmatrix}, \quad i \in N_n^1, \quad r_0 = q_0(0) = 1,$$

$$v_i = p_i(0) = \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} v-n \\ i \end{bmatrix} q^{i^2}, \quad i \in N_n,$$

$$q_i(d) = Q_i(q_1(d)) = r_i \sum_{j=0}^{i} \frac{(-1)^j}{v_j} \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} v+1-i \\ j \end{bmatrix} \begin{bmatrix} d \\ j \end{bmatrix} q^{j(-d+(3j-1)/2)},$$

$$p_i(d) = P_i(p_1(d)) = \sum_{j=0}^{i} (-1)^{i-j} \begin{bmatrix} n-j \\ i-j \end{bmatrix} \begin{bmatrix} n-d \\ j \end{bmatrix} \begin{bmatrix} v-n+j-d \\ j \end{bmatrix} q^{dj+\binom{i-j}{2}},$$

$$|X| = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} v-n \\ i \end{bmatrix} q^{i^2} = \begin{bmatrix} v \\ n \end{bmatrix}.$$

Therefore, using (64) we have

$$M_Q(J(v,n,q),d) = \sum_{j=0}^{d-1} v_j \prod_{i=d}^{n} \frac{\begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} q^{i-j}}{\begin{bmatrix} i \\ 1 \end{bmatrix}}$$

$$= \sum_{j=0}^{d-1} \begin{bmatrix} v-n \\ j \end{bmatrix} \begin{bmatrix} d-1 \\ j \end{bmatrix} q^{j^2} = \begin{bmatrix} v-n+d-1 \\ v-n \end{bmatrix}.$$

Thus, for any code $C \subseteq H(n,r,q)$,

$$\left(q^{(r)}\right)^{d'(C)-1} \leq |C| \leq \left(q^{(r)}\right)^{n-d(C)+1},$$

and for any code $C \subseteq J(v,n,q)$,

$$\frac{\begin{bmatrix} v \\ d'(C)-1 \end{bmatrix}}{\begin{bmatrix} n \\ d'(C)-1 \end{bmatrix}} = \frac{\begin{bmatrix} v \\ n \end{bmatrix}}{\begin{bmatrix} v-d'(C)+1 \\ v-n \end{bmatrix}} \leq |C| \leq \frac{\begin{bmatrix} v \\ n \end{bmatrix}}{\begin{bmatrix} v-n+d(C)-1 \\ v-n \end{bmatrix}} = \frac{\begin{bmatrix} v \\ n+1-d(C) \end{bmatrix}}{\begin{bmatrix} n \\ n+1-d(C) \end{bmatrix}}.$$

7 Solutions of two extremum problems for systems of orthogonal polynomials and two other pairs of bounds

Consider three functions s, w, and m on the set $N_n = \{0, 1, ..., n\}$. Assume that the first function (*change of variable*) $s(d)$ is a decreasing function in $d \in$

N_n, the functions w and m are positive-valued and for the (*weight*) function $w(d)$ the equality $\sum_{d=0}^{n} w(d) = 1$ holds. The orthogonality conditions

$$\sum_{d=0}^{n} U_i(s(d))U_j(s(d))w(d) = m(i)\delta_{i,j} \tag{68}$$

uniquely define a system $U = \{U_i, i = 0, 1, ..., n\} \subset F_n[t]$ of orthogonal polynomials where each polynomial U_i has degree i. We also assume that the system U has the *Krein condition*, i.e., for any $i, j, k \in N_n$, the exist nonnegative numbers $u_{i,j}^k$ such that

$$U_i(t)U_j(t) \equiv \sum_{k=0}^{n} u_{i,j}^k U_k(t) \quad (\bmod U_{n+1}(t)), \tag{69}$$

where $U_{n+1}(t) = \prod_{d=0}^{n}(t - s(d))$. Note that all these assumptions are satisfied for the systems $U = P$ and $U = Q$ of any polynomial graph $|X|$ if put $s(d) = u_1(d), w(d) = |X|^{-1}\overline{u}_d(0)$, and $m(i) = u_i(0)$ (see (13), (20), (45), (46), and (47)).

Now we describe optimal solutions of two (continuous) extremum problems for systems U which give rise to two more pairs of bounds for codes and designs in any polynomial graph. We can consider U as a system of orthogonal polynomials on the interval $I(U) = [s(n), s(0)]$ and for any $h \in F_n[t]$,

$$h = \sum_{j=0}^{n} h_j(U)U_j \text{ where } h_j(U) = \frac{1}{m(j)} \sum_{d=0}^{n} h(s(d))U_j(s(d))w(d),$$

define $\Omega_U(h) = h(s(0))/h_0(U)$ if $h_0(U) \neq 0$. We write $h \in F_n^+[t]$ if $\deg h \leq n$ and $h_j(U) \geq 0$ for $j = 0, 1, ..., \deg h$. Let for any $d, d = 2, ..., n$,

$$K_U(d) = \max \Omega_U(h), \tag{70}$$

where the maximum is taken over all polynomials $h \in F_{d-1}^+[t]$ for which $h(t) \geq 0$ if $s(n) \leq t \leq s(0)$. Let for any real number $s, s(n) \leq s < s(1)$,

$$L_U(s) = \min \Omega_U(h), \tag{71}$$

where the minimum is taken over all polynomials $h \in F_{v(s)}^+[t]$ for which $h(t) \leq 0$ if $s(n) \leq t \leq s$ and the function $v(s)$ will be defined in (73) (we verify that, in a sense, this restriction of degree of h will correspond to that in the first problem). For the systems U (U is either P or Q) of a polynomial graph X, we shall write $K_U(X, d)$ and $L_U(X, s)$ instead of $K_U(d)$ and $L_U(s)$, respectively. Then the definitions (53), (54), (70), and (71) imply that

$$B_U(X, d) \geq K_U(X, d) \text{ and } A_U(X, d) \leq L_U(X, s(d)). \tag{72}$$

In order to describe these values $K_U(d)$ and $L_U(s)$ we consider some systems of orthogonal normalized polynomials which, in a certain sense, are adjacent to U. For any $\alpha, \beta \in \{0,1\}$, the following orthogonality and normalization conditions

$$m^{\alpha,\beta}(i)C^{\alpha,\beta}\sum_{d=0}^{n} U_i^{\alpha,\beta}(s(d))U_j^{\alpha,\beta}(s(d))(s(0)-s(d))^{\alpha}(s(d)-s(n))^{\beta}w(d)=\delta_{i,j},$$

$$U_i^{\alpha,\beta}(s(0))=1,\ i=0,1,...,n-\alpha-\beta,$$

where (for normalization of the new weight function it is required that)

$$C^{\alpha,\beta}\sum_{d=0}^{n}\left(2\frac{s(0)-s(d)}{s(0)-s(n)}\right)^{\alpha}\left(2\frac{s(d)-s(n)}{s(0)-s(n)}\right)^{\beta}w(d)=1,$$

uniquely define a system $U^{\alpha,\beta}$ of polynomials $U_i^{\alpha,\beta}(t)$ of degree $i \in N_{n-\alpha-\beta}$, and a positive-valued function $m^{\alpha,\beta} \in F(N_{n-\alpha-\beta})$. Note that for the systems $U=P$ and $U=Q$ we have $m(i)=u_i(0)$ and hence

$$m^{0,0}(i)=m(i) \text{ and } U_i^{0,0}(t)=\frac{U_i(t)}{U_i(s(0))}.$$

For $k=1,...,n-\alpha-\beta$, denote by $d_k^{\alpha,\beta}(U)$ the number such that $s(d_k^{\alpha,\beta}(U))$ is the largest zero of the polynomial $U_k^{\alpha,\beta}$ and put $d_0^{1,1}(U)=n$, $d_{n-1}^{1,1}(U)=1$, and $d_k(U)=d_k^{0,0}(U)$. It is known (see, for example, [11], [12], [14]) that

$$d_k^{1,1}(U)<d_k^{1,0}(U)<d_{k-1}^{1,1}(U),\ k=1,2,...,n-1.$$

This means that the half-open interval $(1,n]$ is partitioned into the half-open intervals $(d_k^{1,1-\varepsilon}(U), d_{k-\varepsilon}^{1,\varepsilon}(U)]$ where $\varepsilon \in \{0,1\}$, $k=1,2,...,n-1$. Note that the interval $(d_k^{1,1-\varepsilon}(U), d_{k-\varepsilon}^{1,\varepsilon}(U)]$ has the number $2k-\varepsilon$ if enumerate these intervals from the right. For any s, $s(n)\leq s<s(1)$, we put

$$v(s)=2k-\varepsilon \tag{73}$$

if $s\in[s(d_{k-\varepsilon}^{1,\varepsilon}(U)), s(d_k^{1,1-\varepsilon}(U))$, in particular, if $s=s(d_{k-\varepsilon}^{1,\varepsilon}(U))$.

Theorem 6 *For any $\varepsilon \in \{0,1\}$ and $k=1,2,...,n-1$,*

$$K_U(2k+1-\varepsilon)=L_U(s(d_{k-\varepsilon}^{1,\varepsilon}(U)))$$

$$=\left(1-\frac{1}{U_1^{0,0}(s(n))}\right)^{\varepsilon}\sum_{i=0}^{k-\varepsilon}m_i^{0,\varepsilon}=\left(1-\frac{U_{k-1}^{1,0}(s(n))}{U_k^{0,0}(s(n))}\right)^{\varepsilon}\sum_{i=0}^{k-\varepsilon}m_i^{0,0}$$

and $K_U(d)$ is an increasing function in d.

Proof. In fact, this statement is a special case of results in [12] and [14] and here we only give the main ideas of the proof. Let $s_0 = s(n)$ and $s_1, ..., s_{k-\varepsilon}$ be the zeros of $U^{1,\varepsilon}_{k-\varepsilon}$ written in the increasing order and hence $s_{k-\varepsilon} = s(d^{1,\varepsilon}_{k-\varepsilon}(U))$. Consider the polynomial (cf. [14, Lemma 5.38])

$$h(t) = \frac{(t-s_0)^{1+\varepsilon}}{t-s_{k-\varepsilon}}\left(U^{1,\varepsilon}_{k-\varepsilon}(t)\right)^2$$

of degree $v(s_{k-\varepsilon}) = 2k-\varepsilon$. It is clear that $h(t) \le 0$ if $s_0 \le t \le s_{k-\varepsilon}$ and, by [12], [14], $h \in F^+_{2k-\varepsilon}[t]$ and

$$\Omega_U(h) = \left(1 - \frac{1}{U^{0,0}_1(s(n))}\right)^{\varepsilon}\sum_{i=0}^{k-\varepsilon} m_i^{0,\varepsilon} = \left(1 - \frac{U^{1,0}_{k-1}(s(n))}{U^{0,0}_k(s(n))}\right)^{\varepsilon}\sum_{i=0}^{k-\varepsilon} m_i^{0,0}$$

(in [12] and [14] the standard change of variable $\sigma(d) = (2s(d) - s(0) - s(n))/(s(0)-s(n))$ was used). This implies $L_U(s_{k-\varepsilon}) \le \Omega_U(h)$. On the other hand, by [14, Th. 5.39], there exist nonnegative numbers $\rho_0, \rho_1, ..., \rho_{k-\varepsilon}$ such that $\rho_0 = 0$ if $\varepsilon = 0$ and for any $g \in F_{2k-\varepsilon}[t]$,

$$g_0(U) = (\Omega_U(h))^{-1} g(s(0)) + \sum_{i=0}^{k-\varepsilon} \rho_i g(s_i). \tag{74}$$

Therefore, $L_U(s_{k-\varepsilon}) \ge \Omega_U(h)$ and hence $L_U(s_{k-\varepsilon}) = \Omega_U(h)$. From (74) it also follows that for any polynomial $g \in F_{2k-\varepsilon}[t]$ such that $g(t) \ge 0$ if $s(n) \le t \le s(0)$ we have $\Omega_U(g) \le \Omega_U(h)$ and hence $K_U(2k+1-\varepsilon) \le \Omega_U(h)$. Note that the polynomial

$$g(t) = \frac{t-s_{k-\varepsilon}}{t-s_0}h(t) = (t-s_0)^{\varepsilon}\left(U^{1,\varepsilon}_{k-\varepsilon}(t)\right)^2 \tag{75}$$

has the properties: $g(t) \ge 0$ if $s(n) \le t \le s(0)$ and $g \in F^+_{2k-\varepsilon}[t]$ (see [4], [12]). Hence, $K_U(2k+1-\varepsilon) \ge \Omega_U(g)$. Since $g(s_i) = 0$ for $i = 1, ..., k-\varepsilon$ and $\rho_0 = 0$ if $\varepsilon = 0$, from (74) and (75) we get that $\Omega_U(g) = \Omega_U(h)$ and hence $K_U(2k+1-\varepsilon) = \Omega_U(h) = L_U(s_{k-\varepsilon})$. □

Using (57), (58), and (72) we obtain the following two pairs of bounds.

Theorem 7 *For any code C in a polynomial graph X ,*

$$K_Q(X, d'(C)) \le |C| \le \frac{|X|}{K_P(X, d(C))}, \tag{76}$$

where $K_U(X,d) = K_U(d)$ is defined by Theorem 6.

Theorem 8 *Let C be a code in a polynomial graph X and $d(C) \ge d^{1,\varepsilon}_{k-\varepsilon}(Q)$ and $d'(C) \ge d^{1,\theta}_{l-\theta}(P)$ for some integers k, l and $\varepsilon, \theta \in \{0,1\}$. Then*

$$\frac{|X|}{K_P(X, 2l+1-\theta)} \le |C| \le K_Q(X, 2k+1-\varepsilon), \tag{77}$$

where $K_U(X,d) = K_U(d)$ is defined by Theorem 6.

The lower bound in (76) is due to Delsarte [4] in the case of odd $d'(C)$ and to Dunkl [9] in the case of even $d'(C)$. The Rao bound [17] for the Hamming space H_r^n and the Ray-Chaudhuri-Wilson bound [18] for the Johnson space J_n^v are special cases of this bound. The upper bound in (76) is the sphere packing bound (in the case of even $d(C)$ as well) for an arbitrary polynomial graph and is a generalization of the Hamming bound. The upper bound in (77) is a special case of the bound obtained by the author [10]-[12]. The lower bound in (77) seems to be new in the general case. It was published in [13] for the Hamming space.

For the self-dual ($P = Q$) polynomial graphs $X = H(n, r, q)$ it is known [12] that

$$K_Q(X, 2k+1-\varepsilon) = \left(q^{(r)}\right)^{\varepsilon} \sum_{i=0}^{k-\varepsilon} \begin{bmatrix} n-\varepsilon \\ i \end{bmatrix} (q^{(r)}-1)(q^{(r)}-q)\cdots(q^{(r)}-q^{i-1}).$$

One can show (see [12], [14]) that for the polynomial graph $X = J(v, n, q)$,

$$K_Q(X, 2k+1-\varepsilon) = \left(\begin{bmatrix} v \\ 1 \end{bmatrix} / \begin{bmatrix} n \\ 1 \end{bmatrix}\right)^{\varepsilon} \begin{bmatrix} n-\varepsilon \\ k-\varepsilon \end{bmatrix}, \tag{78}$$

$$K_P(X, 2k+1-\varepsilon) = \sum_{i=0}^{k-\varepsilon} \begin{bmatrix} n-\varepsilon \\ i \end{bmatrix} \begin{bmatrix} v-n+\varepsilon \\ i+\varepsilon \end{bmatrix} q^{(i+\varepsilon)^2}. \tag{79}$$

This gives two more pairs of bounds for codes and designs for each of the spaces $H(n, r, q)$ and $J(v, n, q)$ which are attained in some cases. In particular, for the known code $C \subset J_n^v$ with parameters $v = 24$, $n = 8$, $|C| = 759$, $d'(C) = 6$, $d(C) = 4$ we have $d_2^{1,1}(Q) = 4$, $d_1^{1,1}(P) = 6$ and all six bounds of Theorems 5, 7, 8 are attained!

8 Relationships between bounds for codes and designs in the Hamming and Johnson spaces

An important result of coding theory is the Bassalygo-Elias inequality

$$A(H_2^n, 2d) \leq \frac{2^n}{\binom{n}{w}} A(J_w^n, d), \tag{80}$$

which has a simple combinatorial proof. The natural problem is to find a similar relationship between the minimal size $B(H_2^n, 2d)$ of an orthogonal array of strength $2d-1$ and the minimal size $B(J_w^n, d)$ of a block $(d-1)$-design. In this section we present a certain inequality between linear programming bounds for these values which gives rise to new lower bounds for $B(H_2^n, 2d)$ with the help of those for $B(J_w^n, d)$. This inequality is a consequence of Theorem 3 and a result of Rodemich (see [8]) which found an analogue of (80) in terms of linear programming bounds $A_Q(H_2^n, 2d)$ and $A_Q(J_w^n, d)$.

Following Rodemich we consider the polynomial graph J_w^n of diameter w as a subset of H_2^n and take into account that the Johnson distance $d_J(x,y)$ equals the half of the Hamming distance $d(x,y)$ between $x, y \in J_w^n$. We verify that for the systems Q for H_2^n and J_w^n (we also denote this system for J_w^n by Q^w) there exist nonnegative numbers $a_{k,j}$, $k \in N_n$, $j \in N_w$, such that for any $x, y \in J_w^n$,

$$Q_k(q_1(d(x,y))) = \sum_{l=0}^{\min(k,w)} a_{k,l} Q_l^w(q_1^w(d_J(x,y))).$$

This is true since the matrix $Q_k(q_1(d(x,y)))$ is nonnegative definite on the subset J_w^n (see (6), (8) and (46)) and hence all its coefficients in the q-expansion for J_w^n are nonnegative. We should also take into account that $a_{k,j} = 0$ when the degree k of the polynomial $Q_k(t)$ exceeds w and that $a_{0,0} = 1$. For any $h \in F_w(t)$ there exists the unique polynomial $g \in F_n(t)$ such that for any $i \in N_n$,

$$\binom{n}{i} g(q_1(i)) = \begin{cases} \binom{w}{i/2}\binom{n-w}{i/2} h(q_1^w(i/2)) & \text{for } i = 0, 2, \ldots, 2w, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(q_1(0)) = h(q_1^w(0))$ and for any $k \in N_n$,

$$2^n g_k(Q) = \sum_{i=0}^{n} \binom{n}{i} g(q_1(i)) Q_k(q_1(i))$$

$$= \sum_{j=0}^{w} \binom{w}{j}\binom{n-w}{j} h(q_1^w(j)) Q_k(q_1(2j)) = \binom{n}{w} \sum_{l=0}^{\min(k,w)} a_{k,l} h_l(Q^w)$$

and, in particular, $2^n g_0(Q) = \binom{n}{w} h_0(Q^w)$. Therefore, if h has the property $\mathfrak{A}_{Q^w}(d)$ then g has the property $\mathfrak{A}_Q(2d)$ and we get the following result.

Theorem 9 (Rodemich's theorem) *For any d, w, n with $1 \le d \le w \le n/2$,*

$$A_Q(H_2^n, 2d) \le \frac{2^n}{\binom{n}{w}} A_Q(J_w^n, d).$$

Now we apply Theorem 3 for $X = H_2^n$ and $X = J_w^n$ and use that for $X = H_2^n$ the systems Q and P coincide.

Theorem 10 *For any d, w, n with $1 \le d \le w \le n/2$,*

$$A(H_2^n, 2d) \le A_Q(H_2^n, 2d) \le \frac{2^n}{\binom{n}{w}} A_Q(J_w^n, d) = \frac{2^n}{B_P(J_w^n, d)}, \tag{81}$$

$$B(H_2^n, 2d) \ge B_Q(H_2^n, 2d) = B_P(H_2^n, 2d) \ge B_P(J_w^n, d) = \frac{\binom{n}{w}}{A_Q(J_w^n, d)}. \tag{82}$$

In particular, from (72), Theorem 6, (78), and (79) we obtain that for $d \geq d_{k-\varepsilon}^{1,\varepsilon}(Q^w)$,

$$A(H_2^n, 2d) \leq \frac{2^n}{\binom{n}{w}} A_Q(J_w^n, d) \leq \frac{2^n}{\binom{n}{w}} \left(\frac{n}{w}\right)^{\varepsilon} \binom{n-\varepsilon}{k-\varepsilon}, \tag{83}$$

$$B(H_2^n, 2d) \geq \frac{\binom{n}{w}}{A_Q(J_w^n, d)} \geq \frac{\binom{n}{w}}{\left(\frac{n}{w}\right)^{\varepsilon} \binom{n-\varepsilon}{k-\varepsilon}}, \tag{84}$$

and for $d = 2k + 1 - \varepsilon$,

$$A(H_2^n, 2d) \leq \frac{2^n}{B_P(J_w^n, d)} \leq \frac{2^n}{\sum_{i=0}^{k-\varepsilon} \binom{w-\varepsilon}{i} \binom{n-w+\varepsilon}{i+\varepsilon}},$$

$$B(H_2^n, 2d) \geq B_P(J_w^n, d) \geq \sum_{i=0}^{k-\varepsilon} \binom{w-\varepsilon}{i} \binom{n-w+\varepsilon}{i+\varepsilon}.$$

In conclusion, we show that (83) gives the best known asymptotic bound for binary codes and (84) implies a similar bound for binary orthogonal arrays. Consider for any η, $0 \leq \eta \leq \frac{1}{2}$, the decreasing continuous function

$$\xi_\eta(x) = \frac{\eta(1-\eta) - x(1-x)}{1 + 2\sqrt{x(1-x)}}, \tag{85}$$

which maps the interval $[0, \eta]$ onto $[0, \eta(1-\eta)]$. The inverse function $\xi_\eta^{-1}(x)$ can be expressed in the following explicit form:

$$\xi_\eta^{-1}(x) = \frac{1}{2}\left(1 - \sqrt{1 - 4\left(\sqrt{\eta(1-\eta) - x(1-x)} - x\right)^2}\right). \tag{86}$$

In the asymptotic process, as in [16], when $\lim_{n\to\infty} \frac{k}{n} = x$ and $\lim_{n\to\infty} \frac{w}{n} = \eta$ with $0 \leq x \leq \eta,\ 0 < \eta \leq \frac{1}{2}$, then

$$\frac{d_k(Q^w)}{n} \sim \xi_\eta(x). \tag{87}$$

This is also true for $d_k^{1,0}(Q^w)$ (see [14]). Using (83)-(84) for $\varepsilon = 0$ and (85)-(87) we have the following results.

Theorem 11 (The second form of the MRRW bound [16]) *If, with $0 \leq \delta \leq \frac{1}{2}$, the condition $\lim_{n\to\infty} \frac{d}{n} = \delta$ holds, then*

$$\frac{1}{n} \log A(H_2^n, d) \lesssim 1 - \max_{\frac{1}{2}(1-\sqrt{1-2\delta}) \leq \eta \leq \frac{1}{2}} \left(H_2(\eta) - H_2\left(\xi_\eta^{-1}\left(\frac{\delta}{2}\right)\right)\right)$$

where $H_2(x)$ is the binary Shannon entropy.

Theorem 12 *If* $\lim_{n\to\infty} \frac{d}{n} = \delta$, *where* $0 \le \delta \le \frac{1}{2}$, *then*

$$\frac{1}{n}\log B(H_2^n, d) \gtrsim \max_{\frac{1}{2}(1-\sqrt{1-2\delta})\le\eta\le\frac{1}{2}} \left(H_2(\eta) - H_2\left(\xi_\eta^{-1}(\frac{\delta}{2})\right)\right).$$

The last bound improves upon the known bound (see (136) and (129) in [13]) when δ is sufficiently small.

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