Universal Bounds for Codes and Designs

Vladimir I. Levenshtein*

Contents

*The research was partially supported by the Russian Foundation for Basic Research under grant 95-01-01103 and by the International Science Foundation under grant MEF300.

1. Introduction

This chapter deals with packing and some related problems for a class of polynomial metric spaces $X = (X, d(x, y))$ including finite and compact infinite ones, in particular, Hamming space, Johnson space, the unit Euclidean sphere, and real, complex and quaternionic projective spaces. We are interested in bounds on parameters of finite subsets (codes) C of X which are *universal* in the following sense. First the bounds are valid for any code $C \subseteq X$ as distinguished from the *existential* bounds valid only for some codes. Moreover they are valid for any polynomial metric space X, in particular, for the above mentioned metric spaces. The method considered for obtaining universal bounds is based on a description of all nonnegative definite functions $F(x, y)$ on X which depend only on the distance $d(x, y)$. Every such function gives a universal upper bound on the size $|C|$ of a code $C \subseteq X$ with minimal distance $d(C)$ between its distinct points. We want to choose such a nonnegative definite function which gives rise to the best bound.

For a polynomial metric space X there exists a system $Q = \{Q_i(t), i = 0, 1, \ldots\}$ of orthogonal polynomials in a real t and a certain decreasing function $\sigma(d) = \sigma_Q(d)$ such that for a polynomial $f(t)$, the function $f(\sigma(d(x, y)))$ is nonnegative definite on X if and only if $f(t)$ can be expanded over the system Q with nonnegative coefficients. In the framework of the method being considered, this reduces the problem of obtaining the best upper bound on the size of a code with minimal distance d to a certain extremum problem (σ-packing problem, $\sigma = \sigma(d)$) for the system Q of orthogonal polynomials. The author [64] found the optimal solution (over polynomials of restricted degree) of the σ-packing problem and the corresponding objective function $L_Q(\sigma)$ which is an increasing continuous function in σ. This implies the following universal bound for an arbitrary code $C \subseteq X$:

$$|C| \leq L_Q(\sigma(d(C))) \tag{1.1}$$

and gives necessary and sufficient conditions for its attainability which are satisfied in many cases. Any code C with equality in (1.1) is distance-regular and all distances between code points and intersection numbers are uniquely determined by $d(C)$ and some parameters of the whole space X. To a certain extent this chapter is devoted to the description and substantiation of (1.1) for a polynomial metric space X and to calculations of (1.1) for different spaces X of significant interest in coding theory.

Following Delsarte [29], [30] we will also characterize a code $C \subseteq X$ by its dual distance $d'(C)$ and/or the maximal strength $\tau(C) = d'(C) - 1$ of a design formed by C. In order to obtain a universal lower bound on the size of a code with a given dual distance d' we consider one more extremum problem (the τ-design problem, $\tau = d' - 1$) for the system Q. For any $\tau = d' - 1$ using the optimal solution of the σ-packing problem for some special $\sigma = \gamma(d')$, we obtain an optimal solution of the τ-design problem over polynomials of degree at most τ and, as a result, another universal bound for a code $C \subseteq X$:

$$|C| \geq L_Q(\gamma(d'(C))), \tag{1.2}$$

where $\gamma(d) = \gamma_Q(d)$ is a certain increasing function in an integer d. In particular, (1.2) gives the Rao bound [92] for orthogonal arrays in the case of the Hamming space, the Ray-Chaudhuri-Wilson bound [93] for classical block designs in the case of the Johnson space, and the Delsarte-Goethals-Seidel bound [35] for spherical designs in the case of the unit Euclidean sphere. We extend the concept of dual distance and maximal strength of a design to the case of a weighted set and of an arbitrary compact metric space (not necessarily polynomial) and prove the bound (1.2) for the general case. If we denote the maximum size of a code $C \subseteq X$ with $d(C) \geq d$ by $A(X,d)$ and the minimum size of a weighted set $C \subseteq X$ with $d'(C) \geq d$ by $B(X,d)$, we can rewrite (1.1) and (1.2) as follows

$$A(X,d) \leq L_Q(\sigma_Q(d)) \quad \text{and} \quad B(X,d) \geq L_Q(\gamma_Q(d)).$$

For the case of finite polynomial metric spaces X which are P- and Q-polynomial association schemes [30] (called polynomial graphs in this chapter) we can consider the σ-packing and τ-design problems for both finite systems Q and P of orthogonal polynomials. If we denote by $A_Q(d)$ and $B_Q(d)$ respectively objective functions of optimal solutions of the discrete σ-packing ($\sigma = \sigma_Q(d)$) and the discrete τ-design ($\tau = d-1$) problems for the system Q, we have

$$A(X,d) \leq A_Q(d) \leq L_Q(\sigma_Q(d)), \tag{1.3}$$

$$B(X,d) \geq B_Q(d) \geq L_Q(\gamma_Q(d)). \tag{1.4}$$

For the system P we can define the analogous functions $A_P(d)$, $B_P(d)$, $\sigma_P(d)$, $\gamma_P(d)$ and $L_P(\sigma)$ such that

$$A_P(d) \leq L_P(\sigma_P(d)), \tag{1.5}$$

$$B_P(d) \geq L_P(\gamma_P(d)). \tag{1.6}$$

In order to use (1.5)-(1.6) for obtaining universal bounds, we prove the following duality relationship

$$A_Q(d)\, B_P(d) = A_P(d)\, B_Q(d) = |X|. \tag{1.7}$$

This allows us to obtain from (1.3)-(1.7) two other universal bounds for a code C in a P- and Q-polynomial association scheme X:

$$|C| \leq \frac{|X|}{L_P(\gamma_P(d(C)))} \tag{1.8}$$

and

$$|C| \geq \frac{|X|}{L_P(\sigma_P(d'(C)))}. \tag{1.9}$$

The bound (1.8) coincides with the Hamming bound when $d(C)$ is odd. The bound (1.9) seems to be new; it was published by the author in [71] for Hamming space.

Thus for finite polynomial metric spaces we have two pairs (1.1)-(1.2) and (1.8)-(1.9) of universal bounds for codes and designs. For a number of spaces combinatorial proofs of (1.2) and (1.8) are known; however it is not likely that similar proofs of (1.1) and (1.9) can be found.

For polynomial metric spaces under consideration we also have one more universal bound

$$|C| \leq L_Q\left(\gamma_Q\left(2s\left(C\right)+1-\beta\left(C\right)\right)\right) \tag{1.10}$$

where $s(C)$ is the number of different distances between distinct points of C and $\beta(C)=1$ if the diameter of C is equal to the diameter of X and $\beta(C)=0$ otherwise. This bound coincides with the absolute bound of Delsarte [30] when $\beta(C)=0$. For finite polynomial metric spaces a certain dual analog of (1.10) is valid.

The chapter is organized as follows. In Section 2 we define the basic parameters of codes and, in particular, present a concept of a (weighted) design in a compact metric space. For a compact metric space we construct a system Q of polynomials which are orthogonal with respect to a measure on $[-1,1]$ connected with the average measure of metric balls. This system allows us to study some properties of codes and obtain a universal bound for the size of designs in compact metric spaces using the optimal solution of a known extremal problem for systems of orthogonal polynomials. In Section 3 we study inequalities for nonnegative definite functions, introduce a notion of polynomial spaces, and describe their fundamental properties. The σ-packing and τ-design problems for the systems Q corresponding to polynomial spaces are formulated and a preliminary description of a solution of the restricted σ-packing problem is given. Elements of the Delsarte theory of polynomial graphs (P and Q-polynomial association schemes) are included in Section 4. The approach connected with using adjacent systems of orthogonal polynomials to represent annihilating polynomials for codes and designs turned out to be very useful in obtaining basic inequalities for code parameters. In this section a new result on the duality in bounding optimal sizes of codes and designs is presented and a new pair of universal bounds for an arbitrary polynomial graph is given which can be attained for codes which are generalizations of MDS-codes and Steiner systems. Section 5 is devoted to the application of orthogonal polynomials to solve problems of coding theory and is one of the main sections in this chapter. Numerous properties of adjacent systems of orthogonal polynomials are described which are necessary to prove the main theorem on the optimal solution of the restricted σ-packing problem. This solution seems to be new for the theory of orthogonal polynomials as well. We use a non-traditional approach to estimate extreme roots of orthogonal polynomials which are needed for obtaining asymptotic results. In Section 6 we apply the theory described and calculate the universal bounds for metric spaces of essential interest in coding theory: the unit Euclidean sphere, the projective spaces, Hamming space, and Johnson space.

2. Codes and designs in compact metric spaces

2.1. Parameters of codes in compact metric spaces

We consider a finite or compact, infinite metric space X with distance $d(x, y)$. Any finite subset (*code* or *design*) $C, C \subseteq X$, is characterized by its *minimal distance*

$$d(C) = \min_{x,y \in C, x \neq y} d(x, y)$$

when $|C| \geq 2$ and by its *covering radius*

$$\rho(C) = \max_{x \in X} d(x, C),$$

where $d(x, C)$ denotes the minimal distance between x and points of C. Classical problems are to find for any $d > 0$

$$A(X, d) = \max_{C \subseteq X, d(C) \geq d} |C| \tag{2.1}$$

and

$$K(X, d) = \min_{C \subseteq X, \rho(C) \leq d} |C|.$$

We assume that X is endowed with a normalized measure μ, $\mu(X) = 1$, such that for any ρ $(\rho \geq 0)$ and any $x \in X$ the *metric ball*

$$S_\rho(x, X) = \{y : y \in X, \quad d(x, y) \leq \rho\} \tag{2.2}$$

is measurable and

$$\mu(x, \rho) = \mu(S_\rho(x, X)) > 0 \quad \text{ if } \rho > 0. \tag{2.3}$$

In the infinite case we assume that $\mu'_\rho(x, \rho) = \frac{\partial \mu(x,\rho)}{\partial \rho}$ exists and is continuous as a function in two variables. In the case of finite X we assume that μ is the normalized *counting measure*, that is,

$$\mu(A) = \frac{|A|}{|X|} \quad \text{ for any } A \subseteq X.$$

The space X is called *distance invariant* if $\mu(x, \rho)$ does not depend on $x \in X$ for any ρ . In the general case the function

$$\bar{\mu}(\rho) = \int_X \mu(x, \rho) d\mu(x) \tag{2.4}$$

characterizes the *mean measure* of a metric ball of radius ρ. Thus, for distance invariant spaces, $\bar{\mu}(\rho) = \mu(x, \rho)$ for any $x \in X$.

We also consider some other parameters of codes connected with their metric properties. For any $W, W \subseteq X$, let $\Delta(W)$ denote the set of values of $d(x,y)$ when $x, y \in W$ and let $\Delta_0(W)$ be $\Delta(W)\backslash\{0\}$. For any finite set (code) $C, C \subseteq X$, the parameter

$$s(C) = |\Delta_0(C)| \tag{2.5}$$

characterizes the *number of* (distinct) *distances* between distinct points of C. In particular, $s(X)$ is defined in the case of a finite metric space X. In the case of compact infinite metric spaces X, we put $s(X) = \infty$ since then the set $\Delta(X)$ is infinite. Clearly, for any code C,

$$d(C) = \min \Delta_0(C). \tag{2.6}$$

Together with the *diameter* $D(X) = \max \Delta_0(X)$ of the whole space X, we define the *diameter* of a code C by

$$D(C) = \max \Delta_0(C).$$

We also consider the following auxiliary parameter of a code C:

$$\beta(C) = \begin{cases} 1 \text{ if } D(C) = D(X), \\ 0 \text{ otherwise.} \end{cases} \tag{2.7}$$

We call a code C with $\beta(C) = 1$ *diametrical.* For any code $C \subseteq X$, any $x \in X$ and any ρ, the set

$$\overline{S}_\rho(x, C) = \{y : y \in C, \quad d(x,y) = \rho\}$$

is finite. Let

$$B_\rho(x, C) = |\overline{S}_\rho(x, C)|. \tag{2.8}$$

We call a code $C \subseteq X$ *distance invariant* if for any ρ, $B_\rho(x, C)$ does not depend on $x \in C$ (i.e., if C is a distance invariant metric space) and *completely distance invariant* if for any $x \in X$ and for any ρ, $B_\rho(x, C)$ depends only on $d(x, C)$. For a code $C \subseteq X$, let $\Delta(C) = \{d_0 = 0, d_1, ..., d_s\}$ where $s = s(C)$. We call a code $C \subseteq X$ *distance-regular* if for any $i, j \in \{0, 1, ..., s\}$ and for any $x, y \in C$ the numbers

$$p_{i,j}^{x,y} = |\overline{S}_{d_i}(x, C) \bigcap \overline{S}_{d_j}(y, C)| \tag{2.9}$$

depend only on $d(x, y)$. We denote the *intersection numbers* (2.9) for a distance-regular code C by $p_{i,j}^k$ if $d(x, y) = d_k$. This definition of a distance-regular code may be also applied for a finite metric space X if we put $C = X$. Note that for finite spaces X, (2.4) can be rewritten as follows

$$\bar{\mu}\ (\rho) = \frac{1}{|X|^2} \sum_{x \in X} \sum_{d_i \leq \rho} B_{d_i}(x, X), \tag{2.10}$$

and hence $\bar{\mu}\ (\rho)$ is right continuous (i.e., $\bar{\mu}\ (\rho + 0) = \bar{\mu}\ (\rho)$).

Now we introduce one more fundamental parameter of a code C and the concept of a τ-design $(\tau = 0, 1, \ldots)$. For some applications it is useful to consider a finite set (code or design) C as a weighted set $C = (C, m)$ where m is a certain positive-valued function on C (for example, multiplicity or probability). It is convenient to normalize the *weights* $m(x)$ of elements x such that

$$\sum_{x \in C} m\,(x) = |C|\,, \tag{2.11}$$

where $|C|$ as always is the number of distinct elements (the *size*) of a weighted set $C = (C, m)$. Hereafter we consider a *code* C as a special case of a weighted set when $m\,(x) \equiv 1$ for all $x \in C$. The concept of a (weighted) τ-design depends on a continuous strictly monotone real function (*substitution*) $\sigma\,(d)$ defined on the interval $[0, D(X)]$ (for example, $\sigma(d) = d$, d^2 or e^d). A weighted set $C = (C, m)$ will be referred to as a *weighted* τ*-design (in* X *with respect to the substitution* $\sigma\,(d)$*)* if for any polynomial $f\,(t)$ in a real t of degree at most τ,

$$\int_X \int_X f(\sigma(d(x,y)))d\mu\,(x)\,d\mu(y) = \frac{1}{|C|^2} \sum_{x,y \in C} f(\sigma(d(x,y)))m\,(x)\,m(y). \tag{2.12}$$

A τ*-design* is a special case of a weighted τ-design when $m\,(x) \equiv 1$ for all $x \in C$. Such τ-designs are also called *simple*. The maximum integer τ $(\tau \leq s(X))$ such that a (weighted) set C is a (weighted) τ-design is called the *strength of* C and denoted by $\tau(C)$. The value

$$d'(C) = \tau(C) + 1 \tag{2.13}$$

is referred to as the *dual distance* of C. These definitions of a τ-design and of the dual distance are natural extensions to compact metric spaces of the corresponding definitions of Delsarte for association schemes [30] and Delsarte, Goethals and Seidel for the Euclidean sphere [35]. They coincide with the classical definitions for corresponding substitutions (linear for Hamming and Johnson spaces and quadratic for the Euclidean sphere). Some other approaches to extending designs can be found in [46], [83], [36], [84], [25]. The equality (2.12) shows that in a definite sense a (weighted) τ-design C is a good approximation to the whole space X and the parameter (2.13) characterizes the degree of such an approximation. This explains why the problem of finding

$$B(X, d) = \min_{C \subseteq X, d'(C) \geq d} |C| \tag{2.14}$$

is of significant interest in factorial experiments, cryptography, complexity theory and approximation theory ([19], [92], [45], [112], [2], [48], [98], [28]). It should be noted that the minimum in (2.14) is taken over all weighted (not only simple) $(d-1)$-designs $C = (C, m)$.

For values $A(X, d)$ and $K(X, d)$ there exist so-called *sphere packing* bounds based on the facts that the open metric balls of radius $d(C)/2$ circumscribed around all points of a code C do not intersect and hence

$$\sum_{x\in C}\mu(x,\frac{d(C)}{2}-\varepsilon)\leq\mu(X)=1\quad\text{for any }\varepsilon>0,\tag{2.15}$$

and the closed metric balls of radius $\rho(C)$ circumscribed around all points of C cover X and hence

$$\sum_{x\in C}\mu(x,\rho(C))\geq\mu(X)=1.\tag{2.16}$$

In particular, for a distance invariant metric space X, (2.15)-(2.16) imply that

$$A(X,d)\leq(\overline{\mu}(\frac{d}{2}-0))^{-1},\tag{2.17}$$

$$K(X,d)\geq(\overline{\mu}(d))^{-1}.$$

We shall consider a method of obtaining universal bounds based on nonnegative definite functions on X which allow us to improve (2.15) for all codes C with sufficiently large $d(C)$. On the other hand, the inequality (2.16) can be rather precise for codes with both small and large covering radius. The author does not know results improving (2.16) by this method (the interested reader can find bounds on $K(X,d)$ in chapter (Brualdi, Litsyn, Pless)). However, this method gives rise to good lower bounds on $B(X,d)$ and shows that in a certain sense the design problem (similar to the covering problem) is dual to the packing problem.

Example 2.1. *The Hamming space $X=H_v^n$ $(n,v=2,3,...)$ consists of vectors $\mathbf{x}=(x_1,...,x_n)$ where $x_i\in\{0,1,...,v-1\}$ with the distance $d(\mathbf{x},\mathbf{y})$ being equal to the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ. In this case $\Delta(X)=\{0,1,...,n\}$ and hence $s(X)=n$, $D(X)=n$. The Hamming space is distance invariant and*

$$\overline{\mu}(d)=v^{-n}\sum_{0\leq i\leq d}\binom{n}{i}(v-1)^i.$$

The value $A(H_v^n,d)$ is the maximum size of a code $C\subseteq H_v^n$ with $d(C)\geq d$, i.e., a code C capable of correcting $\lfloor d/2\rfloor$ or fewer additive errors. For odd $d=2h+1$ we have $\overline{\mu}(\frac{d}{2}-0)=\overline{\mu}(h)$ and (2.17) gives the well known Hamming bound. It is known [29] that a τ-design C with respect to a linear substitution $\sigma(d)$ is nothing but an orthogonal array of strength τ and of index $\lambda=|C|v^{-\tau}$ with n factors and v levels. This object is defined as a matrix C with rows from H_v^n such that every vector of H_v^τ occurs in any τ columns of C exactly λ times. The value $B(H_v^n,\tau+1)$ is the minimum size of (weighted) τ-designs in the Hamming space H_v^n.

Example 2.2. *The Johnson space J_w^n $(n=2,3,...\ ;w=1,...,\lfloor\frac{n}{2}\rfloor)$ is the set of all w-subsets of the n-set $\{1,...,n\}$, where the distance between two elements $x,y\in J_w^n$*

is defined by $w - |x \cap y|$. In this case $\Delta(X) = \{0, 1, ..., w\}$ and hence $s(X) = w$, $D(X) = w$. The Johnson space is distance invariant and

$$\overline{\mu}(d) = \binom{n}{w}^{-1} \sum_{0 \le i \le d} \binom{w}{i} \binom{n-w}{i}.$$

A set C is a τ-design in J_w^n with respect to a linear substitution $\sigma(d)$ (see [30]) if and only if it is a classical block design $S_\lambda(\tau, w, n)$ and $\lambda = |C|\binom{w}{\tau}/\binom{n}{\tau}$ ($S_\lambda(\tau, w, n)$ is defined as a set S of w-subsets of an n-set such that each τ-subset of the n-set belongs exactly to λ w-subsets of S). The Johnson space J_w^n can also be considered as the subset of the binary Hamming space H_2^n consisting of vectors which have exactly w non-zero coordinates, with distance being equal to half of the Hamming distance.

Example 2.3. *The unit Euclidean sphere in $\mathbf{R}^n$*

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, ..., x_n) \in \mathbf{R^n},\ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

with the Euclidean distance $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ is a compact metric space. In this case $\Delta(X) = [0, 2]$ and hence $D(S^{n-1}) = 2$, $s(X) = \infty$. We can also measure the distance between $\mathbf{x}, \mathbf{y} \in S^{n-1}$ by the angle $\varphi(\mathbf{x}, \mathbf{y})$ where

$$\cos\varphi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i = 1 - \frac{1}{2} d^2(\mathbf{x}, \mathbf{y}). \tag{2.18}$$

The following facts are well-known [26], [95], [42]. The isometry group G of S^{n-1} consists of all orthogonal matrices of order n and acts transitively on S^{n-1}. There exists a unique normalized measure μ on S^{n-1} which is invariant (i.e., $\mu(gA) = \mu(A)$ for any measurable $A \subseteq S^{n-1}$ and any $g \in G$). This measure μ coincides with the normalized Lebesgue measure on S^{n-1} (the normalized surface area). The metric space S^{n-1} is distance invariant and for any d, $0 \le d \le 2$,

$$\overline{\mu}(d) = \frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} \text{ if } \cos\varphi = 1 - d^2/2, \tag{2.19}$$

where $\sigma_{n-1}(\varphi)$ is the surface area of a spherical cap on S^{n-1} of radial angle φ (i.e., of the set $\{\mathbf{y} : \mathbf{y} \in S^{n-1}, \varphi(\mathbf{x}, \mathbf{y}) \le \varphi\}$ with $\mathbf{x} \in S^{n-1}$) and $\sigma_{n-1} = 2\sigma_{n-1}(\frac{\pi}{2})$ is the surface area of S^{n-1}. It is also known (see, for example, [40]) that

$$\frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{\cos\varphi}^{1} (1 - z^2)^{\frac{n-3}{2}} dz \tag{2.20}$$

and $\sigma_{n-1} = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$, where $\Gamma(x)$ is the gamma function ($\Gamma(x) = \int_0^\infty v^{x-1} e^{-v} dv$). In the case of $X = S^{n-1}$ the concept of a (weighted) τ-design $C = (C, m)$ is connected

[35] (see also (3.58)) with the approximation formula for the evaluation of multi-dimensional integrals over S^{n-1} of the following sort

$$\int\limits_{S^{n-1}} u(\mathbf{x})d\mu(\mathbf{x}) \approx \frac{1}{\sum\limits_{x\in C} m(\mathbf{x})} \sum_{\mathbf{x}\in C} u(\mathbf{x})m(\mathbf{x}). \tag{2.21}$$

The (weighted) set $C = (C, m)$ is a (weighted) τ-design in S^{n-1} with respect to the substitution $\sigma(d) = 1 - d^2/2$ if and only if the approximation formula (2.21) becomes equality for all functions $u(\mathbf{x})$ which are polynomials in coordinates of $\mathbf{x} = (x_1, ..., x_n) \in S^{n-1}$ of degree at most τ. Thus $B(S^{n-1}, \tau + 1)$ is the minimum number of nodes in the approximation formula of the sort.

Example 2.4. *Real, complex, and quaternionic projective spaces. For $m = 1, 2$, and 4, consider*

$$\mathbf{T}_m^n = \{\mathbf{u} = (u_1, ..., u_n);\ u_i \in \mathbf{T}_m\},$$

where $\mathbf{T}_1 = \mathbf{R}$ is the real number field, $\mathbf{T}_2 = \mathbf{C}$ is the complex number field, and $\mathbf{T}_4 = \mathbf{H}$ is the associative (but noncommutative) quaternionic algebra. In $\mathbf{H}$ there is the basis 1 (the unity), i, j, k, such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and any $u \in \mathbf{H}$ can be represented in the form $u = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbf{R}$. Thus $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$ and m is the dimension of $\mathbf{T}_m$ over $\mathbf{R}$. For any $u \in \mathbf{H}$ the conjugate element $u^* = x_0 - x_1 i - x_2 j - x_3 k$ is defined. Since $uu^* = u^*u = x_0^2 + x_1^2 + x_2^2 + x_3^2$, one can define a norm $|u|$ of $u \in \mathbf{T}_m$ by means of $|u| = \sqrt{uu^*}$ and verify that $(uv)^* = v^*u^*$ and $|uv| = |u| \cdot |v|$ for any $u, v \in \mathbf{T}_m$. For any vectors (or points) $\mathbf{u} = (u_1, ..., u_n) \in \mathbf{T}_m^n$ and $\mathbf{v} = (v_1, ..., v_n) \in \mathbf{T}_m^n$, define the inner product as follows:

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} u_i v_i^*. \tag{2.22}$$

Points $\mathbf{u} = (u_1, ..., u_n) \in \mathbf{T}_m^n$ and $\mathbf{v} = (v_1, ..., v_n) \in \mathbf{T}_m^n$ are said to be equivalent if there exists a non-zero $\lambda \in \mathbf{T}_m$ such that $v_i = \lambda u_i$ for each $i = 1..., n$. Elements of the projective spaces $\mathbf{T}_m P^{n-1}$ are defined as equivalence classes of non-zero points of $\mathbf{T}_m^n$ and called lines (through the origin in $\mathbf{T}_m^n$). For any $U, V \in \mathbf{T}_m P^{n-1}$, one can define the angle $\varphi = \varphi(U, V)$, $0 \le \varphi \le \frac{\pi}{2}$, between lines U and V by means of

$$\cos\varphi(U, V) = \frac{|(\mathbf{u}, \mathbf{v})|}{\sqrt{(\mathbf{u}, \mathbf{u})(\mathbf{v}, \mathbf{v})}} \text{ for any } \mathbf{u} \in U,\ \mathbf{v} \in V, \tag{2.23}$$

since the right-hand side of (2.23) does not depend on the choice of points $\mathbf{u}, \mathbf{v}$ on these lines, and the distance

$$d(U, V) = \sqrt{1 - \cos\varphi(U, V)} = \sqrt{2} \sin\frac{\varphi(U, V)}{2}. \tag{2.24}$$

For the metric space $X = \mathbf{T}_m P^{n-1}$ we have $\Delta(X) = [0, 1]$ and $D(X) = 1$. The following facts are well-known (see [42], [59], [53]). The isometry group of $\mathbf{T}_m P^{n-1}$

for $m = 1, 2$, and 4, respectively, consists of all orthogonal, unitary, and symplectic matrices of order n and acts transitively on $\mathbf{T}_m P^{n-1}$. As in the case of S^{n-1}, on $\mathbf{T}_m P^{n-1}$ there exists a unique normalized invariant measure μ. The metric space $\mathbf{T}_m P^{n-1}$ is distance invariant and for any $d = \sqrt{2}\sin\frac{\varphi}{2}$, $0 \le \varphi \le \frac{\pi}{2}$,

$$\overline{\mu}(d) = \frac{\Gamma(\frac{m}{2}n)}{\Gamma(\frac{m}{2}(n-1))\Gamma(\frac{m}{2})} \int\limits_{\cos^2\varphi}^{1} (1-z)^{\frac{m}{2}(n-1)-1} z^{\frac{m}{2}-1} dz. \tag{2.25}$$

Example 2.5. *Graphs as metric spaces. Consider an arbitrary connected undirected graph Γ (without loops and multiple edges). The set $X = X(\Gamma)$ of vertices of Γ may be considered as a finite metric space with the path metric $d(x,y)$ equal to the minimum number of edges in a path connecting the vertices x and y of Γ. In this case $\Delta(X) = \{0, 1, ..., D(X)\}$ and $D(X)$ coincides with the diameter of Γ. In particular, if we consider complete, cyclic and linear graphs on a vertex set $\{0, 1, ..., v-1\}$, $v \ge 2$, we obtain respectively metric spaces X with distances*

$$d(x,y) = 1 - \delta_{x,y} \ ,$$

$$d(x,y) = \min(|x-y|, v - |x-y|), \tag{2.26}$$

$$d(x,y) = |x-y|, \tag{2.27}$$

and $D(X) = 1$, $D(X) = \lfloor\frac{v}{2}\rfloor$, $D(X) = v-1$. The first two of these spaces are distance invariant; the third is not for $v \ge 3$. (It should be noted that the concept of a distance invariant metric space is stronger than the usual concept of a regular graph.) Notice that the value $A(X(\Gamma), 2)$ is the maximum size of a set of non-pairwise adjacent vertices (independent set) of Γ.

Example 2.6. *Degrees of metric spaces. For a metric space $X = (X, d(x,y))$ we consider a space $X^{(n)}$ consisting of vectors $\mathbf{x} = (x_1, ..., x_n)$ where $x_i \in X$, $i = 1, ..., n$. We can determine a distance in $X^{(n)}$ in different ways, for example, as*

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} d(x_i, y_i) \tag{2.28}$$

or

$$d(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n} d(x_i, y_i). \tag{2.29}$$

Notice that if $X = X(\Gamma)$ for some graph Γ, then the set $X^{(n)}$ of vertices and a set of edges E such that $(\mathbf{x}, \mathbf{y}) \in E$ if and only if $d(\mathbf{x}, \mathbf{y}) = 1$ form a graph $\Gamma^{(n)} = (X^{(n)}, E)$ with path metric either (2.28) or (2.29). In particular, if we use this construction with the digit-by-digit metric (2.28) in complete and cyclic graphs on v vertices, we obtain the Hamming and Lee metrics and corresponding spaces. For an arbitrary connected graph Γ on v vertices the value $\sqrt[n]{A(X(\Gamma^{(n)}), 2)}$, where the path metric (2.29) is used, has a limit as $n \to \infty$ and this limit is called the Shannon capacity [101] of Γ.

Analogous constructions can also be used in the infinite case. In particular, we consider a metric space $X = (I, d(x,y), \mu)$ with $I = [0,1)$ and distance (2.26), where $v = 1$, and with $I = [0,1]$ and distance (2.27) (μ is the normalized Lebesgue measure on I in the both cases). The degree $X^{(n)}$ with distance (2.29) and with measure equal to the normalized n-dimensional volume gives rise to the n-dimensional torus $T^n = ([0,1)^n, d(\mathbf{x},\mathbf{y}))$ with metric

$$d(\mathbf{x},\mathbf{y}) = \max_{1 \le i \le n} \min(|x_i - y_i|, 1 - |x_i - y_i|) \tag{2.30}$$

and to the n-dimensional cube $I^n = ([0,1]^n, d(\mathbf{x},\mathbf{y}))$ with metric

$$d(\mathbf{x},\mathbf{y}) = \max_{1 \le i \le n} |x_i - y_i|. \tag{2.31}$$

It is clear that T^n is a distance invariant space while I^n is not, and $D(T^n) = \frac{1}{2}$, $D(I^n) = 1$.

2.2. A system Q of orthogonal polynomials for a compact metric space

We endow a compact metric space $X = (X, d(x,y), \mu)$ and a given substitution $\sigma(d)$ with a system of orthogonal polynomials $Q_i(t)$ of degree i, $i = 0, 1, ..., s(X)$. This system $Q = \{Q_i(t)\}$ is in fact defined by the substitution $\sigma(d)$ and the mean measure $\overline{\mu}(d)$ of closed metric balls of radius d. Considering some properties of the system Q we obtain universal bounds on the sizes of codes and designs in compact metric spaces. Notice that the definition of a (weighted) τ-design (with respect to substitution $\sigma(d)$) is invariant under any linear transformation of $\sigma(d)$. This allows us to assume without additional loss of generality that $\sigma(d)$ is a continuous strictly decreasing function on $[0, D(X)]$ such that

$$\sigma(D(X)) = -1 \le \sigma(d) \le \sigma(0) = 1. \tag{2.32}$$

Such a substitution function $\sigma(d)$ is referred to as *standard*. The inverse function of the standard function $\sigma(d)$ is denoted by $\sigma^{-1}(\ )$, i.e., for any t, $-1 \le t \le 1$, $\sigma^{-1}(t) = d$ when $\sigma(d) = t$.

Consider the following inner product for continuous functions in a real t on the interval $[-1, 1]$:

$$f \cdot g = \int_X \int_X f(\sigma(d(x,y)))g(\sigma(d(x,y)))d\mu(x)d\mu(y). \tag{2.33}$$

We verify that the integral on the right-hand side of (2.33) can be rewritten as the Rimann-Stieltjes integral on $[-1, 1]$. To do this we consider the function $\nu(t)$ in a real t on the interval $[-1, 1]$ defined by

$$\nu(t) = 1 - \overline{\mu}(\sigma^{-1}(t)), \tag{2.34}$$

which increases with t from $\nu(-1) = 0$ up to $\nu(1) = 1 - \overline{\mu}(0)$. When X is finite and $\Delta(X) = \{d_0 = 0, d_1, ..., d_s\}$ where $s = s(X)$, $\nu(t)$ is left continuous (i.e., $\nu(t-0) = \nu(t)$) and has $s+1$ steps at the points $t_i = \sigma(d_i)$ with positive step sizes

$$w_i = \nu(t_i + 0) - \nu(t_i) = \overline{\mu}(d_i) - \overline{\mu}(d_i - 0) = \frac{1}{|X|^2} \sum_{x \in X} B_{d_i}(x, X), \tag{2.35}$$

$i = 0, 1, ..., s$, $\sum\limits_{i=0}^{s} w_i = 1$. In the infinite case we assumed that $\mu'_\rho(x, \rho)$ is continuous as a function in two variables $\rho \in [0, D(X)]$, $x \in X$ and hence $\overline{\mu}'(\rho) = \int\limits_X \mu'_\rho(x, \rho) d\mu(x)$. Therefore, for some natural additional assumptions on $\sigma(d)$, $\nu(t)$ is differentiable and $w(t) = \nu'(t) = -\frac{\overline{\mu}'(d)}{\sigma'(d)}$ is continuous on $[-1, 1]$ and positive inside the interval. An arbitrary left continuous function $\nu(t)$ on $[-1, 1]$ generates the Lebesgue-Stieltjes measure ν on the same interval by

$$\nu[a, b] = \nu(b + 0) - \nu(a), \quad \nu(a, b) = \nu(b) - \nu(a + 0), \tag{2.36}$$

$$\nu(a, b] = \nu(b + 0) - \nu(a + 0), \quad \nu[a, b) = \nu(b) - \nu(a). \tag{2.37}$$

In our case the measure ν is normalized ($\nu[-1, 1] = 1$),

$$\nu[a, b] = \overline{\mu}(d_a) - \overline{\mu}(d_b - 0) \quad \text{where } a = \sigma(d_a),\ b = \sigma(d_b),$$

and under the above assumptions, we can rewrite (2.33) as

$$f \cdot g = \int_{-1}^{1} f(t) g(t) d\nu(t). \tag{2.38}$$

Since $|\Delta_0(X)| = s(X)$, the polynomials t^i, $i = 0, ..., s(X)$, for $t = \sigma(d)$, $d \in \Delta(X)$, are linearly independent, and using them in the orthogonalization process with respect to (2.38) we obtain the following statement.

Theorem 2.7. *For a compact metric space $X = (X, d(x, y), \mu)$ with a fixed standard substitution $\sigma(d)$ there exists a unique system Q of orthogonal polynomials $Q_i(t)$ of degree i on the interval $[-1, 1]$ and a unique system of positive constants r_i ($i = 0, 1, ..., s(X)$) such that*

$$r_i \int_{-1}^{1} Q_i(t) Q_j(t) d\nu(t) = \delta_{i,j}, \tag{2.39}$$

$$Q_i(1) = 1, \quad i = 0, 1, ..., s(X), \tag{2.40}$$

where $\nu(t)$ is defined by (2.34).

Notice that

$$Q_0(t) \equiv 1 \quad \text{and} \quad r_0 = 1 \tag{2.41}$$

since the measure is normalized. Under our assumption, the orthogonality condition (2.39) for a finite and infinite X takes respectively the following form:

$$r_i \sum_{k=0}^{s} Q_i(\sigma(d_k))Q_j(\sigma(d_k))w_k = \delta_{i,j}, \quad i,j = 0,1,...,s = s(X), \tag{2.42}$$

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)w(t)dt = \delta_{i,j}, \quad i,j = 0,1,... \quad . \tag{2.43}$$

Example 2.8. *For the Hamming space $X = H_v^n$ with the standard substitution $\sigma(d) = 1 - \frac{2d}{n}$ we have $s = n$, $d_i = i$, $w_i = v^{-n}\binom{n}{i}(v-1)^i$, $i = 0,1,...,n$. Conditions (2.39) and (2.40) uniquely determine constants $r_i = w_i v^n$ and polynomials $Q_i(t)$ which are connected with the Krawtchouk polynomials [61]*

$$K_i^{n,v}(z) = \sum_{j=0}^{i}(-1)^j(v-1)^{i-j}\binom{z}{j}\binom{n-z}{i-j} \tag{2.44}$$

as follows

$$Q_i(t) = (r_i)^{-1}K_i^{n,v}(\frac{n(1-t)}{2}). \tag{2.45}$$

Example 2.9. *For the Johnson space J_w^n ($w \le n/2$) with the standard substitution $\sigma(d) = 1 - \frac{2d}{w}$ we have $s = w$, $d_i = i$, $w_i = \binom{w}{i}\binom{n-w}{i}/\binom{n}{w}$, $i = 0,1,...,w$. Conditions (2.39) and (2.40) uniquely determine constants $r_i = \binom{n}{i} - \binom{n}{i-1}$ and polynomials $Q_i(t)$ which are connected with the Hahn polynomials*

$$J_i(z) = \sum_{j=0}^{i}(-1)^j \frac{\binom{i}{j}\binom{n+1-i}{j}}{\binom{w}{j}\binom{n-w}{j}}\binom{z}{j} \tag{2.46}$$

as follows

$$Q_i(t) = J_i(\frac{w(1-t)}{2}). \tag{2.47}$$

Example 2.10. *For the Euclidean sphere $X = S^{n-1}$ with the standard substitution $\sigma(d) = 1 - \frac{d^2}{2}$ we have $s = \infty$ and $w(t) = \nu'(t) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}(1-t^2)^{\frac{n-3}{2}}$ (see (2.34), (2.19) and (2.20)). It is known [14] that the Jacobi polynomials*

$$P_i^{\alpha,\beta}(t) = \frac{1}{2^i\binom{i+\alpha}{\alpha}} \sum_{j=0}^{i} \binom{i+\alpha}{j}\binom{i+\beta}{i-j}(t-1)^{i-j}(t+1)^j, \tag{2.48}$$

$(\alpha \geq -\frac{1}{2},\ \beta \geq -\frac{1}{2})$, *normalized by* $P_i^{\alpha,\beta}(1) = 1$, *satisfy the following orthogonality condition:*

$$\frac{2i+\alpha+\beta+1}{i+\alpha+\beta+1}\binom{i+\alpha+\beta+1}{i}\frac{\binom{i+\alpha}{i}}{\binom{i+\beta}{i}}c^{\alpha,\beta}\int_{-1}^{1} P_i^{\alpha,\beta}(t)P_j^{\alpha,\beta}(t)(1-t)^{\alpha}(1+t)^{\beta}dt = \delta_{i,j}, \tag{2.49}$$

where the constant

$$c^{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \tag{2.50}$$

normalizes the measure, i.e., $c^{\alpha,\beta}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}d(t) = 1$. *Therefore, in the case* S^{n-1}, $n = 2,3,\ldots$, *we have* $r_i = \frac{2i+n-2}{i+n-2}\binom{i+n-2}{i}$ *and*

$$Q_i(t) = P_i^{\frac{n-3}{2},\frac{n-3}{2}}(t). \tag{2.51}$$

Example 2.11. *For the projective spaces* $X = \mathbf{T}_m P^{n-1}$ *(see Example 2.4) with the standard substitution*

$$\sigma(d) = 2\left(1-d^2\right)^2 - 1 \ (\text{or } \sigma(d) = \cos 2\varphi \text{ if } d = \sqrt{2}\sin\frac{\varphi}{2}), \tag{2.52}$$

we have $s = \infty$ *and* $w(t) = \nu'(t) = \frac{\Gamma(\frac{m}{2}n)}{2^{\frac{m}{2}n-1}\Gamma(\frac{m}{2}(n-1))\Gamma(\frac{m}{2})}(1-t)^{\frac{m}{2}(n-1)-1}\ (1+t)^{\frac{m}{2}-1}$ *using (2.34), (2.25), and the change of variable* $2z = t+1$. *Therefore, in the case* $\mathbf{T}_m P^{n-1}$, $m = 1,2$, *and* 4, $n = 3,4,\ldots$, *with this substitution,*

$$Q_i(t) = P_i^{\frac{m}{2}(n-1)-1,\frac{m}{2}-1}(t). \tag{2.53}$$

Example 2.12. *In the case of the n-dimensional torus* T^n *(see Example 2.6) we have* $\mu(\mathbf{x},\ \rho) = (2\rho)^n$ *for any* $\mathbf{x} \in T^n$ *and any* ρ, $0 \leq \rho \leq D(T^n) = \frac{1}{2}$. *The linear substitution* $\sigma(d) = 1-4d$ *is standard and by (2.34)* $\nu(t) = 1 - \left(\frac{1-t}{2}\right)^n$ *and hence* $w(t) = \nu'(t) = n2^{-n}(1-t)^{n-1}$. *According to Theorem 2.7 and (2.49) in this case* $Q_i(t)$ *coincides with the Jacobi polynomials* $P_i^{n-1,0}(t)$ *and* $r_i = \frac{n+2i}{n}\binom{n-1+i}{i}^2$. *For the n-dimensional cube* I^n, $\mu(\mathbf{x},\rho)$ *in general depends on* $\mathbf{x} \in I^n$. *However it is easy to calculate that* $\overline{\mu}(\rho) = \int_{I^n}\mu(\mathbf{x},\rho)d\mu(\mathbf{x}) = (2\rho-\rho^2)^n$ *for any* ρ, $0 \leq \rho \leq D(I^n) = 1$. *The substitution* $\sigma(d) = 1-2(2d-d^2)$ *is standard and by (2.34)* $\nu(t)$ *also equals* $1-\left(\frac{1-t}{2}\right)^n$. *Hence the polynomials* $Q_i(t)$ *and constants* r_i *coincide with those for the torus* T^n *but under the latter standard substitution.*

2.3. A restricted τ-design problem for systems of orthogonal polynomials

We formulate an extremum problem for a system of orthogonal polynomials. This problem has a unique optimal solution. An application to the system Q considered above gives a universal bound on the size of weighted τ-designs in a compact metric space X.

Let $F[t]$ be the set of polynomials in a real t with real coefficients. Any $f \in F[t]$ of degree at most $s = s(X)$ can be uniquely represented in the form $\sum_{i=0}^{s} f_i Q_i(t)$ where

$$f_i = f_i(Q) = r_i \int_{-1}^{1} f(t) Q_i(t) d\nu(t). \tag{2.54}$$

We also introduce the notation

$$\Omega(f) = \Omega_Q(f) = \frac{f(1)}{f_0} \quad \text{when } f_0 \neq 0. \tag{2.55}$$

Remark 2.13. *We can extend definitions (2.54) and (2.55) to polynomials $f(t)$ of degree more than s when $s = s(X)$ (and X) are finite. It is correct because in this case, the polynomials $f(t)$ and $\sum_{i=0}^{s} f_i Q_i(t)$, where f_i are defined in (2.54), coincide at all $t = \sigma(d)$ when $d \in \Delta(X)$.*

In the sequel, we bear in mind that $f_0(Q) > 0$ for any nonzero polynomial $f \in F[t]$ (of degree at most $s(X)$ in the case of finite X) such that $f(t) \geq 0$ for $-1 \leq t \leq 1$.

Notice that by (2.33), (2.38)-(2.41) and (2.54)

$$f_0 = \int_X \int_X f(\sigma(d(x,y))) d\mu(x) d\mu(y) \tag{2.56}$$

and f_0 is equal to zero when $f(t) = Q_i(t)$, $i = 1, ..., s(X)$. This gives the following corollary.

Corollary 2.14. *A weighted set $C = (C, m)$ is a weighted τ-design if and only if*

$$\sum_{x \in C} \sum_{y \in C} Q_i(\sigma(d(x,y))) m(x) m(y) = 0, \quad i = 1, ..., \tau. \tag{2.57}$$

Consider an arbitrary weighted set $C = (C, m)$. A polynomial $f \in F[t]$ is called *annihilating* for C if $f(\sigma(d(x,y))) = 0$ for any $x, y \in C$, $x \neq y$. A polynomial, annihilating for C, of minimal degree (i.e., $s(C)$) is called *minimal.* Let $f_C(t)$ denote the minimal polynomial for C such that $f_C(1) = 1$. Let

$$\Delta'(C) = \{i : i \in \{0, 1, ..., s(X)\}, \quad B'_i(C) \neq 0\}. \tag{2.58}$$

where

$$B'_i(C) = \frac{r_i}{|C|} \sum_{x,y \in C} Q_i(\sigma(d(x,y))) m(x) m(y), \quad i = 0, 1, ..., s(X) \tag{2.59}$$

(the normalization (2.11) is used). Notice that

$$B'_0(C) = |C| \tag{2.60}$$

and hence $0 \in \Delta'(C)$. Let $\Delta'_0(C) = \Delta'(C)\backslash\{0\}$. From the definition (2.13) of the dual distance $d'(C)$ and Corollary 2.14 it follows (cf. (2.6)) that

$$d'(C) = \min \Delta'_0(C) \tag{2.61}$$

when the set $\Delta'_0(C)$ is non-empty. For codes C in a finite X we consider also two other parameters (cf. (2.5) and (2.7)): the *number of* (nonzero) *dual distances*

$$s'(C) = |\Delta'_0(C)| \tag{2.62}$$

and

$$\beta'(C) = \begin{cases} 1 \text{ if } s(X) \in \Delta'_0(C), \\ 0 \text{ otherwise.} \end{cases} \tag{2.63}$$

For an arbitrary weighted set $C = (C, m)$ and an arbitrary $f \in F[t]$ by (2.59) we have

$$\sum_{x,y\in C} f(\sigma(d(x,y)))m(x)m(y) = |C| \sum_{i=0}^{s(X)} \frac{f_i(Q)}{r_i} B'_i(C). \tag{2.64}$$

Theorem 2.15. *For any weighted τ-design $C = (C, m)$ and any polynomial $f \in F[t]$ of degree at most τ such that $f_0(Q) > 0$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$,*

$$|C| \geq \Omega_Q(f) \tag{2.65}$$

with equality if and only if C is a simple (i.e., $m(x) \equiv 1$ for all $x \in C$) τ-design and $f(t)$ is annihilating for C.

Proof. By Corollary 2.14 and (2.59), $B'_i(C) = 0$ for $i = 1, ..., \tau$ and hence (2.64), (2.60), (2.32) and (2.11) imply the following inequalities

$$f_0 \geq f(\sigma(0)) \frac{\sum_{x\in C} m^2(x)}{\left(\sum_{x\in C} m(x)\right)^2} \geq \frac{f(1)}{|C|}$$

with the above mentioned conditions of attainability. ■

This theorem gives rise to the following extremum problem for the system Q of orthogonal polynomials whose solution ensures the best bound in (2.65).

A restricted[1] **τ-design problem**: In the class of polynomials $f \in F[t]$ of degree at most τ such that $f(t) \geq 0$ for $-1 \leq t \leq 1$ (and hence $f_0(Q) > 0$) find a polynomial which maximizes $\Omega_Q(f)$.

[1]The term "restricted" means that we only consider polynomials of degree at most τ; later we remove the restriction for polynomial metric spaces.

In Section 5 we prove (as a consequence of a result in the theory of orthogonal polynomials) that there exists a unique (up to a constant factor) solution $f = g^\tau$ of the problem. To describe g^τ and calculate $\Omega_Q(g^\tau)$ we define, for any $a, b \in \{0, 1, \dots\ \}$, systems $Q^{a,b}$ of orthogonal polynomials which in a certain sense are *adjacent* to an arbitrary system Q satisfying the conditions (2.39) and (2.40). As before we assume that $\nu(t)$ is a left continuous step function or continuously differentiable, and hence (2.42) or (2.43) holds. First we choose a function $\nu^{a,b}(t)$ and constants $c^{a,b}$ as follows. If $\nu(t)$ is continuously differentiable and $\nu'(t) = w(t)$, then so is $\nu^{a,b}(t)$ and $(\nu^{a,b}(t))' = c^{a,b}(1-t)^a(1+t)^b w(t)$. If $\nu(t)$ is a step function, then so is $\nu^{a,b}(t)$ and it is produced from $\nu(t)$ by multiplying its steps w_i at the points $t_i = \sigma(d_i)$ $(i = 0, 1, \dots, s)$ by $c^{a,b}(1-\sigma(d_i))^a(1+\sigma(d_i))^b$ (thus the number $s^{a,b}+1$ of steps of $\nu^{a,b}(t)$ equals $s-1+\delta_{a,0}+\delta_{b,0}$ since $\sigma(d_0) = 1$ and $\sigma(d_s) = -1$; we put $s^{a,b} = \infty$ in the continuous case). Constants $c^{a,b}$ are chosen so that the Lebesgue-Stieltjes measure $\nu^{a,b}$ on $[-1, 1]$ generated by the function $\nu^{a,b}(t)$ (similar to (2.36)-(2.37)) is also normalized, i.e.,

$$\int_{-1}^{1} d\nu^{a,b}(t) = c^{a,b} \int_{-1}^{1} (1-t)^a(1+t)^b d\nu(t) = 1. \tag{2.66}$$

Similar arguments as in Theorem 2.7 show that there exists a unique system $Q^{a,b}$ of orthogonal polynomials $Q_i^{a,b}(t)$ of degree i on the interval $[-1, 1]$ and a unique system of positive constants $r_i^{a,b}$ $\quad (i = 0, 1, \dots, s^{a,b})$ such that

$$r_i^{a,b} \int_{-1}^{1} Q_i^{a,b}(t) Q_j^{a,b}(t) d\nu^{a,b}(t) = \delta_{i,j}, \tag{2.67}$$

$$Q_i^{a,b}(1) = 1, \quad i = 0, 1, \dots, s^{a,b}. \tag{2.68}$$

It is easily seen that $Q_0^{a,b}(t) = 1$, $r_0^{a,b} = 1$, $Q_i^{0,0}(t) = Q_i(t)$, $r_i^{0,0} = r_i$, $c^{0,0} = 1$. Sometimes we shall omit the indices a, b if they both equal 0. Let

$$T_i^{a,b}(x, y) = \sum_{j=0}^{i} r_j^{a,b} Q_j^{a,b}(x) Q_j^{a,b}(y),\ i = 0, 1, \dots, s^{a,b}. \tag{2.69}$$

We shall see (Lemma 5.24 and Remark 5.27) that

$$Q_i^{a,b+1}(t) = \frac{T_i^{a,b}(t, -1)}{T_i^{a,b}(1, -1)}, \quad Q_i^{a+1,b}(t) = \frac{T_i^{a,b}(t, 1)}{T_i^{a,b}(1, 1)}. \tag{2.70}$$

We let $t_i^{a,b}$ denote the largest root of polynomial $Q_i^{a,b}(t)$ and put

$$d_i^{a,b} = \sigma^{-1}(t_i^{a,b}). \tag{2.71}$$

To describe the optimal polynomial for the restricted τ-design problem we represent τ as an odd or even integer in the form

$$\tau = 2l - \theta \quad \text{where } l = l(\tau) \in \{1, 2, \dots\} \text{ and } \theta = \theta(\tau) \in \{0, 1\}\,.$$

It is clear that $l = \lfloor \frac{\tau+1}{2} \rfloor$ and $\theta = 2\lfloor \frac{\tau+1}{2} \rfloor - \tau$. Let

$$g^{(\tau)}(t) = (1+t)^\theta \left(\sum_{j=0}^{l-\theta} r_j^{0,\theta} Q_j^{0,\theta}(t) \right)^2 . \tag{2.72}$$

Note that

$$\sum_{j=0}^{l-\theta} r_j^{0,\theta} Q_j^{0,\theta}(t) = T_{l-\theta}^{0,\theta}(1,1) Q_{l-\theta}^{1,\theta}(t) = Q_{l-\theta}^{1,\theta}(t) \sum_{j=0}^{l-\theta} r_j^{0,\theta}. \tag{2.73}$$

In accord with (2.67)

$$r_i^{0,\theta} c^{0,\theta} \int_{-1}^{1} Q_i^{0,\theta}(t) Q_j^{0,\theta}(t)(1+t)^\theta d\nu(t) = \delta_{i,j},$$

and, hence, by (2.54) and (2.66)

$$g_0^{(\tau)}(Q) = \int_{-1}^{1} g^{(\tau)}(t) d\nu(t) = (c^{0,\theta})^{-1} \sum_{i=0}^{l-\theta} r_i^{0,\theta},$$

where $c^{0,0} = 1$ and

$$(c^{0,1})^{-1} = \int_{-1}^{1} (1+t) d\nu(t) = \frac{-2Q_1(-1)}{1 - Q_1(-1)}, \tag{2.74}$$

since $1 + t = 2\frac{Q_1(t) - Q_1(-1)}{1 - Q_1(-1)}$. Using the polynomial $g^{(\tau)}$ in Theorem 2.15 and taking into account (2.68)-(2.70), we get the following theorem.

Theorem 2.16. *For any weighted τ-design $C = (C, m)$ in a compact metric space X,*

$$|C| \ge \Omega_Q(g^{(\tau)}) = \left(1 - \frac{1}{Q_1(-1)}\right)^\theta \sum_{i=0}^{l-\theta} r_i^{0,\theta}, \tag{2.75}$$

where $l = \lfloor \frac{\tau+1}{2} \rfloor$ and $\theta = 2l - \tau$ with equality in (2.75) holding if and only if C is a simple τ-design and $(1+t)^\theta Q_{l-\theta}^{1,\theta}(t)$ is annihilating for C.

A τ-design C for which the bound (2.75) is attained is called a *tight* design.

Lemma 2.17. *For any code C in a compact metric space X*

$$\tau(C) \leq 2s(C) - \beta(C). \tag{2.76}$$

Proof. For any code C there exists a polynomial $g(t)$ of degree $s-\beta$ where $s = s(C)$ and $\beta = \beta(C)$, such that the polynomial $f(t) = (1-t)(1+t)^{\beta}(g(t))^2$ is annihilating for C and, moreover, the left-hand side of (2.64) equals zero. The polynomial $f(t)$ is nonnegative for $-1 \leq t \leq 1$ and by (2.54), $f_0(Q) > 0$ (if its degree $2s - \beta + 1$ does not exceed $s(X)$ in the case of finite spaces X). Therefore, the inequalities

$$s(X) \geq \tau(C) \geq 2s(C) - \beta(C) + 1$$

imply that $B_i'(C) = 0$, $i = 1, ..., 2s - \beta + 1$, and contradict the equality (2.64) for the polynomial $f(t)$ of degree $2s - \beta + 1 \leq s(X)$. ∎

Lemma 2.18. *A code C in a compact metric space X is a tight design if and only if $\tau(C) = 2s(C) - \beta(C)$ and $f_C(t) = (\frac{1+t}{2})^{\beta(C)} Q^{1,\beta(C)}_{s(C)-\beta(C)}(t)$.*

Proof. If the polynomial $(1+t)^{\theta} Q^{1,\theta}_{l-\theta}(t)$ where $\theta \in \{0, 1\}$ is annihilating for a code C, then $s(C) - \beta(C) \leq l - \theta$, and $\beta(C) \leq \theta$ since $\sigma(D(X)) = -1$ and $Q_l^{1,0}(-1) \neq 0$. Therefore for a tight $(2l - \theta)$-design C,

$$\tau(C) \geq 2l - \theta \geq 2s(C) - 2\beta(C) + \theta \geq 2s(C) - \beta(C)$$

and hence by Theorem 2.16 and Lemma 2.17 $\tau(C) = 2s(C) - \beta(C)$, $l = s(C)$, $\theta = \beta(C)$. On the other hand, if the conditions of the lemma are satisfied, then using $f(t) = f_C(t)$ in (2.64) shows that $|C| = \Omega(f_C)$. However, taking into account (2.68)-(2.70) again it is easy to verify that $\Omega(f_C) = \Omega(g^{(\tau(C))})$, and we have equality in (2.75). ∎

The optimality and uniqueness of the polynomial $g^{(\tau)}$ for the restricted τ-design problem will be proved in Section 5.

We shall see that the systems $Q^{0,1}$ for the Hamming space H_v^n and for the Johnson space J_w^n coincide with systems Q for the spaces H_v^{n-1} and J_{w-1}^{n-1} respectively. For the Euclidean sphere S^{n-1}, the projective spaces $\mathbf{T}_m P^{n-1}$, the torus T^n, and the cube I^n, the polynomials $Q_i(t)$ are equal to Jacobi polynomials $P_i^{\alpha,\beta}(t)$ with some α and β ; hence, $Q_i^{0,1}(t) = P_i^{\alpha,\beta+1}(t)$ in these cases. It is known (see, for example, [14], [67]) that for normalized Jacobi polynomials $P_i^{\alpha,\beta}(t)$ (see (2.48)-(2.49)),

$$\sum_{j=0}^{i} r_j^{\alpha,\beta} = \binom{i+\alpha+\beta+1}{i}\binom{i+\alpha+1}{i} / \binom{i+\beta}{i} \tag{2.77}$$

and

$$1 - \frac{1}{P_1^{\alpha,\beta}(-1)} = \frac{\alpha+\beta+2}{\beta+1}. \tag{2.78}$$

Therefore, for $Q_i(t) = P_i^{\alpha,\beta}(t)$ the right-hand side of (2.75) equals

$$\binom{l+\alpha+\beta+1}{l}\binom{l+\alpha+1-\theta}{l-\theta}/\binom{l+\beta}{l} \tag{2.79}$$

We can use the examples considered and Theorem 2.16 to obtain the lower bounds on the size of a weighted $(2l-\theta)$-design given in Table 2.1. In the connection with these bounds for the projective spaces the following references are relevant: [30], [34], [35], [57], [82], [53], and [9].

3. Polynomial metric spaces and extremum problems for the system Q of orthogonal polynomials

3.1. Inequalities for nonnegative definite functions

Almost all universal bounds for codes in metric spaces are based on inequalities for functions in two points x and y of the space which are nonnegative definite and depend only on the distance between x and y. The well known universal bounds of Blichfeldt [17] and Rankin [91] are in fact connected with the usual inner product in Euclidean space. Important advances were made in the works of Sidelnikov [103] and Welch [123] which used inequalities for degrees of the inner product or of its modulus. The next step was taken by Delsarte [30] who described all nonnegative definite functions depending on the distance for so-called P- and Q-polynomial association schemes by using a system Q of orthogonal polynomials (this system coincides with that of Section 2.2). An analogous description is known (see, for example, [95], [121]) for some infinite metric spaces including the Euclidean sphere and real, complex and quaternionic projective spaces. This gives rise to some extremum problems for systems of orthogonal polynomials which are linear programs in the case of finite metric spaces. For the Hamming and Johnson spaces McEliece, Rodemich, Rumsey, Jr. and Welch [79] proposed a solution of the linear program which gives the best known asymptotic bound (MRRW-bound) when the code distance grows linearly with length. An analogous solution was also used for the case of the Euclidean sphere and projective spaces by Kabatyanskii and the author [57] in order to obtain the best known asymptotic bound (KL-bound) when the code distance or angle distance is a constant and the dimension tends to infinity. We shall describe a better solution which was found by the author [64] and prove its optimality in a certain sense. This result gives the MRRW- and KL-bounds as a corollary.

For a metric space $X = (X, d(x,y))$ we discuss real- and complex-valued functions $F(x,y)$ in two variables $x, y \in X$. In the case of a finite X, $F(x,y)$ might be considered as an element of a matrix of order $|X| \times |X|$ in row $x \in X$ and column $y \in X$. A function $F(x,y)$ is called *Hermitian if* $F(x,y) = \overline{F(y,x)}$ for any $x, y \in X$ where $\bar{a}$ is the conjugate of a.

A Hermitian function $F(x,y)$ is called *nonnegative definite on* X if for any finite

Space X	Standard substitution $\sigma(d)$	Lower bound on $B(X, 2l-\theta+1)$	References
H_v^n	$1-\frac{2d}{n}$	$v^\theta \sum_{j=0}^{l-\theta} \binom{n-\theta}{j}(v-1)^j$	[92]
J_w^n	$1-\frac{2d}{w}$	$\left(\frac{n}{w}\right)^\theta \binom{n-\theta}{l-\theta}$	[93] when $\theta=0$ [38] when $\theta=1$
S^{n-1}	$1-\frac{d^2}{2}$	$\binom{n+l-2}{l-1}+\binom{n+l-1-\theta}{l-\theta}$	[35]
$\mathbf{R}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\binom{n+2l-1-\theta}{2l-\theta}$	
$\mathbf{C}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\binom{n+l-1}{l}\binom{n+l-1-\theta}{l-\theta}$	
$\mathbf{H}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\frac{1}{l+1}\binom{2n+l-1}{l}\binom{2n+l-2-\theta}{l-\theta}$	
T^n	$1-4d$	$\binom{n+l}{l}\binom{n+l-\theta}{l-\theta}$	
I^n	$1-2d(2-d)$	$\binom{n+l}{l}\binom{n+l-\theta}{l-\theta}$	

Table 2.1: Lower Bounds on the Size of $(2l-\theta)$-Designs

set (code) $C \subseteq X$ and any complex function $\mathbf{v}(x)$, $x \in X$:

$$\sum_{x,y \in C} F(x,y)\mathbf{v}(x)\overline{\mathbf{v}(y)} \geq 0 \tag{3.1}$$

(since $F(x,y)$ is Hermitian, the left-hand side of (3.1) is real). For a finite metric space X, a matrix $F(x,y)$, $x,y \in X$, is nonnegative definite if and only if there exist some number h of functions $\mathbf{w}_1(x), ..., \mathbf{w}_h(x)$ on X such that

$$F(x,y) = \sum_{j=1}^{h} \mathbf{w}_j(x)\overline{\mathbf{w}_j(y)}. \tag{3.2}$$

For our purposes in the case of infinite metric spaces it is also sufficient to consider only nonnegative definite functions of this type. We shall call a function which can be represented in the form (3.2) with continuous nonzero functions $\mathbf{w}_j(x)$, $j = 1, ..., h$ ($h \geq 1$), a *finite dimensional*[2] *nonnegative definite function (FDNDF) on X*. It is significant to note that for any code $C \subseteq X$ and any FDNDF $F(x,y)$ on X we have

$$F(C) = \frac{1}{|C|^2} \sum_{x,y \in C} F(x,y) = \frac{1}{|C|^2} \sum_{j=1}^{h} |\sum_{x \in C} \mathbf{w}_j(x)|^2 \geq 0. \tag{3.3}$$

Example 3.1. *The most significant example of an FDNDF on the Euclidean complex space $\mathbf{C}^n$ is the usual inner product of vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$*

$$(x,y) = \sum_{i=1}^{n} x_i \overline{y_i}. \tag{3.4}$$

The function $|(x,y)|$ is not an FDNDF on $\mathbf{C}^n$ but $|(x,y)|^2$ is because of

$$|(x,y)|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \overline{x_j} \overline{y_i \overline{y_j}} \tag{3.5}$$

The function $K_i^{n,v}(d(x,y))$, where $K_i^{n,v}(z)$ is the Krawtchouk polynomial (2.44) of degree i and $d(x,y)$ is the Hamming distance, can be represented in the form

$$K_i^{n,v}(d(x,y)) = \sum_{a \in H_v^n,\ d(0,a)=i} \xi^{(a,x-y)} \tag{3.6}$$

where ξ is a primitive v-th root of unity. Thus (3.2) holds with $\mathbf{w}_j(x) = \xi^{(a,x)}$ and $h = \binom{n}{i}(v-1)^i$, and $K_i^{n,v}(d(x,y))$ is an FDNDF[3] *on the Hamming space H_v^n.*

[2]Later we consider the function $F(x,y)$ as the kernel of a linear finite dimensional operator on $L_2(X,\mu)$.

[3]Of course, a real FDNDF $F(x,y)$ also has a representation (3.2) with real functions; however, this example shows that the complex representation might be more useful.

Notice the following properties of FDNDFs:

i) A linear combination of FDNDFs with positive coefficients is an FDNDF.

ii) A product of FDNDFs is an FDNDF.

iii) If $F(x,y)$ is an FDNDF on X and α is a (continuous) mapping of a space Y into X, then $H(x,y) = F(\alpha(x), \alpha(y))$ is an FDNDF on Y.

Example 3.2. *It is known that for any v $(v = 2, 3, \dots)$ on S^{v-2} there exists an equidistant code of v points $\pi(0), \pi(1), \dots, \pi(v-1)$ with Euclidean distance $\sqrt{\frac{2v}{v-1}}$ (the vertices of the regular simplex inscribed in S^{v-2}). The mapping $\alpha : H_v^n \to S^{(v-1)n-1}$ such that $\alpha(x) = \frac{1}{\sqrt{n}}(\pi(x_1), \dots, \pi(x_n))$ for $x = (x_1, \dots, x_n) \in H_v^n$ has the property that*

$$1 - \frac{vd(x,y)}{(v-1)n} = (\alpha(x), \alpha(y)) \tag{3.7}$$

and hence this function is an FDNDF on H_v^n. In the case of the Johnson space J_w^n one can map [104] any w-subset x of the n-set $\{0, 1, \dots, n-1\}$ to $\alpha(x) = (z_1, \dots, z_n) \in S^{n-1}$ as follows:

$$z_i = \begin{cases} -\sqrt{\frac{n-w}{wn}} & \text{if } i \in x, \\ \sqrt{\frac{w}{(n-w)n}} & \text{otherwise.} \end{cases}$$

It follows that the function

$$1 - \frac{nd(x,y)}{w(n-w)} = (\alpha(x), \alpha(y)) \tag{3.8}$$

is an FDNDF on J_w^n.

To discuss simultaneously FDNDFs on finite and compact infinite metric spaces X with normalized measure μ we consider the (possibly, finite dimensional) Hilbert space $L_2(X, \mu)$ of complex-valued functions $\mathbf{u}$ on X satisfying

$$\|\mathbf{u}\|^{\mathbf{2}} = \int_X |\mathbf{u}(x)|^{\mathbf{2}} d\mu\,(x) < \infty$$

with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_X \mathbf{u}(x)\overline{\mathbf{v}(x)} d\mu\,(x) \tag{3.9}$$

(Functions are equivalent if they differ on a set A with $\mu(A) = 0$, and hence equivalent continuous functions coincide by our assumption (2.3).) We associate with every FDNDF $F\,(x,y)$ a linear operator F which transforms $\mathbf{u} \in L_2(X, \mu)$ to the function

$$F\mathbf{u}=\int\limits_X F(x,y)\,\mathbf{u}(y)\,d\mu(y)=\sum_{j=1}^{h}\mathbf{w}_j(x)\int\limits_X\overline{\mathbf{w}_j(y)}\mathbf{u}(y)d\mu(y) \tag{3.10}$$

of a subspace $V(F)$ of continuous functions generated by $\mathbf{w}_1(x),\ldots,\mathbf{w}_h(x)$. This finite dimensional operator F is self-conjugate, since

$$\langle F\mathbf{u},\mathbf{v}\rangle=\langle\mathbf{u},F\mathbf{v}\rangle=\sum_{j=1}^{h}\langle\mathbf{u},\mathbf{w}_j\rangle\overline{\langle\mathbf{v},\mathbf{w}_j\rangle}. \tag{3.11}$$

From (3.10) and (3.11) it follows that all eigenvalues of F are nonnegative, all eigenfunctions with positive eigenvalues belong to $V(F)$, and all eigenfunctions with a zero eigenvalue are orthogonal to $V(F)$. Hence, by the Hilbert-Schmidt Theorem (see, for example [58]) we have the following result.

Lemma 3.3. *For any FDNDF $F(x,y)$ on X, in $V(F)$ there exists an orthonormal basis of $m=\dim V(F)$ continuous eigenfunctions $\mathbf{e}_1(x),\ldots,\mathbf{e}_m(x)$ with positive eigenvalues $\lambda_1,\ldots,\lambda_m$ (i.e., $\langle\mathbf{e}_i,\mathbf{e}_j\rangle=\delta_{ij}$, $F\mathbf{e}_i=\lambda_i\mathbf{e}_i$) such that any continuous function $\mathbf{u}(x)$ on X can be uniquely represented as*

$$\mathbf{u}(x)=\sum_{i=1}^{m}u_i\mathbf{e}_i(x)+\mathbf{e}_{\mathbf{u}}(x) \tag{3.12}$$

where $u_i=\langle\mathbf{u},\mathbf{e}_i\rangle$, $\langle\mathbf{e}_{\mathbf{u}},\mathbf{e}_i\rangle=0$, $i=1,...,m$, $F\mathbf{e}_{\mathbf{u}}=0$, and (3.2) is reduced to the following diagonal form:

$$F(x,y)=\sum_{i=1}^{m}\lambda_i\mathbf{e}_i(x)\overline{\mathbf{e}_i(y)}. \tag{3.13}$$

We should only add that (3.2) can be represented in the form (3.13) since $\mathbf{w}_j(x)=\sum_{k=1}^{m}\langle\mathbf{w}_j,\mathbf{e}_k\rangle\mathbf{e}_k(x)$ by (3.12) and $\sum_{j=1}^{h}\langle\mathbf{e}_i,\mathbf{w}_j\rangle\overline{\langle\mathbf{e}_k,\mathbf{w}_j\rangle}=\lambda_i\delta_{i,k}$ for $i,k=1,...,m$, by (3.11).

Now we discuss how an FDNDF $F(x,y)$ can be used for obtaining a (universal) upper bound on the size of a code $C\subseteq X$ with minimal distance $d(C)\geq d$. We know that (3.3) is true. Let

$$F(X)=\int\limits_X\int\limits_X F(x,y)\,d\mu(x)\,d\mu(y)=\sum_{j=1}^{h}\left|\int\limits_X\mathbf{w}_j(x)d\mu(x)\right|^2. \tag{3.14}$$

We shall say that for a given FDNDF $F(x,y)$ the *inequality on the mean* holds if

$$F(C)\geq F(X)\ \text{ for any finite } C\subseteq X. \tag{3.15}$$

It is not true that the inequality on the mean holds for any FDNDF. Therefore it is natural for an FDNDF $F(x,y)$ to find the maximum nonnegative $\lambda = \lambda(F)$ such that $F(x,y) - \lambda J(x,y)$ is FDNDF (here as usual $J(x,y) = \mathbf{j}(x)\mathbf{j}(y)$ where $\mathbf{j}(x) \equiv 1$ for $x \in X$). Since $J(C) = J(X) = 1$ (see (3.3),(3.14)) we have $F(C) \geq \lambda(F)$, $F(X) \geq \lambda(F)$ and that for $F(x,y)$ the inequality on the mean holds when $\lambda(F) = F(X)$.

Example 3.4. *One can check that for any $x, y \in \mathbf{C^n}$,*

$$|(x,y)|^2 - \frac{1}{n}\sum_{i=1}^{n}|x_i|^2\sum_{i=1}^{n}|y_i|^2 = \sum_{i\neq j} x_i\overline{x_j}\overline{y_i}\overline{y_j} + \frac{1}{2n}\sum_{i\neq j}\left(|x_i|^2 - |x_j|^2\right)\left(|y_i|^2 - |y_j|^2\right).$$

It follows that for the FDNDF $F(x,y) = |(x,y)|^2$ (see (3.5)) considered on the unit Euclidean sphere S^{n-1} (and on the unit complex Euclidean sphere $\mathbf{C}S^{n-1}$) we have $\lambda(F) \geq \frac{1}{n}$. The integral (3.14) equals (see Example 2.10)

$$\int_{-1}^{1} t^2 w(t)dt = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} t^2(1-t^2)^{\frac{n-3}{2}}dt = \frac{1}{n}.$$

(We used the following equality

$$\int_{0}^{1} z^{a-1}(1-z)^{b-1}dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ , \tag{3.16}$$

for the beta-function.) Therefore $F(X) = \frac{1}{n}$ and hence $\lambda(F) = F(X) = \frac{1}{n}$ (this is also true for $\mathbf{C}S^{n-1}$).

The value $\lambda(F)$ for a symmetric real matrix $F(x,y)$ was calculated in [80]. We use it and find necessary and sufficient conditions for $\lambda(F) = F(X)$.

Lemma 3.5. *Let an FDNDF $F(x,y)$ be reduced to the form (3.13) and let*

$$\mathbf{j}(x) = \sum_{i=1}^{m} a_i\ \mathbf{e}_i(x) + \mathbf{e_j}(x), \tag{3.17}$$

where $m = \dim V(F) \geq 1$ and $F\mathbf{e_j} = \mathbf{0}$. If $\mathbf{j}(x) \notin V(F)$, then $\lambda(F) = 0 \leq F(X)$ with equality if and only if $a_1 = \ldots = a_m = 0$; if $\mathbf{j}(x) \in V(F)$, then

$$\lambda(F) = \left(\sum_{i=1}^{m}\frac{|a_i|^2}{\lambda_i}\right)^{-1} \leq F(X) \tag{3.18}$$

with equality if and only if all λ_i corresponding to nonzero a_i are equal (and equal to $\lambda(F)$).

Proof. Notice that

$$F(X) = \langle F\mathbf{j},\mathbf{j}\rangle = \sum_{i=1}^{m} \lambda_i |a_i|^2 \tag{3.19}$$

and if $F(x,y) - \lambda J(x,y)$ is an FDNDF then for any function $\mathbf{u}$ (see (3.12)) such that $\langle \mathbf{u},\mathbf{j}\rangle = 1$ we have

$$\langle F\mathbf{u},\mathbf{u}\rangle = \sum_{i=1}^{m} \lambda_i |u_i|^2 \geq \lambda. \tag{3.20}$$

In the case when $\mathbf{j}(x) \notin V(F)$ the continuous function $\mathbf{e_j}(x)$ is not identically zero, and we have $\|\mathbf{e_j}\| \neq 0$. Since $\langle \mathbf{e_j},\mathbf{j}\rangle = \|\mathbf{e_j}\|^2$, for $\mathbf{u} = \mathbf{e_j}/\|\mathbf{e_j}\|^{\mathbf{2}}$ we have $\langle \mathbf{u},\mathbf{j}\rangle = 1$ and $F\mathbf{u} = \mathbf{0}$. By (3.19) and (3.20) the first part of the lemma is proved. In the case when $\mathbf{j}(\mathbf{x}) \in V(F)$,

$$\langle \mathbf{j},\mathbf{j}\rangle = \sum_{i=1}^{m} |a_i|^2 = 1 \tag{3.21}$$

and at least one a_i is not zero. By the Cauchy inequality

$$\left|\sum_{i=1}^{m} u_i\overline{a_i}\right|^2 \leq \sum_{i=1}^{m} \lambda_i |u_i|^2 \cdot \sum_{i=1}^{m} \frac{|a_i|^2}{\lambda_i} \tag{3.22}$$

with equality if and only if the vectors $(u_1,\ldots,u_m)$ and $\left(\frac{a_1}{\lambda_1},\ldots,\frac{a_m}{\lambda_m}\right)$ are proportional. Let

$$\lambda_0 = \left(\sum_{i=1}^{m} \frac{|a_i|^2}{\lambda_i}\right)^{-1}.$$

For $u_i = \frac{\lambda_0}{\lambda_i}a_i$, $i = 1,\ldots,m$, we have $|\sum_{i=1}^{m} u_i\overline{a_i}| = 1$ and equality in (3.22). Therefore (3.20) implies that $\lambda(F) \leq \lambda_0$. On the other hand, as $\mathbf{j}(x) \in V(F)$, the function

$$\sum_{i=1}^{m} \lambda_i \left(\mathbf{e}_i(x) - \frac{\lambda_0}{\lambda_i}\overline{a_i}\mathbf{j}(x)\right)\overline{\left(\mathbf{e}_i(y) - \frac{\lambda_0}{\lambda_i}\overline{a_i}\mathbf{j}(y)\right)}$$

is an FDNDF and coincides with $F(x,y) - \lambda_0 J(x,y)$. Hence $F(X) \geq \lambda(F) \geq \lambda_0$. The last statement of the lemma follows from (3.21) and (3.19) and the necessary and sufficient conditions of attainability of (3.22) for $u_i = a_i$, $i = 1,\ldots,m$. ■

This proof shows that $\lambda(F) = \min \langle F\mathbf{u},\mathbf{u}\rangle$, where the minimum is taken over all functions $\mathbf{u}$ such that $\langle \mathbf{u},\mathbf{j}\rangle = 1$. Furthermore, it can be extended to prove the following statement for the space $X^{(n)} = X \times \ldots \times X$ (see [80]): if $F_i(x,y)$, $i = 1,\ldots,n$, are FDNDFs on X, then

$$\mathbf{F}(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{n} F_i(x_i,y_i), \tag{3.23}$$

where $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in X^{(n)}$, is an FDNDF on $X^{(n)}$ and

$$\lambda(\mathbf{F}) = \prod_{i=1}^{n} \lambda(F_i). \tag{3.24}$$

The general method of using an FDNDF to bound the size of a code is based on the following.

Lemma 3.6. *For any FDNDF $F(x, y)$ on X and any code C, $C \subseteq X$,*

$$\max_{x \in C} F(x, x) - \max_{\substack{x,y \in C \\ d(x,y) \geq d(C)}} ReF(x, y) \geq |C| \left(\lambda(F) - \max_{\substack{x,y \in C \\ d(x,y) \geq d(C)}} ReF(x, y) \right), \tag{3.25}$$

and, in particular, if $\max\limits_{x,y \in C,\ d(x,y) \geq d(C)} ReF(x, y) \leq 0$, *then*

$$\max_{x \in C} F(x, x) \geq |C| \lambda(F). \tag{3.26}$$

Proof. It is immediate from

$$\frac{1}{|C|} \max_{x \in C} F(x, x) + \frac{|C| - 1}{|C|} \max_{\substack{x,y \in C \\ d(x,y) \geq d(C)}} ReF(x, y) \geq \frac{1}{|C|^2} \sum_{x,y \in C} F(x, y) = F(C) \geq \lambda(F).$$

■

Remark 3.7. *Lemma 3.6 gives a universal upper bound on the size of a code with a given minimal distance if the right-hand side of (3.25) or (3.26) is positive. The proof shows that this bound can be improved replacing $\lambda(F)$ by $F(X)$ if for $F(x, y)$ the inequality on the mean holds.*

Example 3.8. *As was remarked in [80], [97], in particular, Lemma 3.6 gives the Lovász bound [77] for Shannon capacity of a graph (see Example 2.6). Indeed, let $M(x, y)$ be the (symmetric) adjacency matrix of a graph Γ with v vertices and N edges and let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of $M(x, y)$. Then $\lambda_{\max} > 0$, $\lambda_{\min} < 0$ and*

$$F(x, y) = M(x, y) - \lambda_{\min}\, I(x, y), \tag{3.27}$$

where $I(x, y)$ is the identity matrix, forms a FDNDF on $X(\Gamma)$. Moreover, $F(x, y)$ possesses the following properties:

$$F(X(\Gamma)) = v^{-2}(2N - v\lambda_{\min}), \tag{3.28}$$

$$F(x, x) = -\lambda_{\min} \text{ for any vertex } x,$$

and

$$F(x, y) = 0 \text{ if } d(x, y) \geq 2. \tag{3.29}$$

By (3.26) $A(X(\Gamma), 2) \leq \frac{-\lambda_{\min}}{\lambda(F)}$. Considering the space $X(\Gamma^{(n)}) = \{\mathbf{x} = (x_1, ..., x_n),$ $x_i \in X\}$ with metric (2.29) and using the facts that the FDNDF (3.23) on $X(\Gamma^{(n)})$ (with $F_i(x,y) = F(x,y)$) has the property

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0 \text{ if } d(\mathbf{x}, \mathbf{y}) \geq 2 \tag{3.30}$$

and (3.24) holds, we obtain the Lovász bound [77]:

$$\sqrt[n]{A(X(\Gamma^{(n)}), 2)} \leq \frac{-\lambda_{\min}}{\lambda(F)}. \tag{3.31}$$

In particular, the adjacency matrix $M(x,y)$ of the cyclic graph with v, $v \geq 3$, vertices can be represented in the following diagonal form (cf. (3.13)):

$$M(x,y) = \sum_{j=0}^{v-1} \lambda_j \mathbf{e}_j(x) \overline{\mathbf{e}_j(y)},$$

where $\lambda_j = v^{-1}(\xi^j + \xi^{-j})$, $\mathbf{e}_j(x) = \xi^{jx}$ and $\xi = e^{\frac{2\pi i}{v}}$ (it is easy to check using $\sum_{j=0}^{v-1} \xi^j = 0$). Since $\lambda_j = 2v^{-1} \cos \frac{2\pi j}{v}$, then $\lambda_{\max} = 2$ and $\lambda_{\min} = -2$ if v is even and $\lambda_{\min} = -2 \cos \frac{\pi}{v}$ if v is odd (the eigenvalues of the matrix $M(x,y)$ are v times more than those of the operator M). Taking into account that $F(x,y) = M(x,y) - \lambda_{\min}\, I(x,y) = \sum_{j=0}^{v-1} (\lambda_j - v^{-1}\lambda_{\min}) \mathbf{e}_j(x)\overline{\mathbf{e}_j(y)}$, $j(x) = \mathbf{e}_0(x)$, $\lambda_0 = 2v^{-1}$, we obtain by Lemma 3.5 that $\lambda(F) = v^{-1}(2 - \lambda_{\min})$ and, hence, for the cyclic graph Γ with v vertices,

$$\sqrt[n]{A(X(\Gamma^{(n)}), 2)} \leq \frac{v}{1 - 2/\lambda_{\min}}. \tag{3.32}$$

Notice that the function $\mathbf{j}(x) \equiv 1$ is an eigenfunction of F if and only if $F\mathbf{j} = \int_X F(x,y)\, d\mu(y)$ does not depend on $x \in X$. In this case the corresponding eigenvalue is $\langle F\mathbf{j}, \mathbf{j} \rangle = F(X)$; however, it may happen that $\lambda(F) < F(X)$. Nevertheless, we verify that the inequality on the mean (3.15) holds for any FDNDF $F(x,y)$ if $\int_X F(x,y)\, d\mu(y)$ does not depend on $x \in X$.

To this end we consider some inequalities for an FDNDF $F(x,y)$ on a finite or infinite set Y which depend on a choice of two measures, two subsets, and two functions on Y. The author found in [62] similar inequalities to explain the nature of the Sidelnikov and Welch inequalities as those on the mean (3.15) and to obtain sufficient conditions under which such inequalities are valid. We present here desirable inequalities in the general form published by Kabatyanskii and the author in [57]. We will tacitly assume the measurability of all sets and the existence of all integrals considered below.

Consider two normalized measures α and β on a finite or infinite set Y ($\alpha(Y) = 1$, $\beta(Y) = 1$) and two complex functions $g(x)$ and $h(x)$. For an FDNDF $F(x,y)$ on Y

and any subsets $A \subseteq Y$ and $B \subseteq Y$ such that $\alpha(A) > 0$, $\beta(B) > 0$, let

$$F_{\alpha,\beta}(A,B,g,h) = \frac{1}{\alpha(A)\beta(B)} \int_A \int_B F(x,y)g(x)\overline{h(y)}d\alpha(x)d\beta(y).$$

Also let $F_{\alpha,\beta}(A,B) = F_{\alpha,\beta}(A,B,1,1)$ and $F_\alpha(A) = F_{\alpha,\alpha}(A,A)$. In particular, for the metric space X with the measure μ, $F(X) = F_\mu(X) = F_{\mu,\mu}(X,X)$ and for any finite set (code) $C \subseteq X$, $F(C) = F_{\chi_C}(X)$ where the measure χ_C is defined as follows

$$\chi_C(A) = \frac{|A \cap C|}{|C|} \text{ for any } A \subseteq X.$$

Theorem 3.9. *For any FDNDF $F(x,y)$ on Y and any α, β, A, B, g and h,*

$$|F_{\alpha,\beta}(A,B,g,h)|^2 \le F_{\alpha,\alpha}(A,A,g,g)F_{\beta,\beta}(B,B,h,h). \tag{3.33}$$

To prove the theorem, twice change the order of summation, and use the Cauchy inequality for sequences and Fubini's Theorem (see [57]).

Corollary 3.10. *For an FDNDF $F(x,y)$ on X the inequality on the mean holds if $\int_X F(x,y)\,d\mu(y)$ does not depend on $x \in X$. In particular, for an FDNDF $F(x,y)$ on a distance invariant space X the inequality on the mean holds if $F(x,y)$ is a function in $d(x,y)$.*

Proof. The condition of the corollary implies that $F_{\alpha,\mu}(A,X) = F_{\mu,\mu}(X,X) = F_\mu(X)$ for any measure α on X and any $A \subseteq X$. Therefore the inequality on the mean (3.15) follows from Theorem 3.9 for $A = C$, $B = Y = X$, $\alpha = \chi_C$, $\beta = \mu$, $g = h = 1$. For a distance invariant space X, $\int_X f(d(x,y))d\mu(y)$ does not depend on $x \in X$ because the function $\mu(x,\rho)$ defined by (2.3) has the same property. ■

Example 3.11 (The inequalities on the mean). *Consider the Sidelnikov inequality [103] for any code $C \subseteq S^{n-1}$:*

$$\frac{1}{|C|^2} \sum_{x,y \in C} (x,y)^{2s} \ge \frac{\Gamma(\frac{n}{2})\Gamma(s+\frac{1}{2})}{\Gamma(\frac{n}{2}+s)\Gamma(\frac{1}{2})}, \; s = 0,1,\dots, \tag{3.34}$$

and the Welch inequality [123] for any code $C \subseteq \mathbf{C}S^{n-1}$:

$$\frac{1}{|C|^2} \sum_{x,y \in C} |(x,y)|^{2s} \ge \frac{1}{\binom{n+s-1}{s}}, \; s = 0,1,\dots, \tag{3.35}$$

which were proved by combinatorial methods. Both these important results can be proved as inequalities on the mean for FDNDFs. We already noticed that the inner *product* $(x,y) = 1 - d^2(x,y)/2$ *and, hence,* $F(x,y) = (x,y)^{2s}$, $s = 0,1,...,$ *is an FDNDF on* S^{n-1}. *Using*

$$F(S^{n-1}) = \int_{S^{n-1}} (x,y)^{2s} d\mu(x) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} t^{2s}(1-t^2)^{\frac{n-3}{2}} dt$$

(see Example 2.10) and (3.16) by Corollary 3.10 we get (3.34) and find that the right-hand side of (3.34) is equal to $F(S^{n-1})$. *(*$F(S^{n-1}) = 0$ *for* $F(x,y) = (x,y)^h$ *if* h *is odd.) It was also proved [62] that (3.35) is the inequality on the mean for FDNDF* $F(x,y) = |(x,y)|^{2s}$ *on* $\mathbf{C}S^{n-1}$ *and the right-hand side of (3.35) is equal to* $F(\mathbf{C}S^{n-1})$. *(The measure on* $\mathbf{C}S^{n-1}$ *coincides with the measure* μ *on* S^{2n-1}, *if one considers any* $x = (x_1, ..., x_n) \in \mathbf{C}S^{n-1}$ *as* $(\mathrm{Re}\, x_1, \mathrm{Im}\, x_1, ..., \mathrm{Re}\, x_n, \mathrm{Im}\, x_n) \in S^{2n-1}$.*) Analogously using FDNDF functions (3.7) and (3.8) we have the following known [104] inequalities: for any code* $C \subseteq H_v^n$ *and any* $h = 0,1,...,$

$$\frac{1}{|C|^2} \sum_{x,y \in C} (1 - \frac{vd(x,y)}{(v-1)n})^h \geq \frac{1}{v^n} \sum_{d=0}^{n} (v-1)^d \binom{n}{d} (1 - \frac{vd}{(v-1)n})^h,$$

and for any code $C \subseteq J_w^n$ *and any* $h = 0,1,...,$

$$\frac{1}{|C|^2} \sum_{x,y \in C} (1 - \frac{nd(x,y)}{w(n-w)})^h \geq \frac{1}{\binom{n}{w}} \sum_{d=0}^{w} \binom{w}{d} \binom{n-w}{d} (1 - \frac{nd}{w(n-w)})^h.$$

Consider also the function $\varphi_{n,v,a,b}(d) = (a-b)^d(a+(v-1)b)^{n-d}$. *From the definition (2.44) of Krawtchouk polynomials* $K_i^{n,v}(z)$ *it follows that for any* $d = 0,1,...,n$,

$$\varphi_{n,v,a,b}(d) = \sum_{i=0}^{n} K_i^{n,v}(d) a^{n-i} b^i.$$

Since $K_i^{n,v}(d(x,y))$ *are FDNDFs on* H_v^n, *so is* $F(x,y) = \varphi_{n,v,a,b}(d(x,y))$ *for any nonnegative* a *and* b. *For any* p, $0 \leq p \leq \frac{v-1}{v}$, $a = \frac{1}{v}$ *and* $b = \frac{1}{v} - \frac{p}{v-1}$, *we have* $a \geq 0$, $b \geq 0$ *and hence by Corollary 3.10 the following inequality on the mean is valid: for any code* $C \subseteq H_v^n$ *and any* p, $0 \leq p \leq \frac{v-1}{v}$,

$$\frac{1}{|C|^2} \sum_{x,y \in C} \left(\frac{p}{v-1}\right)^{d(x,y)} (1-p)^{n-d(x,y)} \geq F(H_v^n) = \frac{1}{v^n}.$$

This inequality turned out to be useful for studying the probability of undetected errors over symmetric channels [75], [63]. Returning to Example 3.8 we can see that for $F(x,y)$ *defined by (3.27) the condition of Corollary 3.10 is fulfilled if and only if the graph* Γ *is regular with some valency* v_1. *Then this condition is also fulfilled for* $\mathbf{F}(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{n} F(x_i, y_i)$. *Moreover* $\mathbf{F}(X(\Gamma^{(n)})) = (F(X(\Gamma)))^n$ *and* $\mathbf{F}(\mathbf{x},\mathbf{x}) =$

$(-\lambda_{\min})^n$. *According to (3.28), (3.30), Lemma 3.6 and Remark 3.7 it follows that for any regular graph* Γ *with* v *vertices and valency* v_1,

$$\sqrt[n]{A(X(\Gamma^{(n)}),2)} \leq \frac{v}{1-v_1/\lambda_{\min}}.$$

In particular, this gives another proof of (3.32) for cyclic graphs ($v_1 = 2$).

Theorem 3.9 also gives rise to many useful inequalities for spaces which are not distance invariant. For example, if we consider finite sets $A = \{x_1, ..., x_M\} \subseteq X$, $B = \{y_1, ..., y_N\} \subseteq X$, measures $\alpha = \chi_A$, $\beta = \chi_B$ and numbers $g(x_i) = g_i$, $h(y_j) = h_j$, we obtain the following inequality for an FDNDF $F(x, y)$ on an arbitrary X [62]:

$$|\sum_{i=1}^{M}\sum_{j=1}^{N} F(x_i, y_j) g_i \overline{h_j}|^2 \leq \sum_{i=1}^{M}\sum_{j=1}^{M} F(x_i, x_j) g_i \overline{g_j} \sum_{i=1}^{N}\sum_{j=1}^{N} F(y_i, y_j) h_i \overline{h_j}. \tag{3.36}$$

In particular, if $X = \mathbf{C}^n, N = 1$, $y = y_1$, $g(x_i) = 1$, $h(y) = 1$ and $F(x, y) = (x, y)$, we have the inequality

$$|\sum_{i=1}^{M}(x_i, y)|^2 \leq \|y\|^2 \sum_{i=1}^{M}\sum_{j=1}^{M}(x_i, x_j),$$

which is equivalent to

$$2\|y\|^2 \sum_{i=1}^{M}\sum_{j=1}^{M} d^2(x_i, x_j) \leq$$

$$4M\|y\|^2 \sum_{i=1}^{M} d^2(x_i, y) - \left(\sum_{i=1}^{M}\|x_i\|^2 - M\|y\|^2 - \sum_{i=1}^{M} d^2(x_i, y)\right)^2.$$

The last inequality implies the Blichfeldt inequality [17]

$$\sum_{i=1}^{M}\sum_{j=1}^{M} d^2(x_i, x_j) \leq 2M \sum_{i=1}^{M} d^2(x_i, y),$$

and for $\|y\| = \|x_i\| = 1$, $i = 1, ..., M$, gives the Rankin inequality [91]

$$2\sum_{i=1}^{M}\sum_{j=1}^{M} d^2(x_i, x_j) \leq \left(4M - \sum_{i=1}^{M} d^2(x_i, y)\right) \sum_{i=1}^{M} d^2(x_i, y).$$

If in (3.36) we put $N = 1$, $y = y_1$, $h(y) = 1$ and use $2|g_i g_j| \leq |g_i|^2 + |g_j|^2$, we obtain for any FDNDF $F(x, y)$ on an arbitrary X, any $A = \{x_1, ..., x_M\} \subseteq X$, and any $y \in X$, the inequality

$$|\sum_{i=1}^{M} F(x_i, y) g_i|^2 \leq F(y, y) \sum_{i=1}^{M} |g_i|^2 \sum_{j=1}^{M} |F(x_i, x_j)|. \tag{3.37}$$

In particular, if we consider in (3.37)

$$g_i = \overline{F(x_i,y)}, \quad g_i = \frac{\overline{F(x_i,y)}}{|F(x_i,y)|}, \quad g_i = \frac{\overline{F(x_i,y)}}{\sum\limits_{j=1}^{M} |F(x_i,x_j)|},$$

we obtain three inequalities

$$\sum_{i=1}^{M} |F(x_i,y)| \le F(y,y) \max_{1\le i\le M} \sum_{j=1}^{M} |F(x_i,x_j)|,$$

$$\left(\sum_{i=1}^{M} |F(x_i,y)|\right)^2 \le F(y,y) \sum_{j=1}^{M}\sum_{j=1}^{M} |F(x_i,x_j)|,$$

$$\sum_{i=1}^{M} \frac{|F(x_i,y)|}{\sum\limits_{j=1}^{M} |F(x_i,x_j)|} \le F(y,y),$$

which in the case $F(x,y) = (x,y)$ are due to Bombieri, Selberg and Halász respectively (see [81]).

Theorem 3.9 and values of known n-tuple integrals (see, for example, [40]) allow us to obtain different generalizations of the Sidelnikov and Welch inequalities. For example, for any code C in an ellipsoid $X = \{x = (x_1, ..., x_n) \in \mathbf{R}^n : \sum\limits_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^2 \le 1\}$ and any $s = 0, 1, \dots$, the following inequality holds [62]:

$$\sum_{x,y\in C} (x,y)^{2s} \ge \frac{\binom{2s}{s}}{l_s} \left(\sum_{x\in C}\left(\sum_{i=1}^{n} x_i^2 a_i^2\right)^s\right)^2 . \tag{3.38}$$

where $l_s = \sum\limits_{k_1+...+k_n=s} a_1^{4k_1} \cdots a_n^{4k_n} \binom{2k_1}{k_1} \cdots \binom{2k_n}{k_n}$ and the sum is taken over all ordered partitions of s into n nonnegative integers (the generating function $\sum\limits_{i=0}^{\infty} l_i z^i$ is equal to $\prod\limits_{i=1}^{n} \left(1 - 4a_i^4 z\right)^{-1/2}$). This inequality follows from Theorem 3.9 when $A = C$, $B = Y = X$, $\alpha = \chi_C$ and $\beta = \mu$ where μ is the normalized n-dimensional volume on the ellipsoid X. In particular, (3.38) implies that for any code $C \subset X = \{x = (x_1, ..., x_n) \in \mathbf{R}^n : \sum\limits_{i=1}^{n} x_i^2 \le 1\}$ and any $s = 0, 1, \dots$,

$$\sum_{x,y\in C} (x,y)^{2s} \ge \frac{\Gamma(\frac{n}{2})\Gamma(s+\frac{1}{2})}{\Gamma(\frac{n}{2}+s)\Gamma(\frac{1}{2})} \left(\sum_{x\in C}\left(\sum_{i=1}^{n} x_i^2\right)^s\right)^2 ,$$

that generalizes (3.34).

3.2. Polynomial metric spaces

In Section 2.2 for a compact metric space $X = \{X, d(x,y)\}$ with a normalized measure μ and with a standard substitution $\sigma(d)$ $(\sigma(D(X)) = -1 \le \sigma(d) \le \sigma(0) = 1)$ we defined by (2.39)-(2.40) the system of orthogonal polynomials $Q = \{Q_i(t),$ $i = 0, 1, ..., s(X)\}$ on the interval $[-1, 1]$ and the system of positive constants r_i, $i = 0, 1, ..., s(X)$. Now we consider a class of compact metric spaces X for which there exists a simple description of all FDNDFs

$$F(x,y) = f(\sigma(d(x,y))) \tag{3.39}$$

where $f(t)$ is a polynomial in real t. Note that in the case of a finite X, the polynomial

$$Q_{s(X)+1}(t) = \prod_{d\in\Delta(X)} (t - \sigma(d)) \tag{3.40}$$

of degree $s(X)+1$ has the property: $Q_{s(X)+1}(\sigma(d(x,y))) = 0$ for any $x, y \in X$. Thus, without loss of generality, we can assume that

$$f(t) = \sum_{i=0}^{s(X)} f_i(Q)Q_i(t). \tag{3.41}$$

The function $F(x,y)$ defined by (3.39) and (3.41) is a real-valued symmetric function, $F(x,x) = f(1)$ for any $x \in X$, and by (2.54) and (2.56)

$$F(X) = \int_X \int_X f(\sigma(d(x,y)))d\mu(x)d\mu(y) = \int_{-1}^{1} f(t)d\nu(t) = f_0(Q). \tag{3.42}$$

It follows, from Corollary 3.10, that for any FDNDF $F(x,y)$ of the form (3.39) on a distance invariant space X, the inequality of the mean $F(C) \ge f_0(Q)$ holds (for any $C \subseteq X$).

We shall call a compact metric space X *FDNDF-polynomial* if $Q_i(\sigma(d(x,y)))$ is an FDNDF on X for any $i = 0, 1, ..., s(X)$. For a FDNDF-polynomial space X (which, in general, is not distance invariant), let $F_i(x,y) = Q_i(\sigma(d(x,y)))$, $V_i = V(F_i)$, and $m_i = \dim V_i$, and let $F_i(x,y)$ have the following diagonal form (3.13):

$$F_i(x,y) = Q_i(\sigma(d(x,y))) = \sum_{j=1}^{m_i} \lambda_{i,j}\mathbf{e}_{i,j}(x)\overline{\mathbf{e}_{i,j}(y)}. \tag{3.43}$$

The functions $\mathbf{e}_{i,j}(x)$, $i = 0, 1, ..., s(X)$, $j = 1, ..., m_i$, form an orthonormal system with respect to the inner product (3.9), since by (2.39), (2.38) and (2.33),

$$\int_{-1}^{1} Q_i(t)Q_j(t)d\nu(t) = \sum_{k=1}^{m_i}\sum_{l=1}^{m_j} \lambda_{i,k}\lambda_{j,l}|\langle \mathbf{e}_{i,k}, \mathbf{e}_{j,l}\rangle|^2 = \frac{1}{r_i}\delta_{i,j}. \tag{3.44}$$

Therefore, for the function $F(x,y)$ defined by (3.39) and (3.41),

$$F\mathbf{e}_{i,j} = f_i(Q)\lambda_{i,j}\mathbf{e}_{i,j}. \tag{3.45}$$

Notice that $F_0(x,y) = J(x,y)$, $m_0 = 1$, $\mathbf{e}_{0,1}(x) = \mathbf{j}(x)$, $\lambda_{0,1}=1$ and hence by (3.45), $F\mathbf{j} = \int\limits_X F(x,y)d\mu(y)$ equals $f_0(Q)$ and does not depend on $x \in X$. This gives the following theorem.

Theorem 3.12. *Let X be an FDNDF-polynomial space with the standard substitution $\sigma(d)$. Then*

(i) the function $F(x,y)$ defined by (3.39) and (3.41) is an FDNDF on X if and only if $f_i(Q) \geq 0$, $i = 0, 1, ..., s(X)$,

(ii) for any FDNDF $F(x,y)$ defined by (3.39) and (3.41) the inequality on the mean holds, and

(iii) for any weighted set $C = (C, m)$ and any $i = 0, 1, ..., s(X)$,

$$B_i'(C) = \frac{r_i}{|C|} \sum_{x,y\in C} Q_i(\sigma(d(x,y)))m(x)m(y)$$

$$= \frac{r_i}{|C|} \sum_{j=1}^{m_i} \lambda_{i,j} |\sum_{x\in C} \mathbf{e}_{i,j}(x)m(x)|^2 \geq 0. \tag{3.46}$$

Property (i) implies the following:

Corollary 3.13. *The system Q of orthogonal polynomials for an FDNDF-polynomial space X satisfies the following condition (Krein condition): for any i, k and h ($0 \leq i, k, h \leq s(X)$) there exist nonnegative (and positive when $i + k = h$) numbers $q_{i,k}^h$ such that*

$$Q_i(t)Q_k(t) = \sum_{h=0}^{s(X)} q_{i,k}^h Q_h(t) \tag{3.47}$$

(for a finite X, (3.47) is considered mod $Q_{s(X)+1}(t)$*).*

For any weighted set $C = (C, m)$ and any polynomial (3.41) we have (2.64). Taking into account that $f(\sigma(d(x,y))) = f(1)$ if $d(x,y) = 0$ and $\sigma(d(x,y)) \leq \sigma(d)$ if $d(x,y) \geq d$, and that $B_0'(C) = |C|$, $B_i'(C) \geq 0$ for $i = 1, ..., s(X)$, we obtain the following refinements of Lemma 3.6 and Theorem 2.15, respectively.

Theorem 3.14. *For any code C with minimal distance $d(C) = d$ in an FDNDF-polynomial space X and any polynomial (3.41) such that $f_0(Q) > 0$, $f_i(Q) \geq 0$ for $i = 1, ..., s(X)$ and $f(t) \leq 0$ for $-1 \leq t \leq \sigma(d)$,*

$$|C| \leq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{3.48}$$

with equality if and only if the polynomial $f(t)$ is annihilating for C and

$$f_i(Q)B_i'(C) = 0 \text{ for } i = 1, ..., s(X). \tag{3.49}$$

Theorem 3.15. *For any weighted τ-design $C = (C, m)$ in an FDNDF-polynomial space X and any polynomial (3.41) such that $f_0(Q) > 0$, $f_i(Q) \leq 0$ for $i = \tau + 1, ..., s(X)$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$,*

$$|C| \geq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{3.50}$$

with equality if and only if C is a simple τ-design, the polynomial $f(t)$ is annihilating for C and

$$f_i(Q)B_i'(C) = 0 \text{ for } i = \tau + 1, ..., s(X). \tag{3.51}$$

Note that by (2.40) and (3.43) for any $x \in X$,

$$1 = Q_i(\sigma(d(x,x))) = \sum_{j=1}^{m_i} \lambda_{i,j} |\mathbf{e}_{i,j}(x)|^2 \tag{3.52}$$

and hence $\sum\limits_{j=1}^{m_i} \lambda_{i,j} = 1$. (We used $\langle \mathbf{e}_{i,j}, \mathbf{e}_{i,j} \rangle = 1$.) Furthermore, from (3.44) it follows that $r_i \sum\limits_{j=1}^{m_i} \lambda_{i,j}^2 = 1$. Since

$$1 = r_i \sum_{j=1}^{m_i} \lambda_{i,j}^2 = \left(\sum_{j=1}^{m_i} \lambda_{i,j} \right)^2 \leq m_i \sum_{j=1}^{m_i} \lambda_{i,j}^2,$$

then $r_i \leq m_i$ with equality if and only if all $\lambda_{i,j}$, $j = 1, ..., m_i$, are equal and, hence, equal to $\frac{1}{r_i}$.

We shall call an FDNDF-polynomial space X *polynomial* (cf. [82], [46]) if for any i, $i = 0, 1, .., s(X)$, all $\lambda_{i,j}$, $j = 1, ..., m_i$, are equal. Thus, for polynomial spaces the constants r_i in (2.39) are integers and equal to the dimensions of the spaces V_i, and (3.43) takes the form

$$Q_i(\sigma(d(x,y))) = \frac{1}{r_i} \sum_{j=1}^{r_i} \mathbf{e}_{i,j}(x) \overline{\mathbf{e}_{i,j}(y)}. \tag{3.53}$$

Furthermore by (3.52),

$$\sum_{j=1}^{r_i} |\mathbf{e}_{i,j}(x)|^2 = r_i \text{ for any } x \in X. \tag{3.54}$$

Fortunately, some metric spaces of significant interest in coding theory, in particular, the Hamming space, the Johnson space and the unit Euclidean sphere, turn out to be polynomial. For the Hamming space the representation (3.6) gives a direct proof of orthogonality of the polynomials (2.44) and the property of the space to be polynomial. However, in the general case a proof of this property is not a simple problem. The usual way to prove that a metric space $X = \{X, d(x, y)\}$ is polynomial is based on the representation of the isometry group of the space by shift operators and the proof that zonal spherical functions of the representation are polynomials in some (substitution) function of distance $d(x, y)$. We return to this problem at the end of the subsection.

Consider an arbitrary code C in a polynomial space X with the standard substitution $\sigma(d)$. Let $\Delta(C) = \{d_0 = 0, d_1, ..., d_s\}$, where $s = s(C)$ is the number of distances between distinct points of C and $0 < d_1 < \cdots < d_s$ (hence, $d(C) = d_1$). Then the minimal polynomial $f_C(t)$ can be represented as follows

$$f_C(t) = \prod_{i=1}^{s} \frac{t - \sigma(d_i)}{1 - \sigma(d_i)}. \tag{3.55}$$

By Corollary 2.14 and (3.53) a weighted set $C = (C, m)$ $(\sum_{x \in C} m(x) = |C|)$ in a polynomial space is a weighted τ-design if and only if for any $i = 1, ..., \tau$,

$$|C|B_i'(C) = r_i \sum_{x,y \in C} Q_i(\sigma(d(x,y)))m(x)m(y) = \sum_{j=1}^{r_i} |\sum_{x \in C} \mathbf{e}_{i,j}(x)m(x)|^2 = 0 \tag{3.56}$$

and hence if and only if

$$\sum_{x \in C} \mathbf{e}_{i,j}(x)m(x) = 0 \text{ for any } i \text{ and } j,\ i = 1, ..., \tau,\ j = 1, ..., r_i. \tag{3.57}$$

Therefore for polynomial spaces X the definition (2.12) may be formulated (see also (2.56)) in the following form: a weighted set $C = (C, m)$ is a weighted τ-design if for any $u(x) \in \bigcup_{i=0}^{\tau} V_i$,

$$\int_X u(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} u(x) m(x) \tag{3.58}$$

or, equivalently, if for any polynomial $f(t)$ in a real t of degree at most τ and for any $y, z \in X$,

$$\int_X f(\sigma(d(x,y))) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(\sigma(d(x,z))) m(x).$$

(We used that all $\mathbf{e}_{i,j}(x)$, $i \geq 1$, are orthogonal to $\mathbf{e}_{0,1}(x) \equiv 1$ with respect to (3.9).) There exists one more equivalent definition in term of orthogonality of the functions $\mathbf{e}_{i,j}(x)$ with weight $m(x)$ on the (finite) set C.

Theorem 3.16. *A weighted set $C = (C, m)$ is a weighted τ-design in a polynomial space X if and only if for any integers i, j, k, l, $0 \leq i + k \leq \tau$, $1 \leq j \leq r_i$, $1 \leq l \leq r_k$,*

$$\frac{1}{|C|} \sum_{x \in C} \mathbf{e}_{i,j}(x) \overline{\mathbf{e}_{k,l}(x)} m(x) = \delta_{i,k} \delta_{j,l}. \tag{3.59}$$

Proof. We use (3.54), (3.53), the Krein condition (3.47) and the equality $r_i q^0_{i,k} = \delta_{i,k}$ to prove the following equalities:

$$\frac{1}{r_i r_k}\sum_{j=1}^{r_i}\sum_{l=1}^{r_k}|\sum_{x\in C}\mathbf{e}_{i,j}(x)\overline{\mathbf{e}_{k,l}(x)}m(x)-|C|\delta_{i,k}\delta_{j,l}|^2$$

$$=\frac{1}{r_i r_k}\sum_{j=1}^{r_i}\sum_{l=1}^{r_k}|\sum_{x\in C}\mathbf{e}_{i,j}(x)\overline{\mathbf{e}_{k,l}(x)}m(x)|^2-\frac{|C|^2}{r_i}\delta_{i,k}$$

$$=\sum_{x,y\in C}\left(Q_i(\sigma(d(x,y)))Q_k(\sigma(d(x,y)))-q^0_{i,k}\right)m(x)\,m(y)$$

$$=\sum_{h=1}^{i+k}q^h_{i,k}\sum_{x,y\in C}Q_h(\sigma(d(x,y)))m(x)\,m(y)=|C|\sum_{h=1}^{i+k}\frac{q^h_{i,k}}{r_h}B'_h(C).$$

This completes the proof by (3.56). ■

Now we extend some results on codes in Q-polynomial association schemes of Delsarte [30] to the case of codes in polynomial spaces.

Theorem 3.17. *If for a code C in a polynomial space X, $f_C(t)=\sum_{i=0}^{s(C)} f_i Q_i(t)$ and $\tau(C)\geq s(C)$, then*

$$f_j|C|=r_j,\quad j=0,1,...,\tau(C)-s(C).$$

If $f_C(t)$ can be expanded over the system Q with positive coefficients and for some τ, $s(C)\leq\tau\leq s(X)$,

$$f_j|C|=r_j,\quad j=0,1,...,\tau-s(C),$$

then $\tau(C)\geq\tau$.

Proof. To prove the theorem it is sufficient to use the polynomials $f(t)=f_C(t)Q_j(t)$ in (2.64), note that $r_j\int_{-1}^{1} f_C(t)Q_j(t)d\nu(t)=f_j$, and consider conditions for which the equalities

$$r_j\sum_{x,y\in C}f_C(\sigma_Q(d(x,y)))Q_j(\sigma_Q(d(x,y)))=f_j|C|^2$$

are valid. ■

Theorem 3.18 (Absolute bound of Delsarte). *For any code C in a polynomial space X,*

$$|C|\leq\sum_{i=0}^{s(C)}r_i \tag{3.60}$$

with equality if and only if $f_C(t)=Q^{1,0}_s(t)$ and C is a tight $2s$-design where $s=s(C)$.

Proof. Let $f_C(t) = \sum\limits_{i=0}^{s} f_i Q_i(t)$. For any $y \in C$ consider the continuous function

$$\mathbf{u}_y(x) = f_C(\sigma(d(x,y))) = \sum_{i=0}^{s} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \overline{\mathbf{e}_{i,j}(y)} \mathbf{e}_{i,j}(x) \tag{3.61}$$

and note that

$$\mathbf{u}_y(x) = \begin{cases} 1 \text{ if } x = y, \\ 0 \text{ if } x \neq y, \ x \in C. \end{cases} \tag{3.62}$$

Since all these $|C|$ functions belong to $\bigcup\limits_{i=0}^{s} V_i$ and are linearly independent, we have (3.60). Equality holds if and only if for any $i = 0, 1, ..., s$, $j = 1, ..., r_i$ and any $y \in C$ there exist constants $c_{i,j}(y)$ such that

$$\mathbf{e}_{i,j}(x) = \sum_{y \in C} c_{i,j}(y) \mathbf{u}_y(x).$$

By (3.62), $c_{i,j}(y) = \mathbf{e}_{i,j}(y)$, and by (3.61) the last equalities are equivalent to the fact that

$$\frac{f_i}{r_i} \sum_{y \in C} \mathbf{e}_{i,j}(y) \overline{\mathbf{e}_{k,l}(y)} = \delta_{i,k} \delta_{j,l}.$$

This completes the proof by taking into account (3.54) and Theorem 3.16. ■

The idea of the proof of Theorem 3.18 was extended to prove many similar results (see the review paper [18] and [7]).

Theorem 3.19. *Any code C in a polynomial space X such that $\tau(C) \geq s(C) - 1$ is distance invariant.*

Proof. For any $x \in X$ and any polynomial $f(t) = \sum\limits_{i=0}^{h} f_i Q_i(t)$ of degree $h \leq \tau(C)$ by (2.8), (3.53), and (3.57) we have

$$\sum_{i=0}^{s(C)} B_{d_i}(x, C) f(\sigma(d_i)) = \sum_{y \in C} f(\sigma(d(x,y)))$$

$$= \sum_{i=0}^{h} \frac{f_i}{r_i} \sum_{l=1}^{r_i} \mathbf{e}_{i,l}(x) \sum_{y \in C} \overline{\mathbf{e}_{i,l}(y)} = f_0 |C|. \tag{3.63}$$

In particular, for any $j = 1, ..., s$, $s = s(C)$, we consider the Lagrange polynomial

$$f_C^{(j)}(t) = \frac{f_C(t)}{(t - \sigma(d_j)) f_C'(\sigma(d_j))} \tag{3.64}$$

of degree $h = s-1$ (here $f'_C(t)$ is the derivative of $f_C(t)$) satisfying the following property:

$$f_C^{(j)}(\sigma(d_i)) = \delta_{i,j}, \quad i = 1, ..., s,$$

and assume that

$$f_C^{(j)}(t) = \sum_{i=0}^{s-1} f_{j,i} Q_i(t). \tag{3.65}$$

Using (3.63) and the fact that $B_0(x, C) = 1$ for all $x \in C$, we obtain

$$B_{d_j}(x, C) = f_{j,0}|C| - f_C^{(j)}(1), \quad j = 1, ..., s, \tag{3.66}$$

and hence $B_{d_j}(x, C)$ does not depend on $x \in C$. ■

Corollary 3.20. *Any code C in a finite polynomial space X such that $\tau(C) = s(X)$ coincides with the whole space X.*

Proof. If we use the polynomial $f_X(t)$ with $s = s(X)$ (see (3.55)) in (3.63) where $h = s(X) = \tau(C)$ and also (2.42), (2.35), we obtain that for any $x \in X$

$$B_0(x, C) = |C| \sum_{i=0}^{s(X)} f_X(\sigma(d_i)) w_i = |C| w_0 = \frac{|C|}{|X|}$$

and hence $C = X$. ■

Theorem 3.21. *Any code C in a polynomial space X with the standard substitution function $\sigma(d)$ such that $\tau(C) \geq 2s(C) - 2$ forms a distance-regular polynomial space with the same substitution function. The intersection numbers $p_{i,j}^k$, $i, j, k \in \{0, 1, ..., s(C)\}$ (see (2.9)) can be defined by means of (3.64) and (3.65) as follows:*

$$p_{i,0}^k = p_{0,i}^k = \delta_{i,k}, \; p_{i,j}^0 = \delta_{i,j} \left(f_{j,0}|C| - f_C^{(j)}(1) \right) \text{ if } j \geq 1, \tag{3.67}$$

and for $i, j, k \in \{1, ..., s(C)\}$,

$$p_{i,j}^k = |C| \sum_{l=0}^{s-1} \frac{f_{i,l} f_{j,l}}{r_l} Q_l(\sigma(d_k)) - f_C^{(i)}(1)\delta_{j,k} - f_C^{(j)}(1)\delta_{i,k}. \tag{3.68}$$

Proof. By Theorem 3.19 $B_{d_j}(x, C)$ does not depend on $x \in C$ (see (3.66)) and we denote it by $B_{d_j}(C)$, $j = 0, 1, ..., s$, $s = s(C)$. To prove that C is a polynomial space we shall construct for any $i = 0, 1, ..., s$ polynomials $\overline{Q}_i(t)$ of degree i, $\overline{Q}_i(1) = 1$, and positive constants $\overline{r}_i$ such that

$$\frac{\overline{r}_i}{|C|} \sum_{j=0}^{s} \overline{Q}_i(\sigma(d_j)) \overline{Q}_k(\sigma(d_j)) B_{d_j}(C) = \delta_{i,k}, \tag{3.69}$$

and also construct an orthonormal basis of functions $\mathbf{v}_{i,j}(x)$ on C, $i = 0, 1, ..., s$, $j = 1, ..., \overline{r}_i$, with respect to the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{|C|} \sum_{x \in C} \mathbf{u}(x) \overline{\mathbf{v}(x)} \tag{3.70}$$

such that

$$\overline{Q}_i(\sigma(d(x,y))) = \frac{1}{\overline{r}_i} \sum_{j=1}^{\overline{r}_i} \mathbf{v}_{i,j}(x) \overline{\mathbf{v}_{i,j}(y)}. \tag{3.71}$$

For any $i = 0, 1, ..., s-1$ we put

$$\overline{r}_i = r_i, \ \overline{Q}_i(t) = Q_i(t), \ \mathbf{v}_{i,j}(x) = \mathbf{e}_{i,j}(x), \ j = 0, 1, ..., \overline{r}_i,$$

and let $\overline{r}_s = |C| - \sum_{i=0}^{s-1} r_i$ and

$$\overline{r}_s \overline{Q}_s(t) = |C| \prod_{j=1}^{s} \frac{t - \sigma(d_j)}{1 - \sigma(d_j)} - \sum_{i=0}^{s-1} \overline{r}_i \ \overline{Q}_i(t). \tag{3.72}$$

By Theorem 3.16 $\mathbf{v}_{i,j}(x)$, $i = 0, 1, ..., s-1$, $j = 0, 1, ..., \overline{r}_i$, are orthonormal with respect to (3.70) and hence $\overline{r}_s \geq 0$. In fact $\overline{r}_s > 0$ since otherwise by Theorem 2.16 C is a tight $(2s-2)$-design which by Lemma 2.18 cannot have s distinct nonzero distances. Using the orthogonalization process we can define the remaining functions $\mathbf{v}_{s,j}(x)$, $j = 0, 1, ..., \overline{r}_s$, so that all $|C|$ functions $\mathbf{v}_{i,j}(x)$, $i = 0, 1, ..., s$, $j = 0, 1, ..., \overline{r}_i$, have the property

$$\frac{1}{|C|} \sum_{x \in C} \mathbf{v}_{i,j}(x) \overline{\mathbf{v}_{k,l}(x)} = \delta_{i,k} \delta_{j,l}. \tag{3.73}$$

By the construction, (3.71) holds for $i = 0, 1, ..., s-1$. Since $\frac{1}{|C|} \sum\limits_{i=0}^{s} \sum_{j=1}^{\overline{r}_i} \mathbf{v}_{i,j}(x) \overline{\mathbf{v}_{i,j}(y)}$ is the identity matrix of order $|C|$ and, hence, coincides with $\prod\limits_{j=1}^{s} \frac{\sigma(d(x,y)) - \sigma(d_j)}{1 - \sigma(d_j)}$, (3.71) is also true for $i = s$ according to (3.72). The orthogonality conditions (3.69) follow from (3.71), (3.73) and the fact that C is distance invariant. Thus C forms a polynomial space with the substitution function $\sigma(d)$ which is standard if and only if C is diametrical. Now we prove that for any $x, y \in C$ and any $i, j \in \{0, 1, ..., s(C)\}$ the intersection numbers $p_{i,j}^{x,y}$ (see (2.9)) depend only on $d(x,y)$ and are defined by means of (3.67) and (3.68). By (3.66) the only case we need to consider is when $i, j \in \{1, ..., s(C)\}$ and $x \neq y$. In this case

$$p_{i,j}^{x,y} = \sum_{z \in C \setminus \{x,y\}} f_C^{(i)}(\sigma(d(x,z))) f_C^{(j)}(\sigma(d(y,z))).$$

On the other hand, if $d(x,y) = d_k$, then by (3.53), (3.65) and Theorem 3.16,

$$\sum_{z \in C} f_C^{(i)}(\sigma(d(x,z))) f_C^{(j)}(\sigma(d(y,z))) = \sum_{z \in C} \sum_{l=0}^{s-1} \sum_{m=0}^{s-1} f_{i,l} f_{j,m} Q_l(\sigma(d(x,z))) Q_m(\sigma(d(z,y)))$$

$$= \sum_{l=0}^{s-1} \sum_{m=0}^{s-1} \frac{f_{i,l} f_{j,m}}{r_l r_m} \sum_{a=1}^{r_i} \sum_{b=1}^{r_m} \mathbf{e}_{l,a}(x) \overline{\mathbf{e}_{m,b}(y)} \sum_{z \in C} \mathbf{e}_{m,b}(z) \overline{\mathbf{e}_{l,a}(z)}$$

$$= |C| \sum_{l=0}^{s-1} \sum_{m=0}^{s-1} \frac{f_{i,l} f_{j,m}}{r_l r_m} \sum_{a=1}^{r_i} \sum_{b=1}^{r_m} \mathbf{e}_{l,a}(x) \overline{\mathbf{e}_{m,b}(y)} \delta_{l,m} \delta_{a,b}$$

$$= |C| \sum_{l=0}^{s-1} \frac{f_{i,l} f_{j,l}}{r_l} Q_l(\sigma(d_k)).$$

This implies (3.68) and completes the proof. ■

The proof of Theorem 3.21 in fact uses the following equivalent definition of a polynomial space which are useful in proving spaces are polynomial. Suppose for a compact metric space $X = (X, d(x, y), \mu)$ with a standard substitution $\sigma(d)$ there exist finite-dimensional subspaces V_i, $i = 0, 1, ..., s(X)$, of continuous functions which are pairwise orthogonal with respect to inner product (3.9) and V_0 consists of constants. Then X is polynomial (with respect to $\sigma(d)$) if there exist polynomials $Q_i(t)$ of degree i $(i = 0, 1, ..., s(X))$ such that for any $x, y \in X$

$$Q_i(\sigma(d(x, y))) = \frac{1}{r_i} \sum_{j=1}^{r_i} \mathbf{e}_{i,j}(x) \overline{\mathbf{e}_{i,j}(y)}. \tag{3.74}$$

where $r_i = \dim V_i$ and $\{\mathbf{e}_{i,j}(x),\ j = 1, ..., r_i\}$ is an orthonormal basis of V_i (the right-hand side of (3.74) does not depend on a choice of the basis). It is easy to verify that the system of polynomials $\{Q_i(t), i = 0, 1, ..., s(X)\}$ satisfies the orthogonality and normalization conditions (2.39)-(2.40) and coincides with the system Q constructed in Section 2.2.

Consider for a finite or infinite connected compact metric space $X = (X, d(x, y))$ an *isometry* group G, i.e., a group of continuous one-to-one mappings X onto X such that $d(gx, gy) = d(x, y)$ for any $x, y \in X$ and $g \in G$. A space X is called *distance-transitive* (with respect to G) if $d(x_1, y_1) = d(x_2, y_2)$ implies the existence of an isometry $g \in G$ with $gx_1 = x_2$ and $gy_1 = y_2$. In particular (the case $x_1 = y_1$ and $x_2 = y_2$), this means that G acts transitively on X, i.e., for any $x, y \in X$ there exists $g \in G$ such that $gx = y$. Therefore, on a distance-transitive space X there exists a unique normalized invariant measure, the Haar measure, μ ($\mu(gA) = \mu(A)$ for any measurable $A \subseteq X$ and any $g \in G$; $\mu(X) = 1$) and X with this measure is distance invariant. In the case of a distance-transitive space X we consider the quasi-regular group G representation $L(g)$ in the Hilbert space $L_2(X, \mu)$ by translations, defined as follows

$$L(g)\mathbf{u}(x) = \mathbf{u}(g^{-1}x). \tag{3.75}$$

The representation is decomposable (see, for example, [120]) into an orthogonal direct sum of pairwise non-equivalent irreducible representations $L_i(g)$ acting on invariant finite dimensional subspaces V_i of continuous functions (including the space V_0 of constant functions). Since the subspaces V_i are invariant (with respect to the action of G) and X is distance-transitive, there exist real (so-called zonal spherical) functions $\Phi_i(d), i = 0, 1, \ldots$, such that for any $x, y \in X$

$$\Phi_i(d(x,y)) = \frac{1}{r_i} \sum_{j=1}^{r_i} \mathbf{e}_{i,j}(x)\overline{\mathbf{e}_{i,j}(y)}, \tag{3.76}$$

where $r_i = \dim V_i$ and $\{\mathbf{e}_{i,j}(x),\ j = 1, \ldots, r_i\}$ is an arbitrary orthonormal basis of V_i. If there exist an ordering of the spaces V_i, polynomials $Q_i(t)$ of degree i, and a (standard) substitution function $\sigma(d)$ such that $\Phi_i(d) = Q_i(\sigma(d))$ for all i and $d \in \Delta(X)$, then X is polynomial with respect to $\sigma(d)$ and $\{Q_i(t), i = 0, 1, \ldots, s(X)\}$ coincides with the system Q constructed in Section 2.2. More detailed description of this approach is contained in [30], [57], [108], [111], [10].

This approach was used to prove that some families of distance-transitive graphs are polynomial [30]-[32], [109], [110]. However, distance-transitive polynomial graphs have not yet been classified (the books [8], [23] contain detailed information about the problem). All infinite connected compact metric spaces which are distance-transitive with respect to their full isometry group (also called "two-point homogeneous") were classified by Wang [121]. Namely, they are the unit sphere S^{n-1} in R^n and the real $\mathbf{R}P^{n-1}$, the complex $\mathbf{C}P^{n-1}$, the quaternionic $\mathbf{H}P^{n-1}$ projective spaces (see Examples 2.3, 2.10, 2.4, 2.11) and Cayley's plane $\mathbf{O}P^2$. In each of these cases there is known [26], [95], [44], [42], [3], [59], [53] to exist an ordering of the spaces V_i and parameters α and β such that

$$\Phi_i(d) = Q_i(\sigma(d)) = P_i^{\alpha,\beta}(\sigma(d)),\ i = 0, 1, \ldots \tag{3.77}$$

where $P_i^{\alpha,\beta}(t)$ are Jacobi polynomials of degree i and $\sigma(d)$ is a continuous decreasing function with property (2.32). So they all are polynomial. The parameters α and β, and the standard substitution $\sigma(d)$ for S^{n-1} and the projective spaces are given in the examples mentioned above.

3.3. A σ-packing and τ-design problem for systems of orthogonal polynomials

Theorems 3.14 and 3.15 show that the problem of obtaining universal bounds for codes and designs in polynomial spaces, in the framework of the method considered, reduce to the following extremum problems for the corresponding systems Q of orthogonal polynomials satisfying the Krein condition.

The σ-packing problem: for any σ, $-1 \le \sigma < 1$, find the infimum of $\Omega_Q(f) = \frac{f(1)}{f_0(Q)}$ over the class of polynomials $f(t) = \sum_{i=0}^{s(X)} f_i(Q)Q_i(t)$ such that $f_0(Q) > 0$, $f_i(Q) \ge 0$ for $i = 1, \ldots, s(X)$ and

$$f(t) \le 0 \text{ for } -1 \le t \le \sigma. \tag{3.78}$$

The τ-design problem: for any integer τ, $\tau = 1, ..., s(X)$, find the supremum of $\Omega_Q(f) = \frac{f(1)}{f_0(Q)}$ over the class of polynomials $f(t) = \sum_{i=0}^{s(X)} f_i(Q)Q_i(t)$ such that $f_0(Q) > 0$, $f_i(Q) \leq 0$ for $i = \tau + 1, ..., s(X)$ and

$$f(t) \geq 0 \text{ for } -1 \leq t \leq 1. \tag{3.79}$$

In Section 2.3 we described the solution of the restricted (in the class of polynomials of degree at most τ) τ-design problem for any orthogonal systems Q. This solution gives rise to the universal bound (2.75) for (weighted) τ-designs in arbitrary compact metric spaces. Now we give a preliminary description of a solution of the restricted (in the class of polynomials of degree at most $h(\sigma)$ where the function $h(\sigma)$ will be defined below) σ-packing problem for orthogonal systems Q satisfying the Krein condition. This solution gives rise to universal bounds for codes in polynomial spaces. We also verify that the solution of the restricted τ-design problem can be obtained from the optimal polynomials for the restricted σ-packing problem for special values of σ.

We consider a finite or countable system $Q = \{Q_i(t), i = 0, 1, ..., s\}$ of orthogonal polynomials on $[-1, 1]$ with the normalization and orthogonality conditions (2.40), (2.42) or (2.43). We use some properties of adjacent systems $Q^{a,b}$ of orthogonal polynomials defined by (2.67) or (2.68), of the largest roots $t_i^{a,b}$ of polynomials $Q_i^{a,b}(t)$, $i = 0, 1, ..., s^{a,b}$, and of the function (kernel) $T_i^{a,b}(x, y)$ defined by (2.69) which will be proved in Section 5.

For any k, $1 \leq k < s$, the following inequalities hold

$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1} \quad \text{where } t_0^{1,1} = -1. \tag{3.80}$$

This means that the half-open interval $[-1, 1)$ in the case of a countable system Q or half-open interval[4] $[-1, t_{s-1}^{1,1})$ in the case of a finite Q is partitioned into half-open intervals $[t_{k-1}^{1,1}, t_k^{1,0})$ and $[t_k^{1,0}, t_k^{1,1})$, $k = 1, ..., s-1$. Enumerate in succession all these half-open intervals from the left to the right by positive integers. For any σ, $-1 \leq \sigma < t_{s-1}^{1,1}$ (we put $t_{s-1}^{1,1} = 1$ if $s = \infty$), denote by $h(\sigma)$ *the number of the (unique) half-open interval containing* σ. Let $k(\sigma) = k$ when $\sigma \in [t_{k-1}^{1,1}, t_k^{1,0})$ or $\sigma \in [t_k^{1,0}, t_k^{1,1})$, and let $\varepsilon(\sigma) = 0$ if $\sigma \in [t_{k-1}^{1,1}, t_k^{1,0})$ for some k and $\varepsilon(\sigma) = 1$ if $\sigma \in [t_k^{1,0}, t_k^{1,1})$ for some k. Then it is clear that

$$h(\sigma) = 2k(\sigma) - 1 + \varepsilon(\sigma). \tag{3.81}$$

For any σ, $-1 \leq \sigma < t_{s-1}^{1,1}$, consider the polynomial

$$f^{(\sigma)}(t) = (t - \sigma)(t + 1)^{\varepsilon} \left(T_{k-1}^{1,\varepsilon}(t, \sigma)\right)^2 \tag{3.82}$$

[4]For a finite metric space X with the standard substitution $\sigma(d)$ and $\Delta(X) = \{0, d_1, ..., d_s\}$ where $0 < d_1 < \cdots < d_s$ and $s = s(X)$, we have $A(X, d_1) = |X|$ and $t_{s-1}^{1,1} = \sigma(d_1)$. Since $\sigma(d)$ decreases with d it is sufficient to consider the σ-packing problem for the system Q for $-1 \leq \sigma < t_{s-1}^{1,1}$.

where $\varepsilon = \varepsilon(\sigma)$ and $k = k(\sigma)$. Note that the polynomial (3.82) has degree $h(\sigma)$ and satisfies property (3.78). We shall prove that $f^{(\sigma)}(1) > 0$ and $f_0^{(\sigma)}(Q) > 0$ and the function

$$L_Q(\sigma) = \Omega(f^{(\sigma)}) = \frac{f^{(\sigma)}(1)}{f_0^{(\sigma)}(Q)}$$

can be represented in the form

$$L_Q(\sigma) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\varepsilon} \left(1 - \frac{Q_{k-1}^{1,\varepsilon}(\sigma)}{Q_k^{0,\varepsilon}(\sigma)}\right) \sum_{i=0}^{k-1} r_i^{0,\varepsilon}. \tag{3.83}$$

The function $L_Q(\sigma)$ grows with σ and takes the following values at the left ends of these half-open intervals

$$L_Q(t_{l-\theta}^{1,\theta}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r_i^{0,\theta} \tag{3.84}$$

where l is an integer and $\theta \in \{0, 1\}$. In particular,

$$L_Q(-1) = 1 - \frac{1}{Q_1(-1)}. \tag{3.85}$$

For $\sigma = t_k^{1,0}$ we have $k(\sigma) = k$, $\varepsilon(\sigma) = 1$ and the polynomial $f^{(\sigma)}(t)$ has factor $t + 1$. For $\sigma = t_{k-1}^{1,1}$ we have $k(\sigma) = k$, $\varepsilon(\sigma) = 0$; however, we shall see that the polynomial $f^{(\sigma)}(t)$ is also divisible by $t+1$. In the both cases the polynomial $f(t) = f^{(\sigma)}(t)/(t+1)$ satisfies property (3.78) and $\Omega(f) = \Omega(f^{(\sigma)}) = L_Q(\sigma)$.

We shall see that the polynomials $f^{(\sigma)}(t)$ have the following extremum property: for any σ, $-1 \le \sigma < t_{s-1}^{1,1}$, and any polynomial $f(t)$ of degree at most $h(\sigma)$ such that $f_0(Q) > 0$ and (3.78) holds,

$$\Omega(f) = \frac{f(1)}{f_0(Q)} \ge L_Q(\sigma) \tag{3.86}$$

with equality if and only if $f(t)$ is proportional to $f^{(\sigma)}(t)$ or $f^{(\sigma)}(t)/(t + 1)$ in the cases when $\sigma = t_l^{1,\theta}$ for some integer l and $\theta \in \{0, 1\}$.

In order to prove that $f^{(\sigma)}(t)$ is the optimal solution of the σ-packing problem in the class of polynomials of degree at most $h(\sigma)$ we should show that all coefficients of $f^{(\sigma)}(t)$ are nonnegative. Analogously to (2.54) for any polynomial $f(t)$ we define coefficients

$$f_i(Q^{a,b}) = r_i^{a,b} \int_{-1}^{1} f(t) Q_i^{a,b}(t) d\nu^{a,b}(t), \ i = 0, 1, ..., s^{a,b}. \tag{3.87}$$

Denote by $F_{\ge}(Q^{a,b})$ (respectively, $F_{>}(Q^{a,b})$) the set of polynomials $f(t) \in F[t]$ such that

$$f_i(Q^{a,b}) \ge 0, \ i = 0, 1, ..., s^{a,b}$$

(respectively, such that

$$f_i(Q^{a,b}) > 0,\ i = 0, 1, ..., h, \text{ and } f_i(Q^{a,b}) = 0,\ i = h + 1, ..., s^{a,b}$$

for some h, $0 \le h \le s^{a,b}$). Then the fact that the system $Q = \{Q_i(t),\, i = 0, 1, ..., s\}$ satisfies the Krein condition can be written as follows

$$Q_i(t)Q_j(t) \in F_{\geq}(Q) \text{ for any } i, j,\ 0 \le i, j \le s. \tag{3.88}$$

Note that

$$t + 1 \in F_{>}(Q) \tag{3.89}$$

since $Q_0(t) \equiv 1$ and $t + 1 = 2\frac{Q_1(t) - Q_1(-1)}{1 - Q_1(-1)}$, and that

$$f(t)g(t) \in F_{>}(Q) \text{ if } f(t) \in F_{>}(Q) \text{ and } g(t) \in F_{>}(Q). \tag{3.90}$$

Moreover, from (2.69)-(2.70) and the Christoffel-Darboux formula (see Theorem 5.4) it follows that

$$Q_i^{1,0}(t) \in F_{>}(Q), \quad (t + 1)Q_i^{0,1}(t) \in F_{\geq}(Q). \tag{3.91}$$

We say that the system $Q = \{Q_i(t),\ i = 0, 1, ..., s\}$ satisfies the *strengthened Krein condition* if together with (3.88) the following holds

$$(t + 1)Q_i^{1,1}(t)Q_j^{1,1}(t) \in F_{>}(Q) \text{ for any } i, j,\ 0 \le i, j \le s^{1,1}. \tag{3.92}$$

Lemma 3.22. *A system $Q = \{Q_i(t),\ i = 0, 1, ...\}$ for any compact metric space satisfies the strengthened Krein condition if for some α and β such that $\alpha \ge \beta \ge -\frac{1}{2}$,*

$$Q_i(t) = P_i^{\alpha,\beta}(t),\ i = 0, 1, ... ,$$

where $P_i^{\alpha,\beta}(t)$ are Jacobi polynomials normalized by $P_i^{\alpha,\beta}(1) = 1$ (see (2.48)).

Proof. By the Gasper Theorem [43] for $\alpha \ge \beta$, $\alpha + \beta + 1 \ge 0$, the system $\{P_i^{\alpha,\beta}(t),$ $i = 0, 1, ...\}$ satisfies the Krein condition. Since $Q_i^{1,1}(t) = P_i^{\alpha+1,\beta+1}(t)$, it is true for the system $\{Q_i^{1,1}(t), i = 0, 1, ...\}$. We shall see in Section 5 (Corollary 5.25) that there exist positive constants l_i and m_i such that

$$Q_i^{1,1}(t) = l_i Q_i^{0,1}(t) + m_i Q_i^{1,0}(t). \tag{3.93}$$

This completes the proof using (3.89)-(3.91). ■

Lemma 3.22 shows that the systems Q for infinite distance-transitive polynomial spaces (in particular, for the Euclidean sphere) satisfy the strengthened Krein condition.

A finite polynomial space $X = (X, d(x, y))$ with a standard substitution $\sigma(d)$ is called *decomposable* [68] if for some h there exist metric subspaces $X_i = (X_i, d(x, y))$, $i = 1, ..., h$, of X such that:

(i)
$$X = \bigcup_{i=1}^{h} X_i,$$
(ii) all subspaces X_i are isometric to a single metric space $\widetilde{X} = (\widetilde{X}, d(x,y))$ which is polynomial with the same $\sigma(d)$,

(iii) for any $x, y \in X$ the number of subspaces X_i containing both x and y is equal to

$$\frac{\sigma(d(x,y)) + 1}{2} h \frac{|\widetilde{X}|}{|X|}. \tag{3.94}$$

Let X be a decomposable space and $X_1, ..., X_h$ be the subspaces mentioned in the definition. For any $C \subseteq X$ we say that $C \cap X_j$ is the *projection* of C onto X_j and, in particular, X_j is the projection of X (onto X_j). Notice that the space $\widetilde{X}$ (and any X_j) is polynomial with respect to $\sigma(d)$, which is not standard for $\widetilde{X}$, since from (2.32) and (3.94) it follows that
$$D(\widetilde{X}) = D(X) - 1.$$
(We shall not standardize the substitution function in $\widetilde{X}$ in order to have a simple formulation of Theorem 3.24.) The parameters of the space $\widetilde{X}$, which are analogous to v_i, r_i, and $Q_i(t)$, $i = 0, 1, ..., D$, of the space X, are denoted by $\widetilde{v}_i$, $\widetilde{r}_i$, and $\widetilde{Q}_i(t)$, $i = 0, 1, ..., D-1$, respectively.

Example 3.23. *For the Hamming space H_v^n with standard substitution $\sigma(d) = 1 - \frac{2d}{n}$ consider $h = nv$ metric subspaces obtained from H_v^n by fixing any value $0, 1, ..., v-1$ in any of the n coordinates. Each of these subspaces is isometric to H_v^{n-1} and hence is polynomial with the same $\sigma(d)$. Any points $x, y \in H_v^n$ such that $d(x,y) = i$ belong to*
$$\frac{\sigma(i) + 1}{2} nv \frac{|H_v^{n-1}|}{|H_v^n|} = n - i$$
subspaces. Hence, the Hamming space H_v^n is decomposable. For the Johnson space J_w^n with standard substitution $\sigma(d) = 1 - \frac{2d}{w}$ consider $h = n$ metric subspaces of J_w^n consisting of w-sets which contain a fixed element of $\{1, ..., n\}$. Each of these subspaces is isometric to J_{w-1}^{n-1} and hence is polynomial with the same $\sigma(d)$. Any points $x, y \in J_w^n$ such that $d(x,y) = i$ belong to

$$\frac{\sigma(i) + 1}{2} n \frac{|J_{w-1}^{n-1}|}{|J_w^n|} = w - i$$

subspaces. Hence, the Johnson space J_v^n is also decomposable.

Theorem 3.24 ([70]). *Let X be a decomposable polynomial space of diameter D. Then $|X| = L_Q(-1)|\widetilde{X}|$ and for $i = 0, 1, ..., D-1$,*

$$\widetilde{v}_i = \frac{\sigma(i) + 1}{2} v_i,$$

$$\widetilde{r}_i = r_i^{0,1}, \quad \widetilde{Q}_i(t) = Q_i^{0,1}(t).$$

Thus for a decomposable polynomial space the system $Q^{0,1}$ satisfies the Krein condition as well. Since $Q_i^{1,1}(t) \in F_>(Q^{0,1})$ (see, for example, (2.73) for $\theta = 1, i = l-1$), this gives rise to the following statement [70].

Lemma 3.25. *The system Q for any decomposable polynomial space satisfies the strengthened Krein condition.*

Thus the systems Q for polynomial spaces which are infinite and distance-transitive or finite and decomposable satisfy the strengthened Krein condition. In Section 5.4 we prove that if for a polynomial space the system Q satisfies the strengthened Krein condition, then for any σ, $-1 \le \sigma < t_{s-1}^{1,1}$, all coefficients of the polynomial $f^{(\sigma)}(t)$ of degree $h(\sigma) = 2k(\sigma) - 1 + \varepsilon(\sigma)$ over Q are positive (i.e., $f^{(\sigma)}(t) \in F_>(Q)$).

Theorem 3.14 and the results mentioned above allow us to obtain the following main bound for any code C in a polynomial space X with a standard substitution $\sigma(d)$ and with the system Q defined by Theorem 2.7 which satisfies the strengthened Krein condition:

$$|C| \le L_Q(\sigma(d(C))). \tag{3.95}$$

We shall also find necessary and sufficient conditions for equality in (3.95) in terms of the values $\tau(C)$ and $f_C(t)$.

The author assumed that $f^{(\sigma)}(t) \in F_>(Q)$ for any system Q corresponding to a polynomial metric space (and, hence, (3.95) is true without the restriction on the strengthened Krein condition) but could not prove this assumption. In the general case it is not known whether $f^{(\sigma)}(t) \in F_>(Q)$ when σ lies inside the half-open interval with even number $h(\sigma) = 2k(\sigma) \ge 4$ (and hence $\varepsilon(\sigma) = 1$). In this connection for any σ, $-1 \le \sigma < t_{s-1}^{1,0}$, we also consider the polynomials

$$\widetilde{f}^{(\sigma)}(t) = (t - \sigma)\left(T_{k-1}^{1,0}(t, \sigma)\right)^2 \tag{3.96}$$

of odd degree $2k - 1$, where k is uniquely determined by $t_{k-1}^{1,0} \le \sigma < t_k^{1,0}$, and the function

$$\widetilde{L}_Q(\sigma) = \Omega(\widetilde{f}^{(\sigma)}) = \left(1 - \frac{Q_{k-1}^{1,0}(\sigma)}{Q_k(\sigma)}\right) \sum_{i=0}^{k-1} r_i. \tag{3.97}$$

It can be shown that $\widetilde{f}^{(\sigma)}(t) \in F_>(Q)$ for any system Q satisfying the Krein condition, and that $\widetilde{L}_Q(\sigma)$ is a continuous (and even differentiable [67]) function of σ. This gives that for any code C in a polynomial space,

$$|C| \le \left(1 - \frac{Q_{k-1}^{1,0}(\sigma)}{Q_k(\sigma)}\right) \sum_{i=0}^{k-1} r_i. \tag{3.98}$$

where $\sigma = \sigma(d(C))$ and $t_{k-1}^{1,0} \le \sigma < t_k^{1,0}$.

The polynomials

$$h^{(\sigma)}(t) = (t-\sigma)\left(T_{k-1}^{0,0}(t,\sigma)\right)^2 \text{ where } t_{k-1}^{0,0} < \sigma < t_k^{0,0} \tag{3.99}$$

of odd degree $2k-1$ were used before in [79] for the Hamming and Johnson spaces and in [57] for the Euclidean sphere and other distance-transitive infinite compact spaces. They have the property $h^{(\sigma)}(t) \in F_>(Q)$ and give rise to the universal bound

$$|C| \leq \left(\frac{1}{Q_{k-1}(\sigma)} - \frac{1}{Q_k(\sigma)}\right)\sum_{i=0}^{k-1} r_i Q_i(\sigma), \tag{3.100}$$

where $\sigma = \sigma(d(C))$. For $\sigma > t_1^{1,0}$ this bound is worse compared to (3.95) and (3.98) and cannot be attained (for $-1 \leq \sigma \leq t_1^{1,0}$ these bounds are obtained by using the polynomial $t-\sigma$ and coincide). However, in a certain asymptotic process these three bounds give rise to the same asymptotic result.

Note the significant special cases of the universal bounds (3.95) and (3.98). If $\sigma = t_{l-\theta}^{1,\theta}$ then $k(\sigma) = l$, $\varepsilon(\sigma) = 1-\theta$, and $h(\sigma) = 2l-\theta$. Using (3.84), the notation $d_i^{a,b} = \sigma^{-1}(t_i^{a,b})$ (see (2.71)), and monotonicity of the functions (3.83) and (3.97), we have that for any code C in a polynomial space,

$$|C| \leq L_Q(t_{l-\theta}^{1,\theta}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r_i^{0,\theta} \quad \text{if} \quad d(C) \geq d_{l-\theta}^{1,\theta}. \tag{3.101}$$

Consider now the polynomials

$$f(t) = \frac{t-\sigma}{t+1} f^{(\sigma)}(t) \text{ where } \sigma = t_{l-\theta}^{1,\theta} \tag{3.102}$$

of degree $h(\sigma) = 2l-\theta$ which by the construction satisfy (3.79). Moreover, we shall see that the polynomials (3.102) up to a constant multiple are equal to polynomials $g^{(2l-\theta)}(t)$ defined by (2.72) and have the property

$$\Omega(f) = \Omega(f^{(\sigma)}) = L_Q(t_{l-\theta}^{1,\theta}).$$

Thus the polynomial (3.102) is a solution of the restricted $(2l-\theta)$-design problem (cf. Theorem 2.16). We shall verify in Section 5 that the optimality of (3.102) and the optimality of $f^{(\sigma)}(t)$ for the restricted $(2l-\theta)$-design and σ-packing problems, respectively, are consequences of the same result in the theory of orthogonal polynomials.

For a polynomial metric space X we define one more function $\gamma(d)$ of an integer d as follows:

$$\gamma(d) = t_{l-\theta}^{1,\theta} \quad \text{if} \quad d = 2l+1-\theta \tag{3.103}$$

where l is an integer and $\theta \in \{0,1\}$. Since $t_{l-\theta}^{1,\theta}$ is the left end of the half-open interval numbered by $d-1 = 2l-\theta$, from (3.80) it follows that $\gamma(d)$ increases with d. Using the function $\gamma(d)$ we can express the bound (2.75) of Theorem 2.16 as follows

$$|C| \geq L_Q(\gamma(d)) \quad \text{if} \quad d'(C) \geq d. \tag{3.104}$$

Furthermore, we can rewrite (3.101) in the form

$$|C| \leq L_Q(\gamma(d)) \quad \text{if} \quad \sigma(d(C)) \leq \gamma(d).$$

and formulate the following known (see, for example [82]) conjecture.

Conjecture 3.26. *For any code C in a polynomial space X,*

$$L_Q(\gamma(d'(C))) \leq |C| \leq L_Q(\gamma(2s(C) - \beta(C) + 1)) \tag{3.105}$$

with equality in either of the bounds if and only if

$$d'(C) = 2s(C) - \beta(C) + 1$$

and

$$f_C(t) = \left(\frac{t+1}{2}\right)^{\beta(C)} Q^{1,\beta(C)}_{s(C)-\beta(C)}(t).$$

This statement is true with respect to the lower bound (see Theorem 2.16 and Lemma 2.18) and with respect to the upper bound when $\beta(C) = 0$ (see Theorem 3.18). Thus one has only to prove the upper bound in (3.105) for diametrical codes ($\beta(C) = 1$) and the fact that the stated conditions of its attainability are necessary in this case (the sufficiency of the conditions follows from Lemma 2.18).

Theorem 3.27. *Conjecture 3.26 is true for all decomposable polynomial spaces.*

Proof. As was noted before, we need to prove only the upper bound (3.105) for diametrical codes and the necessary conditions of its attainability. Let C be a diametrical code in a decomposable space X and let $s(C) = s$. Since the function $\sigma(d)$ is standard, $f_C(t) = \frac{1+t}{2}\, g(t)$, where $g(t)$ is a polynomial of degree $s-1$ such that $g(1) = 1$. From (3.94) and Theorem 3.24 it follows that every point $x \in X$ belongs to $h/L_Q(-1)$ projections among h ones $X_1, ..., X_h$ of the space X. Hence

$$h|C| = L_Q(-1) \sum_{j=1}^{h} |C_j|, \tag{3.106}$$

where $C_j = C \cap X_j$ is the projection of C onto X_j, $j = 1, ..., h$. Since $s(C_j) \leq s(C) - 1$, using Theorem 3.24 and the absolute bound (3.60) for the space X_j, we have

$$|C_j| \leq \sum_{i=0}^{s-1} r_i^{0,1}. \tag{3.107}$$

Moreover, if the bound (3.107) is attained then $g(t) = \widetilde{Q}^{1,0}_{s-1}(t)$ and hence (see (2.70)) $g(t) = Q^{1,1}_{s-1}(t)$. From (3.106) and (3.107) it follows that

$$|C| \leq L_Q(-1) \sum_{i=0}^{s-1} r_i^{0,1} \tag{3.108}$$

and if the bound (3.108) is attained, then for any projection C_j the bound (3.107) is attained and $f_C(t) = \frac{1+t}{2} Q_{s-1}^{1,1}(t)$. Note that (3.108) coincides with the upper bound in (3.105) for the case considered. When the bound (3.108) is attained and $f_C(t) = \frac{1+t}{2} Q_{s-1}^{1,1}(t)$, we consider the polynomial $f(t) = \frac{1+t}{2}\left(Q_{s-1}^{1,1}(t)\right)^2$. By Lemma 3.25 all its coefficients $f_i = r_i \sum_{j=0}^{s(X)} f(\sigma(d_j))Q_i(\sigma(d_j))w_j$, $i = 0, 1, ..., 2s-1$, are positive. Since $f(1) = 1$ and $f_0 = |C|^{-1}$ (see (2.72) and (2.75)), from (2.64) and (2.60) it follows that

$$\sum_{i=1}^{2s-1} \frac{f_i}{r_i} B_i'(C) = 0.$$

This means that $d'(C) \geq 2s(C)$ and hence $d'(C) = 2s(C)$ by Lemma 2.17. ■

This conjecture was also proved for infinite distance-transitive compact metric spaces [35], [34], [54], [53] (in particular, for the Euclidean sphere). The question of whether this conjecture is true for all polynomial spaces is open now (1997). It seems believable because Lemma 2.18 would be a consequence of the statement.

4. Duality in bounding optimal sizes of codes and designs in polynomial graphs

4.1. Polynomial graphs

In this section we study codes and designs in finite polynomial metric spaces which are P- and Q-polynomial association schemes [30], [57], [8], [23]. As we can see later it is convenient to consider such association schemes as graphs with the path metric and call them polynomial graphs. In this connection we use a definition for association schemes which is close to the definition of metric spaces.

A (symmetric) *association scheme* (with D *classes*) $\{X, d(x, y)\}$ is a finite set X with a given function $d(x, y)$ which is defined for any $x, y \in X$, takes values $0, 1, \ldots, D$ and has the following properties :

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for any $x, y \in X$;
3. for any $x, y \in X$ and any $i, j \in \{0, 1, \ldots, D\}$, the number of points z such that $d(x, z) = i, d(z, y) = j$ depends on $d(x, y)$ only (this number is called an *intersection number* and denoted by $p_{i,j}^k$, where $k = d(x, y)$).

Note that any distance-regular code C, in a (finite or infinite) metric space Z with a metric $d_Z(x, y)$ such that $\Delta(C) = \{d_0 = 0, d_1, ..., d_s\}$, is an association scheme with $D = s$ classes if we put $d(x, y) = i$ when $d_Z(x, y) = d_i$. In particular, by Theorem 3.21 all codes C in a polynomial space such that $\tau(C) \geq 2s(C) - 2$ are association schemes and their intersection numbers $p_{i,j}^k$ can be calculated by (3.67) and (3.68).

For association schemes, $p^k_{i,j} = p^k_{j,i}$ and $p^0_{i,i}$ is the number of points $y \in X$ such that $d(x, y) = i$ for a fixed element $x \in X$. The numbers $p^0_{i,i}$ are denoted by v_i and called *valencies.* In general the function $d(x, y)$ does not satisfy the triangle inequality $d(x, y) \le d(x, z) + d(z, y)$. However, for many significant examples of association schemes the function $d(x, y)$ has this property and is a metric. In particular, the Hamming space H^n_v (see Example 2.1), with the metric $d(x, y)$ being equal to the number of places where x and y differ, forms an association scheme with $D = n$ classes. As another example of an association scheme with $D = w$ consider the Johnson space J^n_w, $w \le n/2$ (see Example 2.2) consisting of $\binom{n}{w}$ w-subsets of an n-set with the metric $d(x, y) = w - |x \cap y|$.

Using the adjacency matrices $A_i, i = 0, 1, \ldots, D$, of order $|X|$ defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x, y) = i, \\ 0 & \text{otherwise}, \end{cases} \tag{4.1}$$

the definition of an association scheme can be expressed by

$$A_0 = I, \quad \sum_{i=0}^{D} A_i = J, \quad A_i = A_i^T, \quad A_i A_j = \sum_{k=0}^{D} p^k_{i,j} A_k, \tag{4.2}$$

where I is the identity matrix, J is the matrix with entries all equal to one and A^T is the transpose of A.

The matrices A_i are linearly independent and generate a $(D+1)$-dimensional (over $\mathbb{R}$) commutative algebra $\mathcal{A}$ of symmetric matrices, which is called the Bose-Mesner algebra. We consider the $|X|$-dimensional vector space $V = \{\mathbf{u}(x) : X \to \mathbb{R}\}$ of real functions on X with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{|X|} \sum_{x \in X} \mathbf{u}(x)\mathbf{v}(x).$$

It is known [30], [8], [23] that for an association scheme with D classes (as for the $(D + 1)$-dimensional commutative algebra of symmetric matrices) there exists a decomposition

$$V = V_0 + \ldots + V_D$$

of V into a direct sum of pairwise orthogonal subspaces V_i, $i = 0, 1, \ldots, D$, where each V_i is a maximal common eigenspace of $A_0, A_1, \ldots, A_D$. We can assume that V_0 consists of constants only since the eigenvector of all ones can belong only to a one-dimensional common eigenspace.

Let $r_i = \dim V_i$, $i = 0, 1, \ldots, D$, $r_0 = 1$, and $\{\mathbf{v}_{i,j}(x), j = 1, \ldots, r_i\}$ be any orthonormal basis of V_i. The matrices

$$E_i(x, y) = \frac{1}{|X|} \sum_{j=1}^{r_i} \mathbf{v}_{i,j}(x)\mathbf{v}_{i,j}(y), \quad i = 0, 1, \ldots, D, \tag{4.3}$$

do not depend on the choice of the bases of V_i, possess the properties

$$E_iE_j = E_i\delta_{i,j}, \quad E_0 = \frac{1}{|X|}J, \quad \sum_{i=0}^{D} E_i = I, \tag{4.4}$$

and hence form the basis of irreducible idempotents of $\mathcal{A}$. It follows that there exist two non-degenerate matrices $P = (P_{i,j})$ and $Q = (Q_{i,j})$ of order $D+1$ such that

$$A_j = \sum_{i=0}^{D} P_{i,j}E_i, \; j = 0, 1, \ldots, D, \tag{4.5}$$

$$E_j = \frac{1}{|X|}\sum_{i=0}^{D} Q_{i,j}A_i, \; j = 0, 1, \ldots, D. \tag{4.6}$$

The equations (4.3)-(4.5) show that the column space of E_i is an eigenspace of each A_j, and the corresponding eigenvalue $P_{i,j}$ has the *multiplicity* $r_i = \text{rank } E_i = \; tr\ E_i$. In particular, by (4.1), (4.4)-(4.5) $P_{0,i}$ is equal to the valency $v_i = p^0_{i,i}$, and by (4.3) and (4.6) $Q_{0,i} = r_i$. From (4.5) and (4.6) it follows that

$$PQ = QP = \frac{1}{|X|}I \tag{4.7}$$

where I here is the identity matrix of order $D+1$. Considering $tr(A_jE_i)$ and using $tr(A_iA_j) = v_i|X|\delta_{i,j}$, by (4.1) and (4.2), we also get that

$$P_{i,j}r_i = Q_{j,i}v_j. \tag{4.8}$$

It is clear that an association scheme is a metric space with distance $d(x,y)$ when the triangle inequality holds for this function. Delsarte proved [30] that it holds if there exists polynomials $p_j(\sigma)$ of degree $j, j = 1, \ldots, D$, of a real variable σ such that $A_j = p_j(A_1)$ or, in other words (see (4.5)),

$$P_{i,j} = p_j(P_{i,1}), \quad i = 0, 1, \ldots, D \tag{4.9}$$

Such an association scheme is called *P-polynomial* or *metric.*

There is another description of metric association schemes in terms of graphs. The vertex set X of any undirected graph Γ can be considered as a metric space with the *path metric* $d_\Gamma(x,y)$ equal to the number of edges in the shortest path from x to y. An undirected connected graph Γ with the vertex set X is called *distance-regular* if for any $x, y \in X$ the number of vertices z such that $d_\Gamma(x,z) = 1, d_\Gamma(y,z) = d_\Gamma(x,y) - 1$ and the number of vertices z such that $d_\Gamma(x,z) = 1, d_\Gamma(y,z) = d_\Gamma(x,y) + 1$ depend on $d_\Gamma(x,y)$ only[5]. Delsarte [30] proved that for any distance-regular graph

[5]It is true that a distance-regular graph Γ is a distance-regular metric space with the path metric $d_\Gamma(x,y)$ and its intersection numbers $p^k_{i,j}$ are uniquely defined by $p^k_{1,k-1}$ and $p^k_{1,k+1}$, $i,j,k \in \{0,1,...,D\}$ (see [30], [8] or [23]).

Γ with vertex set X, $\{X, d_\Gamma(x,y)\}$ is a metric association scheme, and for any metric association scheme $\{X, d(x,y)\}$, the graph Γ with the vertex set X and the adjacency matrix A_1 is a distance-regular graph, and $d_\Gamma(x,y) = d(x,y)$. Thus, there is one-to-one correspondence between metric association schemes with D classes and distance-regular graphs of diameter D.

An association scheme $\{X, d(x,y)\}$ with D classes is called *Q-polynomial*, or *cometric*, if there exist polynomials $q_j(\sigma)$ of degree $j, j = 0, 1, \ldots, D$, of a real variable σ such that

$$Q_{i,j} = q_j(Q_{i,1}), \quad i = 0, 1, \ldots, D. \tag{4.10}$$

Hereafter we consider P- and Q-polynomial association schemes (or Q-polynomial distance-regular graphs [23]) which are referred to as *polynomial graphs*. Note that $Q_{i,1}$ (and $P_{i,1}$) are different for different i, $i = 0, 1, \ldots, D$, since otherwise the matrices (4.6) (respectively (4.5)) would be linearly dependent. We introduce an additional restriction that $Q_{i,1}$ and $P_{i,1}$ *decrease with* i, which is fulfilled for many (but not all) polynomial graphs. Let $\sigma_Q(d)$ and $\sigma_P(d)$ be continuous decreasing functions in a real variable d (the *substitutions*) such that for any $d = 0, 1, \ldots, D$,

$$\sigma_Q(d) = 1 - 2\frac{r_1 - Q_{d,1}}{r_1 - Q_{D,1}}, \quad \sigma_P(d) = 1 - 2\frac{v_1 - P_{d,1}}{v_1 - P_{D,1}}$$

By the construction and the assumption

$$\sigma_Q(D) = -1 \leq \sigma_Q(d) \leq \sigma_Q(0) = 1, \ \sigma_P(D) = -1 \leq \sigma_P(d) \leq \sigma_P(0) = 1. \tag{4.11}$$

The following examples are well known [29]-[32].

Example 4.1. *For the Hamming space H_v^n :*

$$v_i = r_i = \binom{n}{i}(v-1)^i, \quad i = 0, 1, \ldots, n,$$

$$Q_{i,k} = P_{i,k} = K_k^{n,v}(i), \quad i, k = 0, 1, \ldots, n,$$

where $K_k^{n,v}(z)$ is the Krawtchouk polynomial of degree k, defined by (2.44) and, in particular,

$$Q_{i,1} = P_{i,1} = (v-1)n - vi.$$

Thus H_v^n is a polynomial graph with

$$\sigma_Q(d) = \sigma_P(d) = 1 - 2\frac{d}{n}.$$

Example 4.2. *For the Johnson space J_w^n with $1 \leq w \leq n/2$:*

$$v_i = \tbinom{n}{i}\tbinom{n-w}{i}, \quad i = 0, 1, \ldots, w$$

$$r_i = \tbinom{n}{i} - \tbinom{n}{i-1}, \quad i = 1, \ldots, w \quad (r_0 = 1),$$

$$Q_{i,k} = r_k \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{n+1-k}{j}}{\binom{w}{j}\binom{n-w}{j}} \binom{i}{j},$$

$$P_{i,k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{w-j}{k-j} \binom{w-i}{j} \binom{n-w+j-i}{j}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{i}{j} \binom{w-i}{k-j} \binom{n-w-i}{k-j},$$

and, in particular,

$$Q_{i,1} = n\left(1 - \frac{ni}{w(n-w)}\right),$$

$$P_{i,1} = w(n-w) - i(n+1-i).$$

Thus J_w^n is a polynomial graph with

$$\sigma_Q(d) = 1 - 2\frac{d}{w}, \quad \sigma_P(d) = 1 - 2\frac{d(n+1-d)}{w(n+1-w)}.$$

Now we introduce systems $\{Q_i(t)\}$ and $\{P_i(t)\}$ of polynomials in a real t, $0 \leq t \leq 1$, which are obtained from polynomials (4.10) and (4.9) by change of variables as follows:

$$r_j Q_j(t) = q_j(\frac{t(r_1 - Q_{D,1}) + r_1 + Q_{D,1}}{2}),$$

$$v_j P_j(t) = p_j(\frac{t(v_1 - P_{D,1}) + v_1 + P_{D,1}}{2}),$$

and hence for any $i, j = 0, 1, \ldots, D$,

$$r_j Q_j(\sigma_Q(i)) = Q_{i,j}, \quad v_j P_j(\sigma_P(i)) = P_{i,j}. \tag{4.12}$$

By (4.5)-(4.12) we have the following equalities

$$r_i \sum_{d=0}^{D} Q_i(\sigma_Q(d)) Q_j(\sigma_Q(d)) v_d = \delta_{i,j}|X|, \quad Q_i(1) = 1, \tag{4.13}$$

$$v_i \sum_{d=0}^{D} P_i(\sigma_P(d)) P_j(\sigma_P(d)) r_d = \delta_{i,j}|X|, \quad P_i(1) = 1, \tag{4.14}$$

$$Q_i(\sigma_Q(d)) = P_d(\sigma_P(i)). \tag{4.15}$$

Furthermore from (4.3), (4.6) and (4.12) it follows that

$$Q_i(\sigma_Q(d(x,y))) = \frac{1}{r_i}\sum_{j=1}^{r_i} \mathbf{v}_{i,j}(x)\mathbf{v}_{i,j}(y). \tag{4.16}$$

Thus any polynomial graph is a (distance-regular) metric space which is polynomial (with respect to the standard substitution $\sigma_Q(d)$) in the sense of the definition in Section 3.2. Herewith the system Q coincides with the system of orthogonal polynomials constructed for an arbitrary (finite) metric space in Section 2.2. For polynomial graphs we have one more system P of orthogonal polynomials with orthogonality conditions (4.14). The system P also satisfies the Krein condition since from (4.2), (4.4), and (4.5) it follows that

$$P_{k,i}P_{k,j} = \sum_{d=0}^{D} p_{i,j}^{d} P_{k,d}. \tag{4.17}$$

The functions v_d and $\sigma(d) = \sigma_Q(d)$ define uniquely all parameters of systems Q and P. It is surprising that by a theorem of Leonard [74] only five of their values do that, for example, v_1, v_2, $\sigma(1)$, $\sigma(2)$, and $\sigma(3)$ (see also Terwilliger [115]). It should be noted by another important result of Leonard [73] (see also [8]) that, in fact, Q and P belong to the class of Askey-Wilson polynomials [4], which was introduced using basic hypergeometric series. There is also an interesting connection of Q and P with systems of orthogonal polynomials introduced by Nikiforov and Uvarov (see [85], [86]) using polynomial solutions of difference equations approximating (up to the second order) the classical hypergeometric differential equations on a lattice with the variable mesh $\sigma(d+1) - \sigma(d)$.

Notice the following formulae for coefficients of expansions of an arbitrary polynomial $f(t) = \sum_{i=0}^{D} f_i(Q)Q_i(t) = \sum_{i=0}^{D} f_i(P)P_i(t)$:

$$f_i(Q) = \frac{r_i}{|X|}\sum_{d=0}^{D} f(\sigma_Q(d))Q_i(\sigma_Q(d))v_d, \tag{4.18}$$

$$f_i(P) = \frac{v_i}{|X|}\sum_{d=0}^{D} f(\sigma_P(d))P_i(\sigma_P(d))r_d. \tag{4.19}$$

In particular,

$$f_0(Q) = \frac{1}{|X|}\sum_{d=0}^{D} f(\sigma_Q(d))v_d \quad ; \quad f_0(P) = \frac{1}{|X|}\sum_{d=0}^{D} f(\sigma_P(d))r_d. \tag{4.20}$$

We fix a code C in a polynomial association scheme X of diameter D (or with D classes). For any point $x \in X$ we consider the vector

$$\mathbf{B}(x) = (B_0(x), B_1(x), \ldots, B_D(x)) \tag{4.21}$$

where (see (2.8))

$$B_i(x) = B_i(x, C) = |\{y : y \in C, d(x, y) = i\}|,$$

which is called the *outer distribution* of C with respect to x. The vector

$$\mathbf{B} = \frac{1}{|C|} \sum_{x \in C} \mathbf{B}(x) = (B_0, B_1, \ldots, B_D) \tag{4.22}$$

is called the *inner distribution* of C. A code C is referred to as *distance invariant* if $\mathbf{B}(x) = \mathbf{B}(y)$ for any $x, y \in C$ (and hence $\mathbf{B}(x) = \mathbf{B}$ for any $x \in C$) and referred to as *completely distance invariant* if $\mathbf{B}(x) = \mathbf{B}(y)$ for any $x, y \in X$ when $d(x, C) = d(y, C)$ (here $d(x, C)$ is the minimum distance between x and the points of C).

For any vector $\mathbf{a} = (a_0, a_1, \ldots, a_D)$ such that not all $a_1, \ldots, a_D$ are equal to 0 (in particular, for $\mathbf{a} = \mathbf{B}$ and $\mathbf{a} = \mathbf{B}(x)$) we introduce the following parameters (cf. [29]) :

$$\begin{aligned}
d(\mathbf{a}) &= \min\{i : i = 1, \ldots, D, \quad a_i \neq 0\}, \\
s(\mathbf{a}) &= |\{i : i = 1, \ldots, D, \quad a_i \neq 0\}|, \\
\alpha(\mathbf{a}) &= \begin{cases} 0 & \text{if } a_0 = 0, \\ 1 & \text{otherwise,} \end{cases} \\
\beta(\mathbf{a}) &= \begin{cases} 0 & \text{if } a_D = 0, \\ 1 & \text{otherwise,} \end{cases}
\end{aligned}$$

We have denoted the values $d(\mathbf{a})$, $s(\mathbf{a})$, $\beta(\mathbf{a})$ for the vector $\mathbf{a} = \mathbf{B}$ by $d(C)$, $s(C)$, $\beta(C)$. They characterize, respectively, the *minimal distance*, the *number of* (nonzero) *distances* and the property of a code C to be *diametrical* (that is, whether or not the diameter of C coincides with the diameter of the whole space X). These values for the vector $\mathbf{a} = \mathbf{B}(x)$ have a similar sense and are denoted by $d(x, C), s(x, C), \beta(x, C)$ ($d(\mathbf{B}(x))$ is really equal to $d(x, C)$). The values $\alpha(\mathbf{a})$ have an auxiliary character. They are introduced to investigate simultaneously both cases because

$$\alpha(\mathbf{B}(x)) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases} \tag{4.23}$$

Now for any vector $\mathbf{a} = (a_0, a_1, \ldots, a_D)$ we consider the vector $\mathbf{a}' = (a'_0, a'_1, \ldots, a'_D)$ defined by

$$a'_i = r_i \sum_{d=0}^{D} a_d Q_i(\sigma_Q(d)), \quad i = 0, 1, \ldots, D. \tag{4.24}$$

The vector $\mathbf{a}'$ is called the *MacWilliams transform* of $\mathbf{a}$ [78], and allows us to determine dual parameters $d'(\mathbf{a}) = d(\mathbf{a}')$, $s'(\mathbf{a}) = s(\mathbf{a}')$, $\alpha'(\mathbf{a}) = \alpha(\mathbf{a}')$, $\beta'(\mathbf{a}) = \beta(\mathbf{a}')$ of the vector **a**. In particular, the parameters $d'(\mathbf{a})$, $s'(\mathbf{a})$, $\beta'(\mathbf{a})$ for the vector $\mathbf{a} = \mathbf{B}$ play a significant role for a code C as well and were denoted by $d'(C)$, $s'(C)$, $\beta'(C)$ in the general case (see (2.61), (2.62), (2.63)). The value $d'(C)$ is called the *dual distance.* If $\tau(C)$ is the *strength of the design* formed by C, that is, the maximum integer τ such that

$$\sum_{x,y\in C} Q_i(\sigma_Q(d(x,y))) = 0 \quad \text{for } i = 1, ..., \tau, \tag{4.25}$$

then from (4.24) it follows that $d'(C) = \tau(C) + 1$. The parameters $d'(\mathbf{a})$, $s'(\mathbf{a})$, $\beta'(\mathbf{a})$ for $\mathbf{a} = \mathbf{B}(x)$ are denoted by $d'(x, C)$, $s'(x, C)$, $\beta'(x, C)$ respectively. Notice that

$$\alpha'(\mathbf{a}) = 1 \quad \text{for } \mathbf{a} = \mathbf{B} \text{ and } \mathbf{a} = \mathbf{B}(x) \text{ for each } x \in X, \tag{4.26}$$

since in the both cases $a_0' = |C|$.

A polynomial $f(t)$ is called *annihilating* or *dual-annihilating* for $\mathbf{a} = (a_0, a_1, \ldots, a_D)$ (and for a code C if $\mathbf{a} = \mathbf{B}$) if respectively

$$a_i f(\sigma_Q(i)) = 0, \quad i = 1, \ldots, D, \tag{4.27}$$

$$a_i' f(\sigma_P(i)) = 0, \quad i = 1, \ldots, D. \tag{4.28}$$

Annihilating and dual-annihilating polynomials for **a** of minimum degree (that is, $s(\mathbf{a})$ and $s'(\mathbf{a})$) are called respectively *minimal* and *dual-minimal.* We denote by $f_C(t)$ and $\widetilde{f}_C(t)$, respectively, the minimal and dual-minimal polynomial for a code C such that $f_C(1) = 1$ and $\widetilde{f}_C(1) = 1$.

The following two theorems follow immediately from (4.13)-(4.15) and (4.24).

Theorem 4.3. *For any vector* $a = (a_0, a_1, \ldots, a_D)$,

$$a_i = \frac{v_i}{|X|} \sum_{d=0}^{D} a_d' P_i(\sigma_P(d)), \quad i = 0, 1, \ldots, D. \tag{4.29}$$

Theorem 4.4. *For any vector* $a = (a_0, a_1, \ldots, a_D)$ *and any polynomial*

$$f(t) = \sum_{i=0}^{D} f_i(Q) Q_i(t) = \sum_{i=0}^{D} f_i(P) P_i(t),$$

the following equalities hold:

$$\sum_{i=0}^{D} a_i f(\sigma_Q(i)) = \sum_{j=0}^{D} a_j' \frac{f_j(Q)}{r_j} \tag{4.30}$$

$$\sum_{i=0}^{D} a_i \frac{f_i(P)}{v_i} = \frac{1}{|X|} \sum_{j=0}^{D} a_j' f(\sigma_P(j)). \tag{4.31}$$

Thus we introduced for a code C six parameters $d(C)$, $s(C)$, $\beta(C)$, $d'(C)$, $s'(C)$, $\beta'(C)$. One more parameter, the *covering radius* $\rho(C)$ of a code C, can be defined as the maximum $d(\mathbf{a})$ over all $\mathbf{a} = \mathbf{B}(x)$, $x \in X$. A code C is called *uniformly packed* [12] if there exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_{\rho(C)}$ such that

$$\sum_{i=0}^{\rho(C)} \alpha_i B_i(x) = 1 \quad \text{for any } x \in X. \tag{4.32}$$

In the next section we prove some inequalities for the parameters. The main tool is to use equalities (4.30) and (4.31) for annihilating and dual-annihilating polynomials and also to use the adjacent (to Q and P) systems of orthogonal polynomials described in Section 2.3.

For arbitrary $a \in \{0,1\}$ and $b \in \{0,1\}$ we define polynomials $Q_j^{a,b}(t)$ and $P_j^{a,b}(t)$ in a real t of degree j, $j = 0, 1, \ldots, D - \delta_{a,1} - \delta_{b,1}$ as follows. First we define positive constants $c^{a,b}(Q)$ and $c^{a,b}(P)$ by equalities

$$c^{a,b}(Q) \sum_{d=0}^{D} (1 - \sigma_Q(d))^a (1 + \sigma_Q(d))^b v_d = |X| \tag{4.33}$$

and

$$c^{a,b}(P) \sum_{d=0}^{D} (1 - \sigma_P(d))^a (1 + \sigma_P(d))^b r_d = |X|, \tag{4.34}$$

respectively. Then the polynomials $Q_j^{a,b}(t)$ and $P_j^{a,b}(t)$ (together with positive constants $r_j^{a,b}(Q)$ and $v_j^{a,b}(P)$) are determined uniquely by the following orthogonality relations

$$r_j^{a,b}(Q) c^{a,b}(Q) \sum_{d=0}^{D} Q_i^{a,b}(\sigma_Q(d)) Q_j^{a,b}(\sigma_Q(d)) (1 - \sigma_Q(d))^a (1 + \sigma_Q(d))^b v_d = \delta_{i,j} \tag{4.35}$$

$$v_j^{a,b}(P) c^{a,b}(P) \sum_{d=0}^{D} P_i^{a,b}(\sigma_P(d)) P_j^{a,b}(\sigma_P(d)) (1 - \sigma_P(d))^a (1 + \sigma_P(d))^b r_d = \delta_{i,j} \tag{4.36}$$

and normalization

$$Q_j^{a,b}(1) = 1, \; P_j^{a,b}(1) = 1. \tag{4.37}$$

Let $t_j^{a,b}(Q)$ and $t_j^{a,b}(P)$ be the largest roots of the polynomials $Q_j^{a,b}(t)$ and $P_j^{a,b}(t)$ respectively. Using the fact that $\sigma_Q(d)$ and $\sigma_P(d)$ decrease with d, we can determine values $d_j^{a,b}(Q)$ and $d_j^{a,b}(P)$ as follows:

$$\sigma_Q(d_j^{a,b}(Q)) = t_j^{a,b}(Q), \quad \sigma_P(d_j^{a,b}(P)) = t_j^{a,b}(P). \tag{4.38}$$

In particular, for the Hamming space H_v^n, $d_k^{0,0}(Q) = d_k^{0,0}(P)$ equals the smallest root $d_k(n)$ of the Krawtchouk polynomial $K_k^{n,v}(z)$

$$\left(d_1(n) = \frac{v-1}{v} n, \quad d_2(n) = \frac{2(v-1)n - v + 2 - \sqrt{4(v-1)n + (v-1)^2}}{2v} \right)$$

and (see, for example, [71])

$$d_k^{a,b}(Q) = d_k^{a,b}(P) = d_k(n-a-b) + a. \tag{4.39}$$

We can apply these orthogonality conditions to find the free coefficient $f_0(Q)$ of the polynomial

$$f(t) = (1-t)^a (1+t)^b \frac{(Q_j^{a,b}(t))^2}{t_j^{a,b}(Q) - t}. \tag{4.40}$$

Using (4.20) and the fact that by (4.35) $Q_j^{a,b}(t)$ is orthogonal with respect to $(1 - \sigma_Q(d))^a (1 + \sigma_Q(d))^b v_d$ to any polynomial of degree at most $j-1$, we obtain

$$f_0(Q) = \frac{1}{|X|} \sum_{d=0}^{D} f(\sigma_Q(d)) v_d = 0. \tag{4.41}$$

4.2. Basic inequalities for code parameters based on annihilating polynomials

Now we obtain a number of inequalities for parameters of an arbitrary code C using (4.30) and (4.31) for annihilating and dual-annihilating polynomials for $\mathbf{a} = \mathbf{B}$ and $\mathbf{a} = \mathbf{B}(x)$. We take into account that by (4.22), (4.24) and (4.16)

$$B_i' = \frac{r_i}{|C|} \sum_{x,y \in C} Q_i(\sigma_Q(d(x,y))) = \frac{1}{|C|} \sum_{j=1}^{r_i} \left(\sum_{x \in C} \mathbf{v}_{i,j}(x) \right)^2 . \tag{4.42}$$

It follows that the vector $\mathbf{B}'$ is nonnegative (that is, it consists of nonnegative coordinates). Furthermore by (4.16), (4.21) and (4.24) for the vector $\mathbf{a} = \mathbf{B}(x)$ we have

$$B_i'(x) = r_i \sum_{j=1}^{r_i} \mathbf{v}_{i,j}(x) \sum_{y \in C} \mathbf{v}_{i,j}(y)$$

and hence

$$B_i' = 0 \quad \text{implies that } B_i'(x) = 0 \text{ for any } x \in X. \tag{4.43}$$

Notice also that

$$B_i = 0 \quad \text{implies that } B_i(x) = 0 \text{ for any } x \in C. \tag{4.44}$$

Theorem 4.5. *For any code C in a polynomial graph of diameter D the following inequalities hold :*

1. $d(C) + d'(C) \leq D + 2$.
2. $d'(C) \leq 2s(C) - \beta(C) + 1$; *equality implies* $|C| = \Omega_Q(f)$ *where* $f(t) = (1 + t)^{-\beta(C)}(f_C(t))^2$.
3. $d(C) \leq 2s'(C) - \beta'(C) + 1$; *equality implies* $|C|\Omega_P(f) = |X|$ *where* $f(t) = (1 + t)^{-\beta'(C)}(\widetilde{f}_C(t))^2$.
4. *If* $d'(C) \geq 2k - \varepsilon + 1$ *where k is an integer and* $\varepsilon \in \{0, 1\}$, *then*
$$d(C) \leq d^{1,\varepsilon}_{k-\varepsilon}(Q)$$
with equality if and only if $k = s(C), \varepsilon = \beta(C)$ *and* $(1 + t)^{\varepsilon} Q^{1,\varepsilon}_{k-\varepsilon}(t)$ *is minimal for C.*
5. *If* $d(C) \geq 2k - \varepsilon + 1$ *where k is an integer and* $\varepsilon \in \{0, 1\}$, *then*
$$d'(C) \leq d^{1,\varepsilon}_{k-\varepsilon}(P)$$
with equality if and only if $k = s'(C), \varepsilon = \beta'(C)$ *and* $(1 + t)^{\varepsilon} P^{1,\varepsilon}_{k-\varepsilon}(t)$ *is dual-minimal for C.*

Remark 4.6. *The first inequality of the theorem seems to be new although it is well known for the Hamming and Johnson spaces and is attained for MDS-codes and Steiner systems respectively [78]. The second and third inequalities improve the corresponding Delsarte's results when $\beta(C) = 1$ and $\beta'(C) = 1$ and are attained only for tight designs and perfect codes respectively [30], [78], [67], [68], [70], [71]. The fourth inequality follows from the author's work [64], [67] as it was noticed in [39], and is attained again for tight designs. The last inequality seems to be new and is attained only for perfect codes. It was proved in [71] for the case of the Hamming space.*

A proof of Theorem 4.5 is based on some auxiliary statements on vectors $\mathbf{a} = (a_0, a_1, \ldots, a_D)$.

Lemma 4.7. *Let $g(t)$ be annihilating for* $\mathbf{a}$, $\alpha'(\mathbf{a}) = 1$ *and* $g(\sigma_Q(i)) \geq 0$ *for* $i = 1, 2, \ldots, D$. *Then* $d'(\mathbf{a}) \leq \deg g(t) + \alpha(\mathbf{a})$. *In the case* $\alpha(\mathbf{a}) = 1$ *equality implies* $a_0 g(1) = a'_0 g_0(Q)$.

Proof. One can assume that $h = \deg\, g + \alpha(\mathbf{a}) \leq D$ because otherwise the statement is trivial. Since $a'_0 \neq 0$ and for the annihilating polynomial $f(t) = (1 - t)^{\alpha(\mathbf{a})} g(t)$ for

$\mathbf{a}$ of degree at most D it holds that $f_0(Q) = \dfrac{1}{|X|}\sum_{d=0}^{D} f(\sigma_Q(d))v_d > 0$ and $a_0 f(1) = 0$, we get from (4.30) that

$$\sum_{j=1}^{h} a'_j \frac{f_j(Q)}{r_j} \neq 0.$$

This completes the proof of the inequality, since not all a'_j, $j = 1, \ldots, h$, are equal to zero, and hence $d'(\mathbf{a}) \leq h$. The remaining part of the statement also follows from (4.30) with $f(t) = g(t)$. ■

Corollary 4.8. *If* $\alpha'(\mathbf{a}) = 1$, *then* $d'(\mathbf{a}) + d(\mathbf{a}) \leq D + 1 + \alpha(\mathbf{a})$.

Proof. Use Lemma 4.7 with the polynomial $g(t) = \prod_{i=d(\mathbf{a})}^{D}(t - \sigma_Q(i))$ satisfying the required properties. ■

Corollary 4.9. *If* $\alpha'(\mathbf{a}) = 1$, *then* $d'(\mathbf{a}) \leq 2s(\mathbf{a}) + \alpha(\mathbf{a}) - \beta(\mathbf{a})$. *In the case* $\alpha(\mathbf{a}) = 1$ *the equality implies* $a_0 g(1) = a'_0 g_0(Q)$ *for the polynomial* $g(t) = (1+t)^{-\beta(\mathbf{a})}(f(t))^2$, *where* $f(t)$ *is minimal for* $\mathbf{a}$.

Proof. Use Lemma 4.7 with this polynomial $g(t)$ satisfying the required properties as well. ■

Lemma 4.10. *Let* $\mathbf{a}$ *be nonnegative,* $\alpha'(\mathbf{a}) = 1$ *and* $d'(\mathbf{a}) \geq 2k - \varepsilon + \alpha(\mathbf{a})$ *where* k *is an integer and* $\varepsilon \in \{0, 1\}$. *Then* $d(\mathbf{a}) \leq d_{k-\varepsilon}^{\alpha(\mathbf{a}),\varepsilon}(Q)$ *with equality if and only if* $k = s(\mathbf{a})$, $\varepsilon = \beta(\mathbf{a})$, *and* $(1+t)^{\beta(\mathbf{a})} Q_{s(\mathbf{a})-\beta(\mathbf{a})}^{\alpha(\mathbf{a}),\beta(\mathbf{a})}(t)$ *is minimal for* $\mathbf{a}$.

Proof. Consider the polynomial

$$f(t) = (1-t)^{\alpha(\mathbf{a})}(1+t)^{\varepsilon} \frac{(Q_{k-\varepsilon}^{\alpha(\mathbf{a}),\varepsilon}(t))^2}{t_{k-\varepsilon}^{\alpha(\mathbf{a}),\varepsilon}(Q) - t}$$

of degree $h = 2k - \varepsilon + \alpha(\mathbf{a}) - 1$ and notice that $f_0(Q) = 0$ by (4.41). Using (4.30) we get

$$\sum_{i=1}^{D} a_i f(\sigma_Q(i)) = 0. \tag{4.45}$$

It follows from Corollary 4.9 that

$$2k - \varepsilon + \alpha(\mathbf{a}) \leq d'(\mathbf{a}) \leq 2s(\mathbf{a}) + \alpha(\mathbf{a}) - \beta(\mathbf{a})$$

and hence $s(\mathbf{a}) \geq k$. All a_i, $i = 1, \ldots, D$, are nonnegative and exactly $s(\mathbf{a})$ among them are positive. Since $f(\sigma_Q(i)) \geq 0$ for $i \geq d_{k-\varepsilon}^{\alpha(\mathbf{a}),\varepsilon}$, it follows that (4.45) implies $d(\mathbf{a}) \leq d_{k-\varepsilon}^{\alpha(\mathbf{a}),\varepsilon}$ with equality if and only if $k = s(\mathbf{a})$, $\varepsilon = \beta(\mathbf{a})$ and the polynomial $(1+t)^{\beta(\mathbf{a})} Q_{s(\mathbf{a})-\beta(\mathbf{a})}^{\alpha(\mathbf{a}),\beta(\mathbf{a})}(t)$ is minimal for $\mathbf{a}$. ■

Remark 4.11. *By Theorems 4.3 and 4.4, Lemmas 4.7, 4.10 and Corollaries 4.8, 4.9 will be valid if we replace in their formulations Q by P, $\mathbf{a}$ by $\mathbf{a}'$, a_0 by a_0', and a_0' by $a_0|X|$.*

Proof of Theorem 4.5. We can use Corollaries 4.8, 4.9 and Lemma 4.10 for the vector $\mathbf{a} = \mathbf{B}$ (see (4.22)) for which $\alpha(\mathbf{a}) = 1$ and $\alpha'(\mathbf{a}) = 1$ by (4.26). This gives the statements 1, 2 and 4. The statements 3 and 5 follow from dual analogues of Corollary 4.9 and Lemma 4.10 (see Remark 4.11). ■

In connection with statement 2 and 3 of Theorem 4.5 it is also worth noting the following consequences of the equalities (4.30) and (4.31).

Corollary 4.12. *If for a code C in a polynomial graph X there exists an annihilating polynomial $f(t)$ which can be expanded over the system Q with positive coefficients, then $|C| \leq \Omega_Q(f)$. Equality implies $d'(C) \geq 1 + \deg f$. If for a code C there exists a dual-annihilating polynomial $f(t)$ which can be expanded over the system P with positive coefficients, then $|C|\Omega_P(f) \geq |X|$. Equality implies $d(C) \geq 1 + \deg f$.*

The duality based on (4.30) and (4.31) for nonnegative vectors $\mathbf{a} = \mathbf{B}$ and $\mathbf{a}' = \mathbf{B}$ and systems Q and P satisfying the Krein conditions and (4.13)-(4.15) can be applied to prove analogs of some statements for polynomial spaces. They are obtained using (4.31) instead of (4.30). In particular, we have the following dual analog of Theorem 3.17.

Theorem 4.13. *If for a code C in a polynomial graph X with dual-minimal polynomial $\widetilde{f}_C(t) = \sum_{i=0}^{s'(C)} \widetilde{f}_i P_i(t)$ the inequality $d(C) \geq s'(C) + 1$ holds, then*

$$\widetilde{f}_j|X| = v_j|C|, \quad j = 0, 1, ..., d(C) - s'(C) - 1.$$

If $\widetilde{f}_C(t)$ can be expanded over the system P with positive coefficients and for some d, $s'(C) \leq d - 1 \leq D(X)$,

$$\widetilde{f}_j|X| = v_j|C|, \quad j = 0, 1, ..., d - s'(C) - 1,$$

then $d(C) \geq d$.

For any vector $\mathbf{a} = \mathbf{B}(x)$ where $x \notin C$, according to (4.23) and (4.26) we have $\alpha'(\mathbf{a}) = 1$, $\alpha(\mathbf{a}) = 0$. Therefore we can use only Corollaries 4.8, 4.9 and Lemma 4.10 and obtain the following results for an arbitrary code C and any $x \in X \setminus C$:

1. $d(x, C) + d'(x, C) \leq D + 1$,
2. $d'(x, C) \leq 2s'(x, C) - \beta'(x, C)$,

3. If $d'(x,C) \geq 2k-\varepsilon$ where k is an integer and $\varepsilon \in \{0,1\}$ then

$$d(x,C) \leq d_{k-\varepsilon}^{0,\varepsilon}(Q)$$

with equality if and only if $k = s(x,C)$, $\varepsilon = \beta(x,C)$ and $(1+t)^{\varepsilon} Q_{k-\varepsilon}^{0,\varepsilon}(t)$ is minimal for $\mathbf{B}(x)$.

Since by (4.43) $d'(C) \leq d'(x,C)$ for any $x \in X$, the last statement gives rise to the following result.

Theorem 4.14. *For a code C in any polynomial graph X the inequality $d'(C) \geq 2k-\varepsilon$ where k is an integer and $\varepsilon \in \{0,1\}$ implies that*

$$\rho(C) \leq d_{k-\varepsilon}^{0,\varepsilon}(Q)$$

with equality if and only if there exists a point $x \in X \setminus C$ such that $(1+t)^{\varepsilon} Q_{k-\varepsilon}^{0,\varepsilon}(t)$ is minimal for $\mathbf{B}(x)$.

The inequality of Theorem 4.14 for the Hamming space is due to Tietäväinen [117], [118]. For an arbitrary polynomial space (in particular, graph) X it was proved together with necessary and sufficient conditions of its attainability in [39]. In particular, it is attained for all binary tight designs of even strength, for example, for the tight 6-design formed by the Golay [23,11,8] code (for this code, by Theorem 4.14, $\rho(C) \leq d_3^{0,1}(23) = d_3(22) = 7$ since $d'(C) \geq 7$).

Corollary 4.15. *A code C in any polynomial graph X is distance invariant if $d'(C) \geq s(C)$ or $d(C) \geq s'(C)$.*

Proof. Given the conditions stated, we prove that the value $B_i(x)$ for any i, $i = 0,1,...,D$, does not depend on $x \in C$. Since we have $B_0(x) = 1$ for any $x \in C$ and by (4.44) $B_i(x) = 0$ for any $x \in C$ if $B_i = 0$, it is sufficient to consider the set H of indices $i \in \{1,...,D\}$ such that $B_i \neq 0$. By the construction $|H| = s(C)$. According to (4.30) for any polynomial $f(z)$ of degree at most $d'(C)-1$,

$$\sum_{i \in H} B_i(x) f(\sigma_Q(i)) = |C| f_0(Q) - f(1).$$

On the other hand, for any $j \in H$ there exists the Lagrange polynomial $F_j(z)$ of degree $s(C)-1$ such that $F_j(\sigma_Q(j)) = 1$ and $F_j(\sigma_Q(i)) = 0$ when $i \in H \setminus \{j\}$. This completes the proof if $d'(C) \geq s(C)$. Analogously using (4.31) one can show that for any i, $i = 0,1,...,D$, the value $B'_i(x)$ does not depend on $x \in C$ if $d(C) \geq s'(C)$. Then by Theorem 4.3 it is also true for values $B_i(x)$. ■

Remark 4.16. *Since $d'(C) = \tau(C)+1$ the first statement of Corollary 4.15 and statement 2 of Theorem 4.5 were proved, respectively, in Theorem 3.19 and Lemma*

2.17 for a code C in an arbitrary polynomial space X (in particular, for a code C on the unit Euclidean sphere). Statement 4 of Theorem 4.5 and the statement of Theorem 4.14 are also true for a code C in an arbitrary polynomial space X as well as Lemma 4.7, Corollary 4.9 , and Lemma 4.10 which are used in the proofs of these statements.

We consider $B_i(x)$ (see (4.21)) as an entry of a matrix of size $|X| \times (D+1)$ with rows $\mathbf{B}(x)$ enumerated by $x \in X$ and columns $\mathbf{B}_i$ enumerated by i, $i = 0, 1, \dots, D$.

Theorem 4.17. *For a code C in any polynomial graph X, $d(x, C) \le s'(C)$ for any $x \in X$ and hence*

$$\rho(C) \le s'(C) \tag{4.46}$$

with equality if and only if C is uniformly packed. The column $\mathbf{B}_{s'}$ where $s' = s'(C)$ is a linear combination of the columns $\mathbf{B}_0, \dots, \mathbf{B}_{s'-1}$ and the column of all ones. Any column $\mathbf{B}_i$, $s' < i \le D$, is a linear combination of preceding columns, any row $\mathbf{B}(x)$ is uniquely determined by its first s' coordinates, and the columns $\mathbf{B}_0, \dots, \mathbf{B}_{s'}$ are linearly independent.

Proof. Let $f(t) = \widetilde{f}_C(t)$ be a dual-minimal polynomial of degree $s' = s'(C)$ for C such that $f(1) = 1$. Using (4.43) and (4.31) for $\mathbf{a} = \mathbf{B}(x)$ we have for any $x \in X$

$$\frac{|X|}{|C|} \sum_{i=0}^{s'} B_i(x) \frac{f_i(P)}{v_i} = 1 \quad \text{where } f_{s'}(P) \neq 0. \tag{4.47}$$

This proves the first two statements of the theorem apart from $\rho(C) = s'(C)$ for a uniformly packed code. The statement about $\mathbf{B}_i$, $s' < i \le D$, is obtained by analogy using the dual-annihilating polynomial $t^{i-s'} f(t)$ of degree i. Therefore any row $\mathbf{B}(x)$ is determined uniquely by its first s' coordinates. The linear independence of the first $s'+1$ columns is a consequence of the Delsarte equality

$$\sum_{x \in X} B_i(x) B_j(x) = |C| \sum_{d=0}^{D} B_d p_{i,j}^d = \frac{|C|}{|X|} \sum_{k=0}^{D} B'_k P_{k,i} P_{k,j}$$

which follows from the definitions of $B_i(x)$, B_d, $p_{i,j}^d$ and from (4.12), (4.14), (4.15), (4.17), and shows that the rank of the matrix under consideration equals $s'+1$. Since $\mathbf{B}_{s'}$ is not a linear combination of preceding columns, from (4.32) and (4.47) it follows that for any uniformly packed code the inequality (4.46) cannot be strict. ∎

Theorem 4.17 belongs to Delsarte [30] except for the necessary and sufficient conditions of the equality in (4.46) which were obtained in [13]. Uniformly packed codes were investigated in [12], [47], [13].

In conclusion we derive from Theorem 4.17 two more results of Delsarte [30].

Corollary 4.18. *A code C in any polynomial graph X is completely distance invariant if $d(C) \ge 2s'(C) - 1$.*

Proof. For any $x \in X$ it follows that $\sum_{i=0}^{s'-1} B_i(x) \leq 1$ (where $s' = s'(C)$), because otherwise, by the triangle inequality, there exist two distinct code points at distance at most $2s' - 2$ from each other. Furthermore by Theorem 4.17 any row $\mathbf{B}(x)$ is determined uniquely by its first s' coordinates. This completes the proof. ■

Corollary 4.19. *For any code C in any polynomial graph X and $s' = s'(C)$,*

$$|C| \geq \frac{|X|}{\sum_{i=0}^{s'} v_i} = \frac{|X|}{L_P(t_{s'}^{1,0}(P))} = \frac{|X|}{L_P(\gamma_P(2s'+1))} \tag{4.48}$$

with equality if and only if $d(C) = 2s'(C) + 1$ and $\widetilde{f}_C(t) = P_{s'}^{1,0}(t)$.

Proof. According to (4.46) the metric balls of radius s' with centers at points of a code C cover X and hence the bound (4.48) holds. If this bound is attained, then $d(C) = 2s'(C) + 1$ and for any i, $i = 0, ..., s'$, there exists $x \in X$ such that $B_i(x) = 1$. By (4.47) this means that $|X| f_i(P) = |C| v_i$ for any i, $i = 0, ..., s'$, and hence

$$\widetilde{f}_C(t) = \sum_{i=0}^{s'} f_i(P) P_i(t) = \frac{|C|}{|X|} \sum_{i=0}^{s'} v_i P_i(t) = P_{s'}^{1,0}(t).$$

The sufficiency of the stated conditions follows from Statement 3 of Theorem 4.5. ■

As in the case of the absolute bound of Delsarte (Theorem 3.18) one can formulate the following conjecture.

Conjecture 4.20. *For any code C in a polynomial graph X,*

$$\frac{|X|}{L_P(\gamma_P(2s'(C) - \beta'(C) + 1))} \leq |C| \leq \frac{|X|}{L_P(\gamma_P(d(C)))} \tag{4.49}$$

with equality in either of the bounds if and only if

$$d(C) = 2s'(C) - \beta'(C) + 1$$

and

$$\widetilde{f}_C(t) = \left(\frac{t+1}{2}\right)^{\beta'(C)} P_{s'(C)-\beta'(C)}^{1,\beta'(C)}(t).$$

This statement is true with respect to the upper bound (it will be proved in Section 5.5) and with respect to the lower bound when $\beta'(C) = 0$ (see Corollary 4.15). Thus one needs to prove only the lower bound in (4.49) when $\beta'(C) = 1$ and the fact that the stated conditions of its attainability are necessary in this case (the sufficiency of the conditions follows also from Statement 3 of Theorem 4.5). Conjecture 4.20 was completely proved for the Hamming space in [71].

4.3. Duality in bounding optimal sizes of codes and designs

In Section 3.3 we considered the σ-packing and τ-design (continuous) problems for the system Q which give universal bounds for codes and designs in polynomial metric spaces. Now we show that for polynomial graphs there exists a duality in bounding the optimal sizes of codes and designs which allows us to use the corresponding extremum problems for the system P as well.

For any code C in a polynomial graph X of diameter D with the distance distribution

$$\mathbf{B}(C) = (B_0(C), ..., B_D(C))$$

where $B_i(C) = \frac{1}{|C|}|\{(x,y) : x, y \in C, d(x,y) = i\}|$ and the dual distribution

$$\mathbf{B}'(C) = (B'_0(C), ..., B'_D(C))$$

where

$$B'_i(C) = r_i \sum_{d=0}^{D} B_d(C) Q_i(\sigma_Q(d)) = \frac{r_i}{|C|} \sum_{x,y \in C} Q_i(\sigma_Q(d(x,y))),$$

by Theorem 4.4 we obtain

$$\sum_{i=0}^{D} B_i(C) f(\sigma_Q(i)) = \sum_{j=0}^{D} B'_j(C) \frac{f_j(Q)}{r_j} \tag{4.50}$$

and

$$\sum_{i=0}^{D} B_i(C) \frac{f_i(P)}{v_i} = \frac{1}{|X|} \sum_{j=0}^{D} B'_j(C) f(\sigma_P(j)). \tag{4.51}$$

Let $U = \{U_0(t), ..., U_D(t)\}$ be either of the systems Q or P of orthogonal polynomials and let

$$f(t) = \sum_{i=0}^{D} f_i(U) U_i(t) \tag{4.52}$$

and

$$\Omega_U(f) = \frac{f(1)}{f_0(U)} \text{ when } f_0(U) \neq 0. \tag{4.53}$$

We say that a polynomial $f(t)$ has the property $\mathfrak{A}_U(d)$, $d = 1, ..., D+1$, if

$$f_0(U) > 0, \ f_i(U) \geq 0 \text{ for } i = 1, ..., D, \tag{4.54}$$

$$f(1) > 0, \ f(\sigma_U(i)) \leq 0 \text{ for } i = d, ..., D, \tag{4.55}$$

and has the property $\mathfrak{B}_U(d)$, $d = 1, ..., D+1$, if

$$f_0(U) > 0, \ f_i(U) \leq 0 \text{ for } i = d, ..., D, \tag{4.56}$$

$$f(1) > 0,\ f(\sigma_U(i)) \geq 0 \ \text{ for } i = 1, ..., D. \tag{4.57}$$

Consider the following discrete extremum problems for the system U by analogy with the continuous ones.

The discrete d-code problem: for any integer d, $d = 1, ..., D+1$, find the minimum $A_U(d)$ of $\Omega_U(f)$ over all polynomials $f(t)$ satisfying the property $\mathfrak{A}_U(d)$.

The discrete $(d-1)$-design problem: for any integer d, $d = 1, ..., D+1$, find the maximum $B_U(d)$ of $\Omega_U(f)$ over all polynomials $f(t)$ satisfying the property $\mathfrak{B}_U(d)$.

Note that both these problems are linear programming ones because $f(1) = \sum_{i=0}^{D} f_i(U)$ and without loss of generality one can assume that $f_0(U) = 1$. We shall see that the second problem for the system U is equivalent to the first problem for the system $\overline{U}$ where $\overline{U} = P$, if $U = Q$, and $\overline{U} = Q$, if $U = P$.

First, compare these discrete problems with the σ-packing problem for $\sigma = \sigma(d)$ and the τ-design problem for $\tau = d - 1$ considered for an arbitrary system U in Section 3.3. Since the conditions (4.55) and (4.57) are weaker than, respectively, (3.78) and (3.79), any permissible solution of the continuous problem is that for the corresponding discrete one. In particular, it follows that for any system U

$$B_U(d) \geq L_U(\gamma_U(d)) \tag{4.58}$$

(see (3.104)) and, if U satisfies the strengthened Krein condition, then

$$A_U(d) \leq L_U(\sigma_U(d)) \tag{4.59}$$

(see (3.95)). Bounding $A(X, d)$ and $B(X, d)$ for a polynomial graph X we can use the inequalities (4.58) and (4.59) for the system $U = Q$ and prove that

$$A(X, d) \leq A_Q(d), \tag{4.60}$$

$$B(X, d) \geq B_Q(d). \tag{4.61}$$

Indeed, using (4.50) and the inequalities $B'_j(C) \geq 0$ we have Delsarte's result which is analogous to Theorems 3.14 and 3.15.

Theorem 4.21. *For any code C in a polynomial graph X and any polynomial $f(t)$ satisfying property $\mathfrak{A}_Q(d(C))$ or $\mathfrak{B}_Q(d'(C))$,*

$$|C| \leq \Omega_Q(f) \tag{4.62}$$

or, respectively,

$$|C| \geq \Omega_Q(f). \tag{4.63}$$

Equality in (4.62) and (4.63) takes place if and only if

$$B_j(C) f(\sigma_Q(j)) = 0, \quad j = 1, ..., D, \tag{4.64}$$

and

$$B'_j(C) f_j(Q) = 0, \quad j = 1, ..., D. \tag{4.65}$$

Remark 4.22. *To prove (4.62) and (4.63) the signs of the values $B_i(C), f(\sigma_Q(i)), B'_j(C)$, and $f_j(Q)$ are only used in (4.50). Additional information allows us to improve these bounds in some cases (see [78]).*

Note that we cannot directly use the inequalities (4.58) and (4.59) for the system $U = P$ because, in general, it is not true that for any code $C \subseteq X$

$$\sum_{x,y \in C} P_i(\sigma_P(d(x,y))) \geq 0, \quad i = 1, ..., D.$$

Nevertheless, it is possible to use these inequalities by introducing a notion of Q- and P-duality of polynomials. For an arbitrary polynomial $f(t)$, the polynomial

$$D_Q(f) = |X|^{-\frac{1}{2}} \sum_{i=0}^{D} r_i f(\sigma_P(i)) Q_i(t) \tag{4.66}$$

is called *Q-dual to $f(t)$* and the polynomial

$$D_P(f) = |X|^{-\frac{1}{2}} \sum_{i=0}^{D} v_i f(\sigma_Q(i)) P_i(t) \tag{4.67}$$

is called *P-dual to $f(t)$*. In particular, for any $d = 1, ..., D$ consider the polynomials

$$m_Q^{(d)}(t) = \prod_{i=d}^{D} \frac{t - \sigma_Q(i)}{1 - \sigma_Q(i)}, \quad m_P^{(d)}(t) = \prod_{i=d}^{D} \frac{t - \sigma_P(i)}{1 - \sigma_P(i)}, \tag{4.68}$$

of degree $D - d + 1$ and note that

$$m_Q^{(d)}(\sigma_Q(j)) > 0 \text{ and } m_P^{(d)}(\sigma_P(j)) > 0 \text{ for } j = 0, 1, ..., d-1 \tag{4.69}$$

under our restriction that $\sigma_Q(i)$ and $\sigma_P(i)$ decrease with i. Let

$$N_Q(d) = \sum_{j=0}^{D} m_Q^{(d)}(\sigma_Q(j)) v_j, \quad N_P(d) = \sum_{j=0}^{D} m_P^{(d)}(\sigma_P(j)) r_j. \tag{4.70}$$

Hence $N_Q(d)$ and $N_P(d)$ are positive-valued increasing functions in d and

$$\Omega_Q(m_Q^{(d)}) = \frac{|X|}{N_Q(d)}, \quad \Omega_P(m_P^{(d)}) = \frac{|X|}{N_P(d)}.$$

Lemma 4.23. *For any d, $d = 1, ..., D$, the polynomial $|X|^{-\frac{1}{2}} N_Q(d) m_P^{(D-d+2)}(t)$ is P-dual to $m_Q^{(d)}(t)$ and belongs to $F_>(P)$, and the polynomial $|X|^{-\frac{1}{2}} N_P(d) m_Q^{(D-d+2)}(t)$ is Q-dual to $m_P^{(d)}(t)$ and belongs to $F_>(Q)$.*

Proof. According to (4.68) and (4.70) the polynomial (4.67), P-dual to $f(t) = m_Q^{(d)}(t)$, is equal to $|X|^{-\frac{1}{2}}N_Q(d)$ at $t = 1$ and has degree $d-1$ and roots $\sigma_P(l)$, $l = D-d+2, ..., D$, since by (4.15) and (4.13),

$$\sum_{i=0}^{D} m_Q^{(d)}(\sigma_Q(i))P_i(\sigma_P(l))v_i = \sum_{i=0}^{D} m_Q^{(d)}(\sigma_Q(i))Q_l(\sigma_Q(i))v_i = 0$$

for $D-d+2 \leq l \leq D$. The polynomial $|X|^{-\frac{1}{2}}N_Q(d)m_P^{(D-d+2)}(t)$ of degree $d-1$ satisfies the same properties and hence is P-dual to $m_Q^{(d)}(t)$. Its coefficients at $P_j(t)$, $j = 0, 1, ..., d-1$, are positive by (4.69). The second part of the statement is proved analogously. ■

Theorem 4.24. *For any polynomial* $f(t)$,

$$D_Q(D_P(f)) = D_P(D_Q(f)) = f, \tag{4.71}$$

$$\Omega_P(f)\Omega_Q(D_Q(f)) = \Omega_Q(f)\Omega_P(D_P(f)) = |X|. \tag{4.72}$$

If $f(t)$ *satisfies property* $\mathfrak{A}_P(d)$ *or* $\mathfrak{B}_P(d)$, *then the* Q*-dual polynomial* $D_Q(f)$ *satisfies, respectively, property* $\mathfrak{B}_Q(d)$ *or* $\mathfrak{A}_Q(d)$ *and conversely, if* $f(t)$ *satisfies property* $\mathfrak{A}_Q(d)$ *or* $\mathfrak{B}_Q(d)$, *then the* P*-dual polynomial* $D_P(f)$ *satisfies, respectively, property* $\mathfrak{B}_P(d)$ *or* $\mathfrak{A}_P(d)$.

Proof. From (4.13)-(4.15), (4.18), (4.19), and (4.52) it follows that

$$\sum_{i=0}^{D}\sum_{j=0}^{D} r_i v_j f(\sigma_Q(j))P_j(\sigma_P(i))Q_i(t) = \sum_{i=0}^{D}\sum_{j=0}^{D} r_j v_i f(\sigma_P(j))Q_j(\sigma_Q(i))P_i(t) = |X|f(t).$$

In accord with definitions (4.66) and (4.67) this gives (4.71) and shows that the value of $D_Q(f)$ at $\sigma_Q(i)$ is equal to $\frac{|X|^{\frac{1}{2}}}{v_i}f_i(P)$ and the value of $D_P(f)$ at $\sigma_P(i)$ is equal to $\frac{|X|^{\frac{1}{2}}}{r_i}f_i(Q)$. Therefore,

$$\Omega_Q(D_Q(f)) = \frac{f_0(P)}{f(1)}|X| \quad \text{and} \quad \Omega_P(D_P(f)) = \frac{f_0(Q)}{f(1)}|X|$$

by (4.53) and (4.20), and (4.72) holds. The remaining statements are obtained if one compares (4.54)-(4.57) with (4.66)-(4.67) and uses (4.71). ■

Remark 4.25. *From (4.28) and the proof of Theorem 4.24 condition (4.65) means that the* P*-dual polynomial* $D_P(f)$ *to an annihilating polynomial* $f(t)$ *for a code* C *is dual-annihilating for* C.

Corollary 4.26. *For any polynomial graph* X *and any* d, $d = 1, ..., D+1$,

$$A_P(d)B_Q(d) = B_P(d)A_Q(d) = |X|. \tag{4.73}$$

Corollary 4.26 and (4.58) allow us to state that together with the bound

$$B(X,d) \geq B_Q(d) \geq L_Q(\gamma_Q(d)) \tag{4.74}$$

valid for any polynomial space X (see Theorem 2.16 and (3.104)), for any polynomial graph X it is also true that

$$A(X,d) \leq A_Q(d) = \frac{|X|}{B_P(d)} \leq \frac{|X|}{L_P(\gamma_P(d))} \tag{4.75}$$

(this is a generalization of the Hamming bound). Analogously, together with the bound

$$A(X,d) \leq A_Q(d) \leq L_Q(\sigma_Q(d)) \tag{4.76}$$

valid for any polynomial space X with the system Q satisfying the strengthened Krein condition (see (3.95)), for any polynomial graph X with the system P satisfying the strengthened Krein condition it is also true that

$$B(X,d) \geq B_Q(d) = \frac{|X|}{A_P(d)} \geq \frac{|X|}{L_P(\sigma_P(d))}. \tag{4.77}$$

The following special cases of bounds (4.76) and (4.77) for an integer l and $\theta \in \{0,1\}$

$$A(X, d_{l-\theta}^{1,\theta}(Q)) \leq A_Q(d_{l-\theta}^{1,\theta}(Q)) \leq L_Q(t_{l-\theta}^{1,\theta}(Q)), \tag{4.78}$$

$$B(X, d_{l-\theta}^{1,\theta}(P)) \geq B_Q(d_{l-\theta}^{1,\theta}(P)) = \frac{|X|}{A_P(d_{l-\theta}^{1,\theta}(P))} \geq \frac{|X|}{L_P(t_{l-\theta}^{1,\theta}(P))} \tag{4.79}$$

hold (see (3.101) and (4.38)) for any polynomial graph X.

Thus in the case of polynomial graphs we have two pairs of universal bounds for codes and designs, namely (4.74)-(4.75) and (4.76)-(4.77). Both these pairs are obtained using the optimal solutions of the restricted (continuous) σ-packing and τ-design problems (see Section 3.3) and the relationship (4.73) valid in the discrete case. There exists one more pair of universal bounds which uses essentially the discrete nature of the problem considered.

Theorem 4.27. *For any code C in a polynomial graph X of diameter D,*

$$\frac{|X|}{N_Q(D - d'(C) + 2)} = N_P(d'(C)) \leq |C| \leq \frac{|X|}{N_Q(d(C))} = N_P(D - d(C) + 2). \tag{4.80}$$

Each of the bounds in (4.80) is attained if and only if $d(C) + d'(C) = D + 2$. In this case $m_Q^{(d(C))}(t)$ is annihilating and $m_P^{(d'(C))}(t)$ is dual-annihilating for C.

Proof. By Theorems 4.21 and 4.24, and Lemma 4.23 the polynomial $m_Q^{(d(C))}(t)$ has property $\mathfrak{A}_Q(d(C))$ and gives rise to the upper bound in (4.80) which is attained if and only if $m_Q^{(d(C))}(t)$ is annihilating and the P-dual (up to a constant factor) polynomial $m_P^{(D-d(C)+2)}(t)$ is dual-annihilating for C. Moreover, if this bound is attained, then by Corollary 4.12 $d'(C) \geq D - d(C) + 2$ since the polynomial $m_Q^{(d(C))}(t)$ of degree $D - d(C) + 1$ belongs to $F_>(Q)$. Analogously, the polynomial $m_P^{(d'(C))}(t)$ has property $\mathfrak{A}_P(d'(C))$ and hence its Q-dual (up to a constant factor) polynomial $m_Q^{(D-d'(C)+2)}(t)$ has property $\mathfrak{B}_Q(d'(C))$ and gives rise to the lower bound in (4.80) which is attained if and only if $m_Q^{(D-d'(C)+2)}(t)$ is annihilating and $m_P^{(d'(C))}(t)$ is dual-annihilating for C. If the lower bound is attained, we have again $d(C) \geq D - d'(C) + 2$ by Corollary 4.12 since the polynomial $m_P^{(d'(C))}(t)$ of degree $D - d'(C) + 1$ belongs to $F_>(P)$. This completes the proof because $N_Q(d)$ and $N_P(d)$ increase with d and hence $d'(C) \leq D - d(C) + 2$. ■

This theorem gives one more pair of universal bounds on $A(X,d)$ and $B(X,d)$ for a polynomial graph X and shows that MDS-codes and Steiner systems for which both bounds (4.80) are attained in Hamming and Johnson space, respectively, belong to the same class of codes with property $d(C) + d'(C) = D + 2$.

5. Applications of orthogonal polynomials

5.1. Properties of orthogonal polynomials

While simultaneously investigating continuous and discrete systems of orthogonal polynomials it is convenient to use the Lebesgue-Stieltjes integral. Let $\nu(t)$ be a non-decreasing left continuous function on the interval $[A,B]$ where maybe $A = -\infty$ and/or $B = \infty$. We consider the following two cases. In the first case $\nu(t)$ is differentiable, the corresponding weight function $\nu'(t) = w(t)$ is continuous on $[A,B]$ and positive inside the interval, and for any nonnegative integer n the integral $\int_A^B t^n w(t)dt$ exists if $A = -\infty$ and/or $B = \infty$. In the second case $\nu(t)$ is a step function with N steps at points denoted by $t_{N,i}$ ($A \leq t_{N,1} < t_{N,2} < \cdots < t_{N,N} \leq B$). Assume the steps to be $w_i > 0$, $i = 1, ..., N$, and at least one of these points to be inside $[A,B]$. In addition suppose that the Lebesgue-Stieltjes measure ν on $[A,B]$ corresponding to the function $\nu(t)$ is normalized. In particular, these properties were assumed if $\nu(t)$ is chosen to be $1 - \overline{\mu}(\sigma^{-1}(t))$ on $[-1,1]$ (see (2.34)) for a compact metric space. Under these assumptions the inner product defined with the help of the Lebesgue-Stieltjes integral

$$(u,v) = \int_A^B u(t)v(t)d\nu(t) \tag{5.1}$$

becomes either an integral of the type

$$\int_A^B u(t)v(t)w(t)dt$$

or represents a sum

$$\sum_{i=1}^N u(t_{N,i})v(t_{N,i})w_i$$

where $\int_A^B w(t)dt = 1$ and $\sum_{i=1}^N w_i = 1$ respectively. In the case of a discrete measure we set

$$U_N(t) = (t - t_{N,1})(t - t_{N,2}) \cdots (t - t_{N,N}). \tag{5.2}$$

In the infinite case we put $N = \infty$.

Below we present (sometimes without proofs) the basic properties (see [14], [113]) of systems of orthogonal polynomials corresponding to the inner product (5.1). We shall normalize polynomials $U(t)$ by $U(B) = 1$ if $B < \infty$ or by $Co(U(t)) = 1$ if $B = \infty$ where $Co(U(t))$ is the highest degree coefficient of the polynomial $U(t)$ (we assume $Co(U(t)) = 1$ if $U(t) \equiv 1$).

Theorem 5.1. *There exist a unique sequence of polynomials $U_i(t)$ of degree i, $0 \le i < N$, and a unique sequence of positive constants r_i, $0 \le i < N$, such that for any i and j, $0 \le i, j < N$,*

$$r_i \int_A^B U_i(t)U_j(t)d\nu(t) = \delta_{i,j} \tag{5.3}$$

and for any $i, 0 \le i < N$,

$$U_i(B) = 1 \text{ if } B < \infty \quad \text{or} \quad Co(U_i(t)) = 1 \text{ if } B = \infty. \tag{5.4}$$

Note that $U_0(t) \equiv 1$ and $r_0 = 1$ because of the normalization of the measure ν and that in the discrete case the polynomial $U_N(t)$ is orthogonal to all polynomials $U_i(t)$, $0 \le i < N$. From Theorem 5.1 it follows that for any polynomial $f(t) = \sum_{i=0}^h f_i U_i(t)$ of degree h, $0 \le h < N$,

$$f_i = r_i \int_A^B f(t)U_i(t)d\nu(t). \tag{5.5}$$

Theorem 5.2. *Any polynomial $U_k(t)$, $1 \le k < N$, has k distinct simple roots inside the interval $[A, B]$.*

Denote by $t_{k,i}$ $(i = 1, ..., k)$ the roots of the polynomial $U_k(t)$, $1 \le k < N$, arranged in increasing order. In the discrete case this agrees with previous notation introduced for the roots of the polynomial $U_N(t)$. Furthermore, we denote by t_k the largest zero $t_{k,k}$ of the polynomial $U_k(t)$. Note that by Theorem 5.2 and the normalization (5.4) the highest coefficients of polynomials $U_k(t)$ are positive and, hence,

$$\text{sgn } U_k(A) = (-1)^k \quad \text{for } 0 \le k < N \ . \tag{5.6}$$

Since $(tU_i(t), U_j(t)) = (U_i(t), tU_j(t)) = 0$ for any j, $0 \le j \le i-2$, we have

Theorem 5.3 (recurrence). *For any i, $0 \le i < N$, there exist real numbers a_i, b_i, and c_i $(c_0 = 0)$ such that*

$$(t - a_i)U_i(t) = b_i U_{i+1}(t) + c_i U_{i-1}(t). \tag{5.7}$$

We can see that

$$b_i = \frac{Co(U_i(t))}{Co(U_{i+1}(t))}, \tag{5.8}$$

$$c_i = \frac{(U_i(t), tU_{i-1}(t))}{(U_{i-1}(t), U_{i-1}(t))} = \frac{r_{i-1}}{r_i} b_{i-1} \tag{5.9}$$

and, hence, all numbers b_i, $i \ge 0$, and c_i, $i \ge 1$, are positive in accord with the chosen normalization (5.4).

Theorem 5.4 (Christoffel-Darboux formulae). *For any k, $0 \le k < N$, and any real numbers x and y,*

$$\sum_{i=0}^{k} r_i U_i(x) U_i(y) = r_k b_k \frac{U_{k+1}(x)U_k(y) - U_k(x)U_{k+1}(y)}{x - y} \quad \text{if } x \ne y, \tag{5.10}$$

$$\sum_{i=0}^{k} r_i U_i(x) U_i(x) = r_k b_k (U'_{k+1}(x)U_k(x) - U'_k(x)U_{k+1}(x)). \tag{5.11}$$

Corollary 5.5 (monotonicity). *For any k, $0 \le k < N$,*

$$\frac{U_{k+1}(t)}{U_k(t)} \quad \text{increases with } t \text{ if } U_k(t) \ne 0. \tag{5.12}$$

Corollary 5.6 (separation of roots). *For any k and j such that $1 \le j-1 \le k-1 < N$,*

$$t_{k,j-1} < t_{k-1,j-1} < t_{k,j}. \tag{5.13}$$

Corollary 5.7. *If $t_{k-1} < t < t_k$, $0 \le k-1 < N$, then*

$$U_k(t) < 0 \text{ and } U_i(t) > 0 \text{ for any } i, \quad i = 0, 1, ..., k-1.$$

If $t \ge t_k$, then $U_k(t) \ge 0$.

It is convenient to introduce a special notation for the left-hand side of (5.10). For $0 \le k < N$ let

$$T_k(x,y) = \sum_{i=0}^{k} r_i U_i(x) U_i(y). \tag{5.14}$$

For a polynomial $g(t)$ of degree k having k simple roots $x_1, ..., x_k$ denote by $l_i(g;t)$, $i = 1, ..., k$, the Lagrange polynomials of degree $k-1$ such that $l_i(g; x_j) = \delta_{i,j}$. It is clear that

$$l_i(g;t) = \frac{g(t)}{(t - x_i) g'(x_i)}. \tag{5.15}$$

In particular, from (5.10) and (5.11) it follows that for $1 \le k < N$ and for $k = N$ if $N < \infty$,

$$l_i(U_k;t) = \frac{T_{k-1}(t, t_{k,i})}{T_{k-1}(t_{k,i}, t_{k,i})}. \tag{5.16}$$

One of fruitful ideas of the theory of orthogonal polynomials consists in the fact that for any polynomial $f(t)$ of degree at most $2k-1$, $1 \le k < N$, the polynomial

$$f(t) - \sum_{i=1}^{k} f(t_{k,i}) l_i(U_k; t) \tag{5.17}$$

is divisible by $U_k(t)$ and, hence, orthogonal to $U_0(t) \equiv 1$. In particular, this gives the following well-known theorem.

Theorem 5.8 (Gauss-Jacobi formula). *For any polynomial $f(t)$ of degree at most $2k-1$, $1 \le k < N$,*

$$f_0 = \int_A^B f(t) d\nu(t) = \sum_{i=1}^{k} f(t_{k,i}) \lambda_{k,i}$$

where

$$\lambda_{k,i} = \int_A^B l_i(U_k;t) d\nu(t) = \frac{1}{T_{k-1}(t_{k,i}, t_{k,i})} > 0.$$

Note that for discrete systems the orthogonality condition (5.3) takes the form

$$r_i \sum_{l=1}^{N} U_i(t_{N,l}) U_j(t_{N,l}) w_l = \delta_{i,j} \tag{5.18}$$

where $\sum\limits_{l=1}^{N} w_l = 1$. Since $l_i(U_N; t_{N,l}) = \delta_{i,l}$ and $\sum\limits_{l=1}^{N} U_i(t_{N,l}) w_l = \delta_{i,0}$, from (5.16) where $k = N$, it follows that for any i, $i = 1, 2, ..., N$,

$$w_i = \sum_{l=1}^{N} l_i(U_N; t_{N,l}) w_l = \frac{1}{T_{N-1}(t_{N,i}, t_{N,i})}$$

and, hence,

$$w_i \sum_{l=0}^{N-1} U_l(t) U_l(t_{N,i}) r_l = l_i(U_N; t).$$

In particular, we have the following statement.

Lemma 5.9 (the second orthogonality condition for discrete systems). *If $N < \infty$, then for any i and j, $1 \le i, j \le N$,*

$$w_i \sum_{l=0}^{N-1} U_l(t_{N,i}) U_l(t_{N,j}) r_l = \delta_{i,j}. \tag{5.19}$$

If there exists a substitution function $\sigma(l)$ such that for any i, $i = 1, ..., N$, the value $U_l(t_{N,i})$ is a polynomial of degree $i-1$ in $\sigma(l)$, then (5.19) gives the orthogonality condition for these polynomials. In particular, this holds for polynomial graphs (see (4.13)-(4.15)).

The next lemma follows from (5.14), (5.3), and (5.10).

Lemma 5.10. *For any k and l, $0 \le k, l < N$,*

$$\int_A^B T_k(t,x) T_l(t,y) d\nu(t) = T_{\min(k,l)}(x,y), \tag{5.20}$$

$$\int_A^B (t-x) T_k(t,x) T_l(t,y) d\nu(t) = -\delta_{k,l} r_k b_k U_k(x) U_{k+1}(y), \tag{5.21}$$

and if x and y are distinct roots of the equation $(t-\sigma)T_k(t,\sigma) = 0$ for a fixed σ, then

$$T_k(x,y) = 0. \tag{5.22}$$

We further introduce the function

$$R_k(x,y,z) = T_{k-1}(x,y) U_k(z) - T_{k-1}(y,z) U_k(x) \tag{5.23}$$

for $1 \le k < N$ and note that because of (5.14)

$$R_k(x,y,z) = T_k(x,y) U_k(z) - T_k(y,z) U_k(x). \tag{5.24}$$

From (5.10), (5.14), and (5.23) it follows that for $1 \le k < N$

$$(z-y) R_k(x,y,z) = (z-x) R_k(y,x,z) \tag{5.25}$$

and

$$(y-x)(z-y) R_k(x,y,z) \left(r_{k-1} b_{k-1}\right)^{-1} = -(z-x) U_{k-1}(y) U_k(x) U_k(z)$$

$$+U_k(y)\left((z-y)U_{k-1}(x)U_k(z)+(y-x)U_k(x)U_{k-1}(z)\right). \tag{5.26}$$

On the other hand, from (5.10), (5.14), and (5.24) it follows that for $1 \le k < N$

$$(y-x)(z-y)R_k(x,y,z)\left(r_k b_k\right)^{-1} = (z-x)U_{k+1}(y)U_k(x)U_k(z)$$

$$-U_k(y)\left((z-y)U_{k+1}(x)U_k(z)+(y-x)U_k(x)U_{k+1}(z)\right). \tag{5.27}$$

Using (5.3), (5.20), (5.23), (5.24) and then the equalities

$$(z-t)R_k(x,t,z) = (z-x)R_k(t,x,z),$$

$$(t-x)R_k(x,t,z) = (x-t)R_k(z,t,x) = (x-z)R_k(t,z,x),$$

we get we following statement.

Lemma 5.11. *For any k and l $(1 \le k,l < N)$,*

$$r_k \int_A^B R_k(t,x,z)R_l(t,z,x)d\nu(t) = \delta_{k,l}T_{k-1}(x,z)T_k(x,z), \tag{5.28}$$

$$r_k \int_A^B (t-x)(z-t)R_k(x,t,z)R_l(x,t,z)d\nu(t) = -\delta_{k,l}(x-z)^2 T_{k-1}(x,z)T_k(x,z). \tag{5.29}$$

5.2. Bounds on extreme roots of orthogonal polynomials

Theorems 5.3 and 5.8 allow us to give the following useful expressions for the extreme (the smallest and the largest) roots $t_{k,1}$ and $t_{k,k} = t_k$ of the polynomials $U_k(t)$ of degree k which form the system of orthogonal polynomials considered. Note that $t_1 = a_0$.

Theorem 5.12. *For any k, $1 \le k < N$,*

$$t_{k,1} = \min\left(\sum_{i=0}^{k-1} a_i x_i^2 - 2\sum_{i=0}^{k-2} x_i x_{i+1}\sqrt{b_i c_{i+1}}\right), \tag{5.30}$$

$$t_{k,k} = \max\left(\sum_{i=0}^{k-1} a_i x_i^2 + 2\sum_{i=0}^{k-2} x_i x_{i+1}\sqrt{b_i c_{i+1}}\right), \tag{5.31}$$

where the extrema are taken over all (or only over positive) $x_0, x_1, \ldots, x_{k-1}$ such that

$$\sum_{i=0}^{k-1} x_i^2 = 1. \tag{5.32}$$

Proof. By (5.4), (5.6), and (5.13) for any i and k, $0 \le i < k < N$,

$$(-1)^i U_i(t_{k,1}) > 0 \quad \text{and} \quad U_i(t_{k,k}) > 0.$$

To prove (5.30) we consider the polynomial $f(t) = (t - t_{k,1})(g(t))^2$ of degree $2k-1$, where

$$g(t) = \sum_{i=0}^{k-1} x_i \sqrt{r_i}\,(-1)^i\, U_i(t)$$

and x_i, $i = 0, 1, ..., k-1$, satisfy (5.32). Using (5.7) and (5.9) we have

$$f_0 = \int_A^B f(t) d\nu(t) = \int_A^B (t - t_{k,1})\,(g(t))^2\, d\nu(t)$$

$$= \int_A^B g(t) \left(\sum_{i=0}^{k-1} x_i \sqrt{r_i}\,(-1)^i\,(b_i U_{i+1}(t) + (a_i - t_{k,1})\,U_i(t) + c_i U_{i-1}(t)) \right) d\nu(t)$$

$$= \sum_{i=0}^{k-1} a_i x_i^2 - \sum_{i=1}^{k-1} x_i x_{i-1} \sqrt{\frac{r_{i-1}}{r_i}} b_{i-1} - \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{\frac{r_{i+1}}{r_i}} c_{i+1} - t_{k,1} \sum_{i=0}^{k-1} x_i^2$$

$$= \sum_{i=0}^{k-1} a_i x_i^2 - 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_i c_{i+1}} - t_{k,1}.$$

Note that in the special case when

$$x_i = \frac{\sqrt{r_i}\,(-1)^i\, U_i(t_{k,1})}{\sqrt{T_{k-1}(t_{k,1}, t_{k,1})}}, \quad i = 0, 1, ..., k-1, \tag{5.33}$$

(and all $x_i > 0$) the equality (5.32) holds and we have

$$g(t) = \frac{T_{k-1}(t, t_{k,1})}{\sqrt{T_{k-1}(t_{k,1}, t_{k,1})}}$$

and, hence, by (5.16) $f(t_{k,i}) = 0$ for all $i = 1, ..., k$. Using Theorem 5.8 for the polynomial $f(t) = (t - t_{k,1})(g(t))^2$ we get $f_0 \ge 0$ with equality in the special case (5.33). This completes the proof of (5.30). Analogously considering the polynomial $f(t) = (t_{k,k} - t)(g(t))^2$ of degree $2k-1$, where

$$g(t) = \sum_{i=0}^{k-1} x_i \sqrt{r_i} U_i(t)$$

and x_i, $i = 0, 1, ..., k-1$, satisfy (5.32), we prove that (5.31) holds and the maximum is attained when

$$x_i = \frac{\sqrt{r_i} U_i(t_{k,k})}{\sqrt{T_{k-1}(t_{k,k}, t_{k,k})}}, \quad i = 0, 1, ..., k-1. \ \blacksquare \tag{5.34}$$

Remark 5.13. *Though (5.33) and (5.34) are solutions of these extremum problems they are expressed by functions in values we would like to find. Nevertheless relationships (5.30) and (5.31) turn out to be very useful for bounding extreme roots with the help of the known extrema of some simple quadratic forms and the monotonicity of some sequences. The author has not found this elementary and general statement in the literature on orthogonal polynomials (cf. [56]) and published it with applications to coding theory in [71].*

Remark 5.14. *As should be expected Theorem 5.12 is invariant with respect to a choice of normalization (5.4) of polynomials $U_k(t)$ and the measure ν. Indeed, if $\widetilde{U} = \left\{\widetilde{U}_k(t),\ 0 \le k < N\right\}$ where $\widetilde{U}_k(t) = d_k U_k(t)$ and $d_k \neq 0$, then the roots of $\widetilde{U}_k(t)$ and $U_k(t)$ coincide and coefficients $\widetilde{a}_i$, $\widetilde{b}_i$, $\widetilde{c}_i$ of the recurrence for the system $\widetilde{U}$ are connected with those a_i, b_i, c_i for the system U (see (5.7)) as follows*

$$\widetilde{a}_i = a_i,\ \widetilde{b}_i = b_i d_i/d_{i+1},\ \widetilde{c}_i = c_i d_i/d_{i-1} \text{ and, hence, } \widetilde{b}_i\widetilde{c}_{i+1} = b_i c_{i+1}.$$

Thus, applying Theorem 5.12 we can use any normalization of polynomials and measure.

Consider the Hermite polynomials $P_k(t)$ of degree k, $k = 0, 1, \dots$, which are orthogonal on $(-\infty, \infty)$ with the respect to the weight function $\exp(-t^2)$ and which can be defined by the following recurrence:

$$P_{k+1}(t) - 2tP_k(t) + 2kP_{k-1}(t) = 0, \quad P_0(t) = 1$$

(see [14], [113]). Then the largest root h_k of $P_k(t)$, $k = 1, \dots$, by Theorem 5.12 can be represented as follows

$$h_k = \max \sqrt{2} \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{i+1}, \tag{5.35}$$

where the maximum is taken over positive $x_0, x_1, \dots, x_{k-1}$ satisfying (5.32). This root and, hence, the maximum of the corresponding quadratic form were calculated with high accuracy. It is known that $h_1 = 0$, $\sqrt{2}h_2 = 1$, $\sqrt{2}h_3 = \sqrt{3}$, $\sqrt{2}h_4 = \sqrt{3+\sqrt{6}}$, $\sqrt{2}h_5 = \sqrt{5+\sqrt{10}}$, and (see [113])

$$h_k = \sqrt{2k} - c(2k)^{-1/6} + o(k^{-1/6}) \quad \text{as } k \to \infty \tag{5.36}$$

where $c = 1.85575\dots$. We can use Theorem 5.12 and these bounds on the quadratic form (5.35) for estimating the extreme roots of other systems of orthogonal polynomials. We formulate the following statement only for the largest roots; the corresponding statement for the smallest roots is also valid.

Lemma 5.15. *Given k, $2 \le k < N$, suppose that $a_i \le a_{i+1}$, $i = 0, ..., k-2$, and there exists a sequence m_i, $i = 1, ..., k-1$, such that $a_i + \sqrt{b_{i-1}c_i} + \sqrt{b_{i-2}c_{i-1}} \le m_i$ and $m_i \le m_{i+1}$. Then*

$$t_{k,k} \le m_{k-1}. \tag{5.37}$$

If additionally to these properties, $m_i \le a_{i+1} + \sqrt{b_i c_{i+1}} + \sqrt{b_{i-1}c_i} + C$, where C does not depend on i, then for any l ($l = 2, ..., k$),

$$t_{k,k} \ge \frac{l-2}{l}(m_{k-l+1} - C) + \frac{2}{l}a_{k-l}. \tag{5.38}$$

Proof. Using (5.31), the inequality $2x_i x_{i+1} \le x_i^2 + x_{i+1}^2$, and the monotonicity of a_i and m_i we have

$$t_{k,k} \le \sum_{i=0}^{k-2} x_i^2 m_{i+1} + x_{k-1}^2 \left(a_{k-1} + \sqrt{b_{k-2}c_{k-1}}\right) \le m_{k-1}.$$

If we put $x_{k-1} = ... = x_{k-l} = l^{-1/2}$ in (5.31), we get

$$t_{k,k} \ge \frac{1}{l}\left(\sum_{i=k-l}^{k-1} a_i + 2\sum_{i=k-l}^{k-2} \sqrt{b_i c_{i+1}}\right) \tag{5.39}$$

and use the motonicity of a_i and m_i and the upper bound for m_i to obtain (5.38). ∎

We shall see that the proper choice of $l = l(k)$ in Lemma 5.15 can prove that the inequalities (5.37) are asymptotically tight. In particular, this lemma for $m_i = \sqrt{\frac{i}{2}} + \sqrt{\frac{i-1}{2}}$ shows that $\sqrt{2}h_k \le \sqrt{k-1} + \sqrt{k-2}$ and (put $l = l(k) = \lfloor\sqrt{k}\rfloor$) that $h_k = \sqrt{2k} + O(1)$ as $k \to \infty$ (cf. (5.36)).

Apply Theorem 5.12 and Lemma 5.15 to the Jacobi polynomials $P_k^{\alpha,\beta}(t)$ of degree $k = 1, 2, \ldots$, defined by (2.48) and orthogonal on $[-1, 1]$ with respect to weight function $c^{\alpha,\beta}(1-t)^\alpha(1+t)^\beta$ where the normalizing constant $c^{\alpha,\beta}$ is defined by (2.50). Let $p_k^{\alpha,\beta}$ be the largest root of $P_k^{\alpha,\beta}(t)$. In particular,

$$p_1^{\alpha,\beta} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \; p_2^{\alpha,\beta} = \frac{2\sqrt{(\alpha+\beta+3)(2+\alpha)(2+\beta)} - (\alpha+\beta+3)(\alpha-\beta)}{(\alpha+\beta+3)(\alpha+\beta+4)}.$$

For these polynomials the recurrence (5.7) holds for the following coefficients:

$$a_k = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}, \; b_k = \frac{2(k+\alpha+1)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)},$$

$$c_k = \frac{2k(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}.$$

One can verify that $a_{i+1} \ge a_i$, if $\alpha \ge \beta$, and $b_i c_{i+1} \ge b_{i-1}c_i$, if $\alpha + \beta \ge 1$ (otherwise, in general, the latter is not true). Therefore the conditions of Lemma 5.15 are fulfilled for $m_i = a_i + 2\sqrt{b_{i-1}c_i}$ (and for $m_i = a_i + \sqrt{b_{i-1}c_i} + \sqrt{b_{i-2}c_{i-1}}$ as well) with $C = 0$.

Corollary 5.16. *If $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha+\beta \geq 1$, then for every k, $k=2,\dots$,*

$$p_k^{\alpha,\beta} \leq \frac{\beta^2-\alpha^2}{(2k+\alpha+\beta-2)(2k+\alpha+\beta)}$$
$$+4\sqrt{\frac{(k-1)(k+\alpha-1)(k+\beta-1)(k+\alpha+\beta-1)}{(2k+\alpha+\beta-3)(2k+\alpha+\beta-2)^2(2k+\alpha+\beta-1)}}.$$

Moreover, if $\lim\limits_{k\to\infty} \frac{\alpha}{k}=\zeta$ and $\lim\limits_{k\to\infty} \frac{\beta}{k}=\eta$, where $\zeta \geq \eta \geq 0$, then

$$p_k^{\alpha,\beta} = \frac{4\sqrt{(1+\zeta)(1+\eta)(1+\zeta+\eta)}-\zeta^2+\eta^2}{(2+\zeta+\eta)^2}+o(1). \tag{5.40}$$

The asymptotic formula (5.40) was obtained in [57] with the help of Sturm-Liouville arguments.

In the case $\alpha=\beta$ we can also use (5.35) and the fact that $b_i c_{i+1}/(i+1)$ decreases with i to obtain a lower bound.

Corollary 5.17. *If $\alpha=\beta \geq \frac{1}{2}$, then for every k, $k=2,\dots$,*

$$\sqrt{2}h_k \leq p_k^{\alpha,\alpha}\sqrt{\frac{(2k+2\alpha-3)(2k+2\alpha-1)}{k+2\alpha-1}} \leq 2\sqrt{k-1}.$$

Considering systems Q and P for polynomial graphs and using the fact that $\sigma_Q(d)$ and $\sigma_P(d)$ decrease with d, we define values $d_k(Q)$ and $d_k(P)$ by $\sigma_Q(d_k(Q))=t_k(Q)$ and $\sigma_P(d_k(P))=t_k(P)$ (cf. (4.38)). If $\sigma_Q(d)$ (respectively $\sigma_P(d)$) is a linear function, then $d_k(Q)$ (respectively, $d_k(P)$) is the smallest root of a polynomial of degree k. In particular, this is true for Hamming and Johnson spaces with respect to the system Q and the corresponding polynomials are Krawtchouk and Hahn polynomials. Note the following useful restrictions which follow from equalities $Q_k(\sigma_Q(j))=P_j(\sigma_P(k))$ (see (4.15)) and monotonicity of $\sigma_Q(d)$ and $\sigma_P(d)$.

Lemma 5.18. *For a polynomial graph,*

$$d_k(Q) \leq 1 \quad \textit{if} \quad k \geq d_1(P), \tag{5.41}$$

$$d_k(P) \leq 1 \quad \textit{if} \quad k \geq d_1(Q). \tag{5.42}$$

Proof. If $k \geq d_1(P)$, then $\sigma_P(k) \leq \sigma_P(d_1(P)) = t_1(P)$ and hence $Q_k(\sigma_Q(1)) = P_1(\sigma_P(k)) \leq 0$. Therefore $\sigma_Q(1) \leq t_k(Q) = \sigma_Q(d_k(Q))$ and hence (5.41) holds. Analogously (5.42) is proved. ■

For the Hamming space, $P=Q$ and $d_k(P)=d_k(Q)$ is the smallest root $d_k(n)=d_k(n,v)$ of the Krawtchouk polynomial $K_k^n(z)=K_k^{n,v}(z)$ of degree k defined by (2.44). Since $d_1(n)=\frac{v-1}{v}n$, (5.41) means that $d_k(Q)\leq 1$ if $k \geq \frac{v-1}{v}n$. We use Theorem 5.12, Remark 5.14 and the known recurrence

$$(k+1)K_{k+1}^n(z) = ((v-1)n-(v-2)k-vz)\,K_k^n(z)-(v-1)(n-k+1)K_{k-1}^n(z).$$

Corollary 5.19. *For every k, $k = 1, \dots, n$,*

$$d_k(n) = \frac{v-1}{v}n - \frac{1}{v}\max\left((v-2)\sum_{i=0}^{k-1} ix_i^2 + 2\sqrt{v-1}\sum_{i=0}^{k-2} x_i x_{i+1}\sqrt{(i+1)(n-i)}\right) \tag{5.43}$$

where the maximum is taken over positive $x_0, x_1, \dots, x_{k-1}$ satisfying (5.32).

Since $\left(\sum_{i=0}^{k-2} x_i x_{i+1}\sqrt{i+1}\right)^2 \le \sum_{i=0}^{k-2} x_i^2 \sum_{i=1}^{k-1} ix_i^2$, using (5.35) we have

Corollary 5.20. *For every k, $k = 1, \dots, n$,*

$$d_k(n) \le \frac{v-1}{v}n - \frac{v-2}{2v}h_k^2 - \frac{1}{v}\sqrt{2(v-1)(n-k+2)}h_k. \tag{5.44}$$

On the other hand, from Corollary 5.19 and (5.35) it follows that for any k, $k = 1, \dots, n$,

$$vd_k(n) \ge (v-1)n - (v-2)(k-1) - \sqrt{2(v-1)n}h_k. \tag{5.45}$$

It is also true that for any k, $2 \le k \le n(v-1)/v$,

$$\begin{aligned} vd_k(n) \ge (v-1)n - (v-2)(k-1) \\ -\sqrt{v-1}\left(\sqrt{(k-1)(n-k+2)} + \sqrt{(k-2)(n-k+3)}\right) \end{aligned} \tag{5.46}$$

(to estimate from above the maximum in (5.43) one can use, as in the proof of Lemma 5.15, that $m_i \le m_{i+1}$ for $m_i = (v-2)i + \sqrt{v-1}\left(\sqrt{i(n-i+1)} + \sqrt{(i-1)(n-i+2)}\right)$, when $i = 1, ..., k-2$).

By virtue of (5.36), (5.44), and (5.46) the following holds.

Corollary 5.21. *For $n \to \infty$ and any $k = k(n)$, $1 \le k \le n(v-1)/v$,*

$$\frac{d_k(n)}{n} = \gamma_v\left(\frac{k}{n}\right) + o(1), \tag{5.47}$$

where

$$\gamma_v(x) = \frac{1}{v}\left(v - 1 - (v-2)x - 2\sqrt{(v-1)x(1-x)}\right). \tag{5.48}$$

For the Johnson space the system $\{Q_i(t), i = 0, 1, ..., w\}$, defined by (2.47) with the help of Hahn polynomials $J_i(z) = J_i^{n,w}(z)$ (see (2.46)), satisfies (5.7) with coefficients

$$a_i = \frac{(n-2w)(2i(n-i+1) - w(n+2))}{w(n-2i)(n-2i+2)}, \tag{5.49}$$

$$b_i = \frac{2(w-i)(n-w-i)(n-i+1)}{w(n-2i)(n-2i+1)},\ c_i = \frac{2i(w-i+1)(n-w-i+1)}{w(n-2i+1)(n-2i+2)}.$$

The largest root $t_k(Q)$ of $Q_k(t)$ and the smallest root $d_k(Q)$ of $J_i(z)$ are connected as follows:

$$t_k(Q) = 1 - \frac{2}{w} d_k(Q). \tag{5.50}$$

Since $t_1(P) = -\frac{n-w-1}{n-w+1}$ and $d_1(P) = \frac{1}{2}\left(n+1-\sqrt{(n+1)^2-4w(n-w)}\right)$(see a_0 below in (5.56) and (5.55)), by Lemma 5.18, $d_k(Q) \le 1$ if $k \ge d_1(P)$, and, hence, we can consider $k < \frac{1}{2}\left(n+1-\sqrt{(n+1)^2-4w(n-w)}\right) < w$. It is clear that $a_i \le a_{i+1}$, when $0 \le i \le k-2$. For $i = 1, ..., k-1$, define

$$m_i = 1 - \frac{2\left(w(n-w) - i(n-i)\right)}{w\left(n + 2\sqrt{i(n-i)}\right)} + C \text{ with } C = \frac{6\sqrt{k(n-k)}}{w(n-2k)}. \tag{5.51}$$

One can check that $a_i + \sqrt{b_{i-1}c_i} + \sqrt{b_{i-2}c_{i-1}} \le m_i \le a_{i+1} + \sqrt{b_i c_{i+1}} + \sqrt{b_{i-1}c_i} + C$. Moreover, $m_i \le m_{i+1}$, $1 \le i \le k-2$, because the derivative with respect to x of the function $\frac{x-\eta(1-\eta)}{1+2\sqrt{x}}$, where $x = \frac{i(n-i)}{n^2}$ and $\eta = \frac{w}{n}$, is nonnegative in the range $0 \le x \le \eta(1-\eta)$, $0 < \eta \le \frac{1}{2}$. Therefore, for any k, $2 \le k < w$, the conditions of Lemma 5.15 are fulfilled for the sequence m_i and the value C defined by (5.51). In particular, this proves that $t_k(Q) \le m_{k-1}$ and gives the following asymptotic equalities for $t_k(Q)$ and $d_k(Q)$ (see (5.50)).

Corollary 5.22. *If* $\lim\limits_{n\to\infty} \frac{k}{n} = \zeta$ *and* $\lim\limits_{n\to\infty} \frac{w}{n} = \eta$, *where* $0 \le \zeta \le \eta$ *and* $0 < \eta \le \frac{1}{2}$, *then*

$$t_k(Q) = 1 - \frac{2\left(\eta(1-\eta) - \zeta(1-\zeta)\right)}{\eta(1+2\sqrt{\zeta(1-\zeta)})} + o(1), \tag{5.52}$$

$$\frac{d_k(Q)}{n} = \frac{\eta(1-\eta) - \zeta(1-\zeta)}{(1+2\sqrt{\zeta(1-\zeta)})} + o(1). \tag{5.53}$$

We shall use below that for $0 \le i \le k-1 \le w-2$ the coefficients (5.49) have the following properties:

$$c_{i+1} - c_i \le \frac{2(w-i)n}{w(n-2i)(n-2i-1)},$$

$$\sqrt{b_i c_{i+1}} - \sqrt{b_{i+1}c_{i+1}} \le \frac{(n-w-i)n}{w(n-2i)(n-2i-1)},$$

and, hence, provided $\sum_{i=0}^{k-1} x_i^2 = 1$,

$$\sum_{i=0}^{k-1} x_i^2 \left(c_{i+1} - c_i\right) + 2\sum_{i=0}^{k-2} x_i x_{i+1}\left(\sqrt{b_i c_{i+1}} - \sqrt{b_{i+1}c_{i+1}}\right) \le \frac{2n}{w(n-2k+1)}. \tag{5.54}$$

In conclusion we consider the largest zeros $t_k(P) = t_k(n, w)$ of the polynomials $P_k(t) = P_k^{n,w}(t)$ defined for the Johnson space (see Example 4.2 and (4.12)) and the corresponding values $d_k(P)$ such that

$$t_k(P) = 1 - 2\frac{d_k(P)(n+1-d_k(P))}{w(n+1-w)}. \tag{5.55}$$

For the polynomials $P_k(t)$ the recurrence (5.7) holds with

$$a_i = \frac{2in - w(n-w-1) - 4i^2}{w(n+1-w)},\ b_i = \frac{2(w-i)(n-w-i)}{w(n+1-w)},\ c_i = \frac{2i^2}{w(n+1-w)}. \tag{5.56}$$

Since $d_1(Q) = \frac{w(n-w)}{n}$ (see a_0 in (5.49) and (5.49)), by Lemma 5.18, $d_k(P) \le 1$ when $k \ge \frac{w(n-w)}{n}$, and hence we can consider $k < \frac{w(n-w)}{n} \le \frac{n}{4}$. We again have $a_i \le a_{i+1}$, when $0 \le i \le k-2$. For $i = 1, ..., k-1$, define

$$m_i = 1 - \frac{2\left(\sqrt{w(n-w) - i(n-i)} - i\right)^2}{w(n-w+1)} + C \text{ with } C = \frac{3}{w}. \tag{5.57}$$

One can check that $a_i + \sqrt{b_{i-1}c_i} + \sqrt{b_{i-2}c_{i-1}} \le m_i \le a_{i+1} + \sqrt{b_i c_{i+1}} + \sqrt{b_{i-1}c_i} + C$. Moreover, $m_i \le m_{i+1}$, $1 \le i \le k-2$, because the derivative with respect to x of the function $-\left(\sqrt{\tau - x(1-x)} - x\right)^2$, where $x = \frac{i}{n}$ and $\tau = \frac{w(n-w)}{n^2}$, is nonnegative in the range $0 \le x \le \tau \le \frac{1}{4}$. Therefore, for any k, $2 \le k < \frac{w(n-w)}{n}$, the conditions of Lemma 5.15 are fulfilled for the sequence m_i and the value C defined by (5.57). In particular, this proves that $t_k(P) \le m_{k-1}$ and gives the following asymptotic equalities for $t_k(P)$ and $d_k(P)$ (see (5.55)).

Corollary 5.23. *If* $\lim_{n\to\infty} \frac{k}{n} = \zeta$ *and* $\lim_{k\to\infty} \frac{w}{n} = \eta$, *where* $0 \le \zeta \le \eta(1-\eta)$ *and* $0 < \eta \le \frac{1}{2}$, *then*

$$t_k(P) = 1 - \frac{2\left(\sqrt{\eta(1-\eta) - \zeta(1-\zeta)} - \zeta\right)^2}{\eta(1-\eta)} + o(1), \tag{5.58}$$

$$\frac{d_k(P)}{n} = \frac{1}{2}\left(1 - \sqrt{1 - 4\left(\sqrt{\eta(1-\eta) - \zeta(1-\zeta)} - \zeta\right)^2}\right) + o(1). \tag{5.59}$$

5.3. Properties of adjacent systems of orthogonal polynomials

We continue investigation of systems $U = \{U_k(t),\ 0 \le k < N\}$ of orthogonal polynomials introduced in Section 5.1. However now we shall assume that they are orthogonal on the interval $[-1, 1]$ (i.e., $A = -1$, $B = 1$) with the normalization

$U_k(1) = 1$ (see (5.4)). For this normalization by Theorem 5.1 any system U and the corresponding system of positive constants r_i are uniquely determined by the weight function $w(t)$ in the continuous case or by the steps w_i at the points $t_{N,i}$, $i = 1, ..., N$, in the discrete case. For any nonnegative integers a and b we can use the fact to determine *adjacent* (to U) systems $U^{a,b} = \left\{U_k^{a,b}(t),\ 0 \le k < N^{a,b}\right\}$ of orthogonal polynomials (and positive constants $r_i^{a,b}$) considering, respectively, weight functions $w^{a,b}(t) = c^{a,b}(1-t)^a(1+t)^b w(t)$ or steps $w_i^{a,b} = c^{a,b}(1-t_{N,i})^a(1+t_{N,i})^b w_i$ at the same points $t_{N,i}$, $i = 1, ..., N$, where the positive constants $c^{a,b}$ are chosen as follows

$$c^{a,b}\int_{-1}^{1}(1-t)^a(1+t)^b w(t)dt = 1, \quad c^{a,b}\sum_{i=1}^{N}(1-t_{N,i})^a(1+t_{N,i})^b w_i = 1 \tag{5.60}$$

to normalize the corresponding measure. Thus the orthogonality and normalization conditions for the system $U^{a,b}$ may be rewritten in the following form

$$r_i^{a,b}\int_{-1}^{1} U_i^{a,b}(t)U_j^{a,b}(t)c^{a,b}(1-t)^a(1+t)^b d\nu(t) = \delta_{i,j}, \quad U_i^{a,b}(1) = 1. \tag{5.61}$$

In particular, the systems of Jacobi polynomials with integer parameters are adjacent to the system of Legendre polynomials (with $\nu(t) = t$).

Our immediate goal is to establish a connection between the parameters of the adjacent systems $U^{a,b}$ of orthogonal polynomials and those of the original system U. All the content of Sections 5.1 and 5.2 remains valid if we replace the parameters $w(t)$, w_i, $U_k(t)$, N, r_k, a_k, b_k, c_k, $t_{k,i}$, t_k, $T_k(x,y)$, $R_k(x,y,z)$ by the corresponding parameters $w^{a,b}(t)$, $w_i^{a,b}$, $U_k^{a,b}(t)$, $N^{a,b}$, $r_k^{a,b}$, $a_k^{a,b}$, $b_k^{a,b}$, $c_k^{a,b}$, $t_{k,i}^{a,b}$, $t_k^{a,b}$, $T_k^{a,b}(x,y)$, $R_k^{a,b}(x,y,z)$. Note that $U_0^{a,b}(t) \equiv 1$ and $r_0^{a,b} = 1$, the same as for the system U. In the discrete case the number $N^{a,b}$ of positive steps may be less than N. In particular, if $t_{N,1} = -1$ and $t_{N,N} = 1$, then $N^{a,b} = N - 2 + \delta_{a,0} + \delta_{b,0}$.

We shall often take into account that by (5.6), (5.10) and (5.23),

$$\operatorname{sgn} T_k(1,-1) = (-1)^k \quad \text{for } 0 \le k < N^{0,1} \tag{5.62}$$

($T_{N-1}(1,-1) = 0$ if $t_{N,N} = 1$ and $t_{N,1} = -1$; however, $N^{0,1} = N-1$ in this case) and

$$\operatorname{sgn} R_{k+1}(-1,1,1) = (-1)^k \quad \text{for } 1 \le k+1 < N. \tag{5.63}$$

In particular, using (5.61)-(5.63), Theorem 5.1 and Lemmas 5.10 and 5.11 we have the following statement.

Lemma 5.24. *For any k ($0 \le k < N^{a,b}$, respectively)*

$$U_k^{0,1}(t) = \frac{T_k(t,-1)}{T_k(1,-1)}, \; r_k^{0,1} = \frac{(T_k(1,-1))^2}{-c^{0,1}r_k b_k U_k(-1)U_{k+1}(-1)}; \tag{5.64}$$

$$U_k^{1,0}(t) = \frac{T_k(t,1)}{T_k(1,1)},\ r_k^{1,0} = \frac{(T_k(1,1))^2}{c^{1,0} r_k b_k U_k(1) U_{k+1}(1)}; \tag{5.65}$$

$$U_k^{1,1}(t) = \frac{R_{k+1}(-1,t,1)}{R_{k+1}(-1,1,1)},\ r_k^{1,1} = \frac{r_{k+1}(R_{k+1}(-1,1,1))^2}{-4c^{1,1} T_k(1,-1) T_{k+1}(1,-1)}. \tag{5.66}$$

If in the discrete case $N^{1,1} = N - 2$ *and, hence,* $t_{N,N} = 1$ *and* $t_{N,1} = -1$, *then*

$$\frac{U_{N-2}^{1,1}(t)}{U_{N-2}^{1,1}(1)} = \frac{R_{N-1}(-1,t,1)}{R_{N-1}(-1,1,1)} = \frac{(t - t_{N,2}) \cdots (t - t_{N,N-1})}{(1 - t_{N,2}) \cdots (1 - t_{N,N-1})}.$$

Now we deduce some consequences of Lemma 5.24 and previous results.

Corollary 5.25. *For any* k, $0 \le k < N^{1,1}$,

$$U_k^{1,1}(t) = l_k U_k^{0,1}(t) + m_k U_k^{1,0}(t)$$

where

$$l_k = \frac{U_{k+1}(1) T_k(-1,1)}{R_{k+1}(-1,1,1)} U_k^{1,1}(1) > 0 \text{ and } m_k = \frac{-U_{k+1}(-1) T_k(1,1)}{R_{k+1}(-1,1,1)} U_k^{1,1}(1) > 0.$$

In the discrete case this is also true for $k = N^{1,1}$ *when* $N^{1,1} = N - 2$.

Lemma 5.24 and (5.8), (5.9), (5.61) allow us to express the coefficients $a_k^{a,b}$, $b_k^{a,b}$, $c_k^{a,b}$ via parameters of the original system U. In particular,

$$b_k^{1,0} = c_{k+1} \frac{T_{k+1}(1,1)}{T_k(1,1)}, \quad c_k^{1,0} = b_k \frac{T_{k-1}(1,1)}{T_k(1,1)}, \quad b_k^{1,0} c_{k+1}^{1,0} = b_{k+1} c_{k+1},$$

$$a_k^{1,0} = 1 - b_k^{1,0} - c_k^{1,0} = 1 - b_k - c_{k+1} = a_k + c_k - c_{k+1}.$$

Using Theorem 5.15 we have the following.

Corollary 5.26. *For any* k, $2 \le k + 1 < N$,

$$t_k^{1,0} = \max \left(\sum_{i=0}^{k-1} (a_i + c_i - c_{i+1}) x_i^2 + 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_{i+1} c_{i+1}} \right),$$

where the maximum is taken over all $x_0, x_1, \ldots, x_{k-1}$ *such that* $\sum_{i=0}^{k-1} x_i^2 = 1$.

Remark 5.27. *Any adjacent system of orthogonal polynomials* $U^{a,b} = \left\{ U_k^{a,b}(t),\ 0 \le k < N^{a,b} \right\}$ *can serve as the original system* U *and thus the results of Sections 5.1 and 5.2 may be applied to this system. In particular, Lemma 5.24 enables us to express the polynomials* $U_k^{a+1,b}(t)$ *and* $U_k^{a,b+1}(t)$ *in terms of polynomials* $U_k^{a,b}(t)$. *For example,*

$$U_k^{0,2}(t) = \frac{T_k^{0,1}(t,-1)}{T_k^{0,1}(1,-1)}, \qquad U_k^{2,0}(t) = \frac{T_k^{1,0}(t,1)}{T_k^{1,0}(1,1)}, \tag{5.67}$$

$$U_k^{1,1}(t) = \frac{T_k^{1,0}(t,-1)}{T_k^{1,0}(1,-1)} = \frac{T_k^{0,1}(t,1)}{T_k^{0,1}(1,1)}. \tag{5.68}$$

The next lemma follows from this remark, Lemma 5.24, (5.10), (5.14), (5.23), and (5.24).

Lemma 5.28. *For any k , $2 \le k+1 < N$, and any real x and σ*

$$\frac{(t-\sigma)T_{k-1}^{0,1}(t,\sigma)}{(x-\sigma)T_{k-1}^{0,1}(x,\sigma)} = \frac{U_k(t)T_{k-1}(\sigma,-1) - U_k(\sigma)T_{k-1}(t,-1)}{U_k(x)T_{k-1}(\sigma,-1) - U_k(\sigma)T_{k-1}(x,-1)}, \tag{5.69}$$

$$\frac{(t-\sigma)T_{k-1}^{1,0}(t,\sigma)}{(x-\sigma)T_{k-1}^{1,0}(x,\sigma)} = \frac{U_k(t)T_{k-1}(\sigma,1) - U_k(\sigma)T_{k-1}(t,1)}{U_k(x)T_{k-1}(\sigma,1) - U_k(\sigma)T_{k-1}(x,1)}, \tag{5.70}$$

$$\frac{(t-\sigma)T_{k-1}^{1,1}(t,\sigma)}{(x-\sigma)T_{k-1}^{1,1}(x,\sigma)} = \frac{T_k(t,1)T_k(\sigma,-1) - T_k(\sigma,1)T_k(t,-1)}{T_k(x,1)T_k(\sigma,-1) - T_k(\sigma,1)T_k(x,-1)}, \tag{5.71}$$

when the denominators in expressions (5.69)-(5.71) differ from zero.

The following two statements are based on analysis of the equations $(1+t)U_k^{0,1}(t) = 0$, $(1-t^2)U_{k-1}^{1,1}(t) = 0$, and $(1-t)U_k^{1,0}(t) = 0$ by using Lemma 5.24, Theorems 5.2 and 5.4, Corollaries 5.5 and 5.6, and the equalities (5.27) and (5.26), respectively.

Lemma 5.29. *For any j and k such that $1 \le j \le k+1 < N$,*

$$t_{k,j-1} < t_{k,j-1}^{0,1} < t_{k-1,j-1}^{1,1} < t_{k,j}^{1,0} < t_{k,j} \tag{5.72}$$

in the cases when the corresponding entries in (5.72) are defined.

Lemma 5.30. *For any j and k such that $1 \le j \le k-1$, $2 \le k < N$,*

$$t_{k,j} < t_{k-1,j}^{1,0} < t_{k-1,j}^{1,1} < t_{k-1,j}^{0,1} < t_{k,j+1}. \tag{5.73}$$

These lemmas on the separation of roots of adjacent polynomials can be used to prove some results on the monotonicity of the ratio of adjacent polynomials (see [68]).

Lemma 5.31. *The functions*

$$\frac{U_k(t)}{U_k^{1,0}(t)}, \qquad \frac{U_k^{0,1}(t)}{U_k(t)}, \qquad \frac{U_k^{0,1}(t)}{U_k^{1,0}(t)}$$

increase in t in those cases where denominators differ from zero.

Lemma 5.32. *The functions*

$$\frac{U_{k-1}^{0,1}(t)}{U_k(t)}, \qquad \frac{U_{k-1}^{1,1}(t)}{U_k(t)}, \qquad \frac{U_{k-1}^{1,0}(t)}{U_k(t)}$$

decrease in t if $U_k(t) \neq 0$.

One more statement on separation of roots follows from analysis of the equations $(1+t)U_{k-1}^{0,2}(t)=0$ and $(1-t)U_{k-1}^{2,0}(t)=0$ by using (5.67) and Lemma 5.32.

Lemma 5.33. *For any j and k such that $1 \le j \le k-1$, $2 \le k < N$,*

$$t_{k,j} < t_{k-1,j}^{2,0} < t_{k-1,j}^{1,0} < t_{k-1,j}^{0,1} < t_{k-1,j}^{0,2} < t_{k,j+1}. \tag{5.74}$$

From the equalities (5.23), (5.24), and (5.68)-(5.70) we obtain the following statements.

Lemma 5.34. *For any k, $2 \le k < N$, the equation $(t-1)U_{k-1}^{1,1}(t)=0$ is equivalent to either of the equations*

$$U_{k-1}^{0,1}(t) = U_k(t) \text{ and } U_k^{0,1}(t) = U_k(t),$$

and the equation $(t+1)U_{k-1}^{1,1}(t)=0$ is equivalent to either of the equations

$$U_{k-1}^{1,0}(t)U_k(-1) = U_k(t)U_{k-1}^{1,0}(-1) \text{ and } U_k^{1,0}(t)U_k(-1) = U_k(t)U_k^{1,0}(-1).$$

Corollary 5.35. *For any k, $2 \le k < N$,*

$$U_k(t_{k-1}^{1,1}) = U_{k-1}^{0,1}(t_{k-1}^{1,1}) = U_k^{0,1}(t_{k-1}^{1,1})$$

$$= U_{k-1}^{1,0}(t_{k-1}^{1,1})\frac{U_k(-1)}{U_{k-1}^{1,0}(-1)} = U_k^{1,0}(t_{k-1}^{1,1})\frac{U_k(-1)}{U_k^{1,0}(-1)}.$$

Corollary 5.36.

$$\frac{(t-\sigma)T_{k-1}^{1,0}(t,\sigma)}{(1-\sigma)T_{k-1}^{1,0}(1,\sigma)} = \begin{cases} \frac{t+1}{2}U_{k-1}^{1,1}(t) & \text{if } \sigma = t_{k-1}^{1,1},\ 1 \le k < N^{1,1}, \\ U_k^{1,0}(t) & \text{if } \sigma = t_k^{1,0},\ 1 \le k < N^{1,0}, \end{cases} \tag{5.75}$$

$$\frac{(t-\sigma)T_{k-1}^{1,1}(t,\sigma)}{(1-\sigma)T_{k-1}^{1,1}(1,\sigma)} = \begin{cases} U_k^{1,0}(t) & \text{if } \sigma = t_k^{1,0},\ 1 \le k < N^{1,0}, \\ U_k^{1,1}(t) & \text{if } \sigma = t_k^{1,1},\ 1 \le k < N^{1,1}. \end{cases} \tag{5.76}$$

5.4. Main theorem and consequences

Now for a system U of orthogonal polynomials we consider extremum problems of essential interest in coding and design theory of which solutions are expressed in parameters of adjacent systems $U^{a,b}$. In the discrete case we shall assume additionally that $t_{N,N}=1$ (and, hence, $N^{1,0}=N-1$, $N^{1,1}=N^{0,1}-1$). This is fulfilled for the systems Q (and P) for finite polynomial spaces with standard substitutions (see Section 2.1 and 5.1). We denoted by $F[t]$ the set of polynomials in real t with real coefficients. Any polynomial $f \in F[t]$ can be uniquely represented (mod $U_N(t)$ in the discrete case, see (5.2)) in the form $\sum\limits_{i=0}^{N-1} f_i U_i(t)$ where

$$f_i = f_i(U) = r_i \int\limits_{-1}^{1} f(t)U_i(t)d\nu(t). \tag{5.77}$$

We also introduced the notation

$$\Omega(f) = \Omega_U(f) = \frac{f(1)}{f_0(U)} \quad \text{when } f_0(U) \neq 0. \tag{5.78}$$

(Note that in the discrete case polynomials which are equivalent modulo $U_N(t)$ have the same values $f(1)$, $f_0(U)$ and, hence, $\Omega_U(f)$ because of $t_{N,N} = 1$). First for any σ, $-1 \leq \sigma < 1$, we study the problem of finding the infimum of $\Omega_U(f)$ over the class of polynomials $f(t) \in F[t]$ such that $f_0(U) > 0$ and

$$f(t) \leq 0 \quad \text{for } -1 \leq t \leq \sigma. \tag{5.79}$$

We present a solution of this problem with the additional restriction that the degree of $f(t)$ is bounded by some function $h(\sigma)$. For special cases of σ, this solution gives rise to the known solution [96] of the following problem: for any integer τ, $1 \leq \tau < N$, find the maximum of $\Omega_U(f)$ over the class of polynomials $f(t) \in F[t]$ of degree at most τ such that $f_0(U) > 0$ and

$$f(t) \geq 0 \quad \text{for } -1 \leq t \leq 1. \tag{5.80}$$

While describing the results, the following polynomials

$$f_{2k-1}^{(\sigma)}(t) = (t-\sigma)\left(T_{k-1}^{1,0}(t,\sigma)\right)^2 \tag{5.81}$$

and

$$f_{2k}^{(\sigma)}(t) = (t-\sigma)(t+1)\left(T_{k-1}^{1,1}(t,\sigma)\right)^2 \tag{5.82}$$

play the key role. The polynomial $f^{(\sigma)}(t)$ is chosen as one of (5.81) or (5.82) for the proper choice of k. Consider

$$M_{2k-1}(\sigma) = \Omega_U(f_{2k-1}^{(\sigma)}) \text{ for } t_{k-1}^{1,0} \leq \sigma < t_k$$

($1 \leq k < N^{1,0}$, $t_0^{1,0} = -1$) and

$$M_{2k}(\sigma) = \Omega_U(f_{2k}^{(\sigma)}) \text{ for } t_{k-1}^{1,1} \leq \sigma < t_k^{0,1}$$

($1 \le k < N^{1,1}$, $t_0^{1,1} = -1$). Since by (5.70) and (5.14) $f_{2k-1}^{(\sigma)}(t)/f_{2k-1}^{(\sigma)}(1)$ can be represented as follows:

$$\left(\frac{U_k(t)T_{k-1}(\sigma,1) - U_k(\sigma)T_{k-1}(t,1)}{U_k(1)T_{k-1}(\sigma,1) - U_k(\sigma)T_{k-1}(1,1)}\right)\frac{\sum_{i=0}^{k-1} r_i^{1,0}U_i^{1,0}(t)U_i^{1,0}(\sigma)}{\sum_{i=0}^{k-1} r_i^{1,0}U_i^{1,0}(1)U_i^{1,0}(\sigma)},$$

using (5.65) and (5.20) we have

$$\frac{1}{\Omega_U(f_{2k-1}^{(\sigma)})} = \frac{-U_k(\sigma)}{U_k(1)T_{k-1}(\sigma,1) - U_k(\sigma)T_{k-1}(1,1)}$$

and, hence,

$$M_{2k-1}(\sigma) = \left(1 - \frac{U_{k-1}^{1,0}(\sigma)}{U_k(\sigma)}\right)\sum_{i=0}^{k-1} r_i. \tag{5.83}$$

Thus, $M_{2k-1}(\sigma)$ is positive-valued and increases with σ in the half-open interval being considered, by Lemmas 5.30 and 5.32. From definitions (5.78), (5.81), (5.82), and (5.61) it follows that $\Omega_U(f_{2k}^{(\sigma)})$ may be obtained from $\Omega_U(f_{2k-1}^{(\sigma)})$ by replacing all parameters of the system U by those of $U^{0,1}$ and by multiplying by $2c^{0,1}$. Since $2c^{0,1} = 1 - 1/U_1(-1)$ (see (2.74)), this gives

$$M_{2k}(\sigma) = \left(1 - \frac{1}{U_1(-1)}\right)\left(1 - \frac{U_{k-1}^{1,1}(\sigma)}{U_k^{0,1}(\sigma)}\right)\sum_{i=0}^{k-1} r_i^{0,1} \tag{5.84}$$

and implies that $M_{2k}(\sigma)$ is positive-valued and increases with σ in the half-open interval. On the other hand, by (5.71) and (5.14) $f_{2k}^{(\sigma)}(t)/f_{2k}^{(\sigma)}(1)$ can be represented as follows:

$$\left(\frac{T_k(t,1)T_k(\sigma,-1) - T_k(\sigma,1)T_k(t,-1)}{T_k(1,1)T_k(\sigma,-1) - T_k(\sigma,1)T_k(1,-1)}\right)\frac{(t+1)\sum_{i=0}^{k-1} r_i^{1,1}U_i^{1,1}(t)U_i^{1,1}(\sigma)}{2\sum_{i=0}^{k-1} r_i^{1,1}U_i^{1,1}(1)U_i^{1,1}(\sigma)},$$

and by (5.68), (5.10), and (5.65)

$$\begin{aligned}\frac{t+1}{2}U_i^{1,1}(t) = \frac{(t+1)T_i^{1,0}(t,-1)}{2T_i^{1,0}(1,-1)} &= \frac{U_{i+1}^{1,0}(t)U_i^{1,0}(-1) - U_i^{1,0}(t)U_{i+1}^{1,0}(-1)}{U_{i+1}^{1,0}(1)U_i^{1,0}(-1) - U_i^{1,0}(1)U_{i+1}^{1,0}(-1)} \\ &= \frac{T_{i+1}(t,1)T_i(1,-1) - T_i(t,1)T_{i+1}(1,-1)}{T_{i+1}(1,1)T_i(1,-1) - T_i(1,1)T_{i+1}(1,-1)}.\end{aligned}$$

Using (5.20) we have

$$\frac{1}{\Omega_U(f_{2k}^{(\sigma)})} = \frac{T_k(\sigma,-1)}{T_k(1,1)T_k(\sigma,-1) - T_k(\sigma,1)T_k(1,-1)}$$

and obtain another expression

$$M_{2k}(\sigma) = \left(1 - \frac{U_k^{1,0}(\sigma)}{U_k^{0,1}(\sigma)}\right) \sum_{i=0}^{k} r_i. \tag{5.85}$$

From (5.83)-(5.85), Corollary 5.35, and the equality

$$\left(1 - \frac{U_{k-1}^{1,0}(\sigma)}{U_k(\sigma)}\right) \sum_{i=0}^{k-1} r_i = \left(1 - \frac{U_k^{1,0}(\sigma)}{U_k(\sigma)}\right) \sum_{i=0}^{k} r_i \tag{5.86}$$

(see (5.64), (5.23), and (5.24)) it follows that

$$M_{2k-1}(t_k^{1,0}) = M_{2k}(t_k^{1,0}) = M_{2k+1}(t_k^{1,0}) = \sum_{i=0}^{k} r_i, \tag{5.87}$$

$$\begin{aligned} & M_{2k}(t_k^{1,1}) = M_{2k+1}(t_k^{1,1}) = M_{2k+2}(t_k^{1,1}) \\ = & \left(1 - \frac{1}{U_1(-1)}\right) \sum_{i=0}^{k} r_i^{0,1} = \left(1 - \frac{U_k^{1,0}(-1)}{U_{k+1}(-1)}\right) \sum_{i=0}^{k} r_i. \end{aligned} \tag{5.88}$$

Herewith, by lemmas on separation of roots of adjacent polynomials the corresponding functions are defined at the points $t_k^{1,0}$ and $t_k^{1,1}$.

To define the optimal polynomial $f^{(\sigma)}(t)$ we note that by Lemmas 5.29 and 5.30 for any k, $1 \leq k < N-1$, the following inequalities hold

$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1} \text{ where } t_0^{1,1} = -1. \tag{5.89}$$

In the discrete case when $t_{N,N} = 1$ and $t_{N,1} = -1$ we have $N^{1,1} = N-2$ and $t_{N-2}^{1,1} = t_{N,N-1}$. We assume that $t_{N-2}^{1,1} = 1$ in the infinite case. As was remarked before, the inequalities (5.89) mean that the half-open interval $[-1, t_{N-2}^{1,1})$ is partitioned into half-open intervals $[t_{k-1}^{1,1}, t_k^{1,0})$ and $[t_k^{1,0}, t_k^{1,1})$, $k = 1, ..., N-2$. Let $k(\sigma) = k_U(\sigma) = k$ when either $\sigma \in [t_{k-1}^{1,1}, t_k^{1,0})$ or $\sigma \in [t_k^{1,0}, t_k^{1,1})$, and let $\varepsilon(\sigma) = \varepsilon_U(\sigma) = 0$ if $\sigma \in [t_{k-1}^{1,1}, t_k^{1,0})$ for some k and $\varepsilon(\sigma) = \varepsilon_U(\sigma) = 1$ if $\sigma \in [t_k^{1,0}, t_k^{1,1})$ for some k. Any σ, $-1 \leq \sigma < t_{N-2}^{1,1}$, belongs to the unique half-open interval,

$$t_{k(\sigma)-1+\varepsilon(\sigma)}^{1,1-\varepsilon(\sigma)} \leq \sigma < t_{k(\sigma)}^{1,\varepsilon(\sigma)}, \tag{5.90}$$

with the number

$$h(\sigma) = 2k(\sigma) + \varepsilon(\sigma) - 1. \tag{5.91}$$

We also consider the increasing function $\gamma(d) = \gamma_U(d)$ which, in a certain context, is the inverse to $h(\sigma)$ and maps an integer d ($d \geq 2$) to the left end of the half-open interval with the number $d-1$, i.e.,

$$\gamma(d) = t_{l-\theta}^{1,\theta} \quad \text{if} \quad d - 1 = 2l - \theta \tag{5.92}$$

where l is an integer and $\theta \in \{0,1\}$. ($k(t^{1,\theta}_{l-\theta}) = l$, $\varepsilon(t^{1,\theta}_{l-\theta}) = 1-\theta$, and hence $h(t^{1,\theta}_{l-\theta}) = 2l-\theta$.) For any σ, $-1 \le \sigma < t^{1,1}_{N-2}$, we define the polynomial

$$f^{(\sigma)}(t) = (t-\sigma)(t+1)^{\varepsilon}\left(T^{1,\varepsilon}_{k-1}(t,\sigma)\right)^2 \tag{5.93}$$

where $\varepsilon = \varepsilon(\sigma)$ and $k = k(\sigma)$. Thus, given σ we choose as $f^{(\sigma)}(t)$ (5.81) when $\varepsilon(\sigma) = 0$ or (5.82) when $\varepsilon(\sigma) = 1$ with $k = k(\sigma)$ in the both cases. Note that the polynomial (5.93) has degree $h(\sigma) = 2k(\sigma) + \varepsilon(\sigma) - 1$ and satisfies the property (5.79). For any σ, $-1 \le \sigma < t^{1,1}_{N-2}$, we consider the function

$$L_U(\sigma) = \Omega(f^{(\sigma)}) = \frac{f^{(\sigma)}(1)}{f^{(\sigma)}_0(U)}. \tag{5.94}$$

Since $t^{1,0}_{k-1} \le t^{1,1}_{k-1} < t^{1,0}_k < t_k$ and $t^{1,1}_{k-1} < t^{1,0}_k < t^{1,1}_k < t^{0,1}_k$ for $1 \le k < N-1$, we obtain

$$L_U(\sigma) = \begin{cases} M_{2k-1}(\sigma) & \text{if } \ t^{1,1}_{k-1} \le \sigma < t^{1,0}_k \\ M_{2k}(\sigma) & \text{if } \ t^{1,0}_k \le \sigma < t^{1,1}_k \end{cases} \tag{5.95}$$

and, hence, $L_U(\sigma)$ can be represented in the form

$$L_U(\sigma) = \left(1 - \frac{1}{U_1(-1)}\right)^{\varepsilon}\left(1 - \frac{U^{1,\varepsilon}_{k-1}(\sigma)}{U^{0,\varepsilon}_k(\sigma)}\right)\sum_{i=0}^{k-1} r^{0,\varepsilon}_i. \tag{5.96}$$

where $\varepsilon = \varepsilon(\sigma)$ and $k = k(\sigma)$. Summarizing the properties of the function $L_U(\sigma)$, we have the following statement.

Lemma 5.37. *The function $L_U(\sigma)$ (see (5.94) and (5.96)) is a positive-valued strictly increasing continuous function which takes the following values at the left ends of the half-open intervals*

$$L_U(t^{1,\theta}_{l-\theta}) = \left(1 - \frac{1}{U_1(-1)}\right)^{\theta}\sum_{i=0}^{l-\theta} r^{0,\theta}_i \tag{5.97}$$

where l is an integer and $\theta \in \{0,1\}$.

In particular,

$$L_U(-1) = 1 - \frac{1}{U_1(-1)}. \tag{5.98}$$

Using (5.93) and Corollary 5.36 we get the following.

Lemma 5.38.

$$(t-\sigma)f^{(\sigma)}(t) = \begin{cases} c(t+1)^2\left(U^{1,1}_{k-1}(t)\right)^2 & \textit{if } \sigma = t^{1,1}_{k-1} \\ 2c(t+1)\left(U^{1,0}_k(t)\right)^2 & \textit{if } \sigma = t^{1,0}_k \end{cases} \tag{5.99}$$

where, for the corresponding σ, $c = \frac{1}{4}(1-\sigma)f^{(\sigma)}(1) > 0$.

The polynomials $f_{2k-1}^{(\sigma)}(t)$, $f_{2k}^{(\sigma)}(t)$, and $f^{(\sigma)}(t)$ were introduced and the corresponding upper bounds on the size of d-codes in polynomial metric spaces were presented by the author in [64] (see also [67], [68]). Some extremum properties of the polynomials $f_{2k-1}^{(\sigma)}(t)$ and $f_{2k}^{(\sigma)}(t)$ were found by Sidelnikov in [105]. These results are summarized in the following statement [68],[71] which, to a certain extent, is a generalization of the Gauss-Jacobi formula.

Theorem 5.39. *For any σ, $-1 \le \sigma < t_{N-2}^{1,1}$, the polynomial*

$$(t-\sigma)(t+1)^{\varepsilon} T_{k-1}^{1,\varepsilon}(t,\sigma) \tag{5.100}$$

with $k = k(\sigma)$ and $\varepsilon = \varepsilon(\sigma)$ has $k+\varepsilon$ simple roots $\alpha_0, \alpha_1, \ldots, \alpha_{k+\varepsilon-1}$ ($\alpha_0 < \alpha_1 < \ldots < \alpha_{k+\varepsilon-1}$) where $\alpha_{k+\varepsilon-1} = \sigma$ and $\alpha_0 \ge -1$ with equality holding if and only if $\varepsilon = 1$ or $\varepsilon = 0$ and $\sigma = t_{k-1}^{1,1}$. Moreover, for any polynomial $f(t)$ of degree at most $h(\sigma) = 2k-1+\varepsilon$ the following equality holds

$$f_0(U) = \int_{-1}^{1} f(t)d\nu(t) = (L_U(\sigma))^{-1} f(1) + \sum_{j=0}^{k+\varepsilon-1} \rho_j^{(\sigma)} f(\alpha_j) \tag{5.101}$$

where (defined below by (5.112) and (5.113)) coefficients $\rho_j^{(\sigma)}$, $j = 1, \ldots, k+\varepsilon-1$, are positive and $\rho_0^{(\sigma)} \ge 0$ with equality holding if and only if $\sigma = t_k^{1,0}$.

Proof. Using Lemmas 5.29 and 5.30, and the definition of $k = k(\sigma)$ and $\varepsilon = \varepsilon(\sigma)$, we have

$$t_{k-1}^{1,\varepsilon} < t_{k-1+\varepsilon}^{1,1-\varepsilon} \le \sigma < t_k^{1,\varepsilon}. \tag{5.102}$$

Moreover, in accord with Remark 5.27 and Lemma 5.33,

$$t_{k-1}^{1,1+\varepsilon} \le t_{k-1+\varepsilon}^{1,1-\varepsilon}. \tag{5.103}$$

Hence by Corollary 5.7,

$$U_{k-1}^{1,\varepsilon}(\sigma) > 0 \tag{5.104}$$

and

$$U_{k-1}^{1,1+\varepsilon}(\sigma) = \frac{T_{k-1}^{1,\varepsilon}(\sigma,-1)}{T_{k-1}^{1,\varepsilon}(1,-1)} \ge 0 \tag{5.105}$$

It is easy to see that equality in (5.105) holds if and only if it takes place in (5.102) and (5.103), i.e., when $\varepsilon = 0$ and $\sigma = t_{k-1}^{1,1}$. Since by Theorem 5.4 (for the system $U^{1,\varepsilon}$),

$$(t-\sigma)T_{k-1}^{1,\varepsilon}(t,\sigma) = r_{k-1}^{1,\varepsilon} b_{k-1}^{1,\varepsilon} \left(U_k^{1,\varepsilon}(t) U_{k-1}^{1,\varepsilon}(\sigma) - U_{k-1}^{1,\varepsilon}(t) U_k^{1,\varepsilon}(\sigma) \right), \tag{5.106}$$

the equation $(t-\sigma)T_{k-1}^{1,\varepsilon}(t,\sigma)=0$ is equivalent to

$$\frac{U_k^{1,\varepsilon}(t)}{U_{k-1}^{1,\varepsilon}(t)}=\frac{U_k^{1,\varepsilon}(\sigma)}{U_{k-1}^{1,\varepsilon}(\sigma)}.$$

The equality (5.106) for $t=-1$ and (5.105) show that

$$\frac{U_k^{1,\varepsilon}(-1)}{U_{k-1}^{1,\varepsilon}(-1)}\leq\frac{U_k^{1,\varepsilon}(\sigma)}{U_{k-1}^{1,\varepsilon}(\sigma)} \tag{5.107}$$

with equality if and only if $\varepsilon=0$ and $\sigma=t_{k-1}^{1,1}$. Let $t_{k-1,0}^{1,\varepsilon}=-\infty$ and $t_{k-1,k}^{1,\varepsilon}=\infty$. Since the ratio $U_k^{1,\varepsilon}(t)/U_{k-1}^{1,\varepsilon}(t)$ increases in each interval $(t_{k-1,i}^{1,\varepsilon},t_{k-1,i+1}^{1,\varepsilon})$, $i=0,1,...,k-1$, from -∞ to ∞, the polynomial $(t-\sigma)T_{k-1}^{1,\varepsilon}(t,\sigma)$ has a zero (which it is convenient to denote) $\alpha_{i+\varepsilon}$ in each such interval. Moreover, according to (5.107), $\alpha_\varepsilon\geq-1$ with equality if and only if $\varepsilon=0$ and $\sigma=t_{k-1}^{1,1}$, and we have $\alpha_{k-1+\varepsilon}=\sigma$, since $t_{k-1}^{1,1}<\sigma$. Denoting $\alpha_0=-1$ when $\varepsilon=1$ we complete the proof of the theorem concerning the zeros of (5.100).

Consider for the polynomial $g(t)=(t-1)(t-\sigma)(t+1)^\varepsilon T_{k-1}^{1,\varepsilon}(t,\sigma)$ having $k+1+\varepsilon$ simple roots $\alpha_0,\alpha_1,\ldots,\alpha_{k+\varepsilon}$ $(-1\leq\alpha_0<\alpha_1<\ldots<\alpha_{k+\varepsilon-1}=\sigma<\alpha_{k+\varepsilon}=1)$ the Lagrange polynomials $l_j(g;t)$, $j=0,1,...,k+\varepsilon$, of degree $k+\varepsilon$ such that $l_j(g;\alpha_i)=\delta_{i,j}$, $0\leq i,j\leq k+\varepsilon$ (see (5.15)). Using the last statement of Lemma 5.10 we have

$$l_{k+\varepsilon}(g;t)=\frac{(t+1)^\varepsilon(t-\sigma)T_{k-1}^{1,\varepsilon}(t,\sigma)}{2^\varepsilon(1-\sigma)T_{k-1}^{1,\varepsilon}(1,\sigma)}, \tag{5.108}$$

$$l_{i+\varepsilon}(g;t)=\frac{(t+1)^\varepsilon(t-1)T_{k-1}^{1,\varepsilon}(t,\alpha_{i+\varepsilon})}{(\alpha_{i+\varepsilon}+1)^\varepsilon\,(\alpha_{i+\varepsilon}-1)T_{k-1}^{1,\varepsilon}(\alpha_{i+\varepsilon},\alpha_{i+\varepsilon})},\ i=0,1,...,k-1, \tag{5.109}$$

$$l_0(g;t)=\frac{(t-1)(t-\sigma)T_{k-1}^{1,1}(t,\sigma)}{2(1+\sigma)T_{k-1}^{1,1}(-1,\sigma)}\quad\text{when } \varepsilon=1 \text{ and } \alpha_0=-1. \tag{5.110}$$

For any polynomial $f(t)$ of degree at most $2k-1+\varepsilon$, the polynomial

$$f(t)-\sum_{j=0}^{k+\varepsilon}f(\alpha_j)l_j(g;t) \tag{5.111}$$

equals zero at all (simple) roots of $g(t)$ and hence can be represented in the form $g(t)h(t)$ where $h(t)$ is a polynomial of degree at most $k-2$. Using the Christoffel-Darboux formulae (5.10) for $(t-\sigma)T_{k-1}^{1,\varepsilon}(t,\sigma)$ and the orthogonality relations (5.61) for the system $U^{1,\varepsilon}$ as well, we can see that $\int_{-1}^{1}g(t)h(t)d\nu(t)=0$. This gives

$$f_0(U)=\sum_{j=0}^{k+\varepsilon}\rho_j^{(\sigma)}f(\alpha_j)\quad\text{where}\quad\rho_j^{(\sigma)}=\int_{-1}^{1}l_j(g;t)d\nu(t).$$

Using (5.70), (5.71), (5.83), (5.85), Theorem 5.4, and (5.61) we find that

$$\rho_{k+\varepsilon}^{(\sigma)} = (L_U(\sigma))^{-1} ,$$

$$\rho_{i+\varepsilon}^{(\sigma)} = \frac{1}{c^{1,\varepsilon}\,(1+\alpha_{i+\varepsilon})^{\varepsilon}\,(1-\alpha_{i+\varepsilon})\,T_{k-1}^{1,\varepsilon}(\alpha_{i+\varepsilon},\alpha_{i+\varepsilon})} > 0,\ i=0,...,k-1, \tag{5.112}$$

$$\rho_0^{(\sigma)} = \frac{T_k(\sigma,1)}{T_k(-1,-1)T_k(\sigma,1)-T_k(-1,1)T_k(\sigma,-1)} \ \text{ for } \varepsilon = 1 \ . \tag{5.113}$$

Since $U_k^{1,0}(\sigma) \geq 0$ with equality if and only if $\sigma = t_k^{1,0}$, it remains to prove that for $\sigma > t_k^{1,0}$,

$$\frac{T_k(-1,-1)T_k(1,1)}{(T_k(1,-1))^2} - \frac{U_k^{0,1}(\sigma)}{U_k^{1,0}(\sigma)} > 0$$

and, hence, $\rho_0^{(\sigma)} > 0$. This is true because by Lemma 5.31 $U_k^{0,1}(\sigma)/U_k^{1,0}(\sigma)$ increases for $t_k^{1,0} < \sigma \leq 1$ from $-\infty$ to 1 and by the Cauchy inequality $(T_k(1,-1))^2 \leq T_k(-1,-1)T_k(1,1)$. ■

Corollary 5.40. *For any σ, $-1 \leq \sigma < t_{N-2}^{1,1}$,*

$$L_U(\sigma) = \min \Omega_U(f) \tag{5.114}$$

where the minimum is taken over the class of polynomials $f(t) \in F[t]$ of degree at most $h(\sigma)$ such that $f_0(U) > 0$ and condition (5.79) is fulfilled. Equality in (5.114) takes place if and only if $f(t)$ is proportional to $f^{(\sigma)}(t)$ or to $f^{(\sigma)}(t)/(t+1)$ when $\sigma = t_{l-\theta}^{1,\theta}$ for some integer l and $\theta \in \{0,1\}$.

Proof. By Theorem 5.39 for any polynomial $f(t)$ of degree at most $h(\sigma) = 2k(\sigma) + \varepsilon(\sigma) - 1$ satisfying conditions (5.79) and $f_0(U) > 0$,

$$L_U(\sigma) \leq \Omega_U(f) \tag{5.115}$$

with equality if and only if all roots of (5.100) are roots of $f(t)$ except the root $\alpha_0 = -1$ when $\rho_0^{(\sigma)} = 0$ and, hence, $\sigma = t_k^{1,0}$. However, condition (5.79) implies that all roots $\alpha_1, \ldots, \alpha_{k+\varepsilon-2}$ and α_0 when $\alpha_0 \neq -1$ must be double. It follows that equality in (5.115) is attained only for $f^{(\sigma)}(t)$ and for polynomials obtained from $f^{(\sigma)}(t)$ removing the second root $\alpha_0 = -1$ for $\sigma = t_{k-1}^{1,1}$ and the same root for $\sigma = t_k^{1,0}$ (see (5.93) and (5.99)). ■

Corollary 5.41 ([96]). *For any $\tau = 2l - \theta$ where $l \in \{1, ..., N-1\}$ and $\theta \in \{0,1\}$,*

$$L_U(t_{l-\theta}^{1,\theta}) = \left(1 - \frac{1}{U_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r_i^{0,\theta} = \max \Omega_U(f) \tag{5.116}$$

where the maximum is taken over the class of polynomials $f(t) \in F[t]$ of degree at most τ such that $f_0(U) > 0$ and condition (5.80) is fulfilled. The maximum in (5.116) takes place if and only if $f(t)$ is proportional to $(t+1)^{\theta}\left(U_{l-\theta}^{1,\theta}(t)\right)^2$.

Proof. Let $\sigma = t_{l-\theta}^{1,\theta}$ and, hence, $k(\sigma) = l$, $\varepsilon(\sigma) = 1-\theta$, and $h(\sigma) = 2k(\sigma) - \varepsilon(\sigma) + 1 = 2l - \theta = \tau$. By Theorem 5.39 for any polynomial $f(t)$ of degree at most τ satisfying the conditions (5.80) and $f_0(U) > 0$,

$$L_U(t_{l-\theta}^{1,\theta}) \geq \Omega_U(f) \tag{5.117}$$

with equality if and only if all roots of (5.100) (with $\sigma = t_{l-\theta}^{1,\theta}$) are roots of $f(t)$ except for the root $\alpha_0 = -1$ when $\rho_0^{(\sigma)} = 0$ and, hence, $\sigma = t_l^{1,0}$. However, in this case condition (5.80) implies that all roots $\alpha_1, \ldots, \alpha_{k+\varepsilon-2}$ and $\alpha_{k+\varepsilon-1} = \sigma$ must be double and $\alpha_0 = -1$ must be of multiplicity at least $\theta = 1 - \varepsilon(\sigma)$. This means that the polynomial $f(t)$ of degree at most $\tau = h(\sigma)$ for which (5.80) is satisfied must be proportional to $\frac{t-\sigma}{t+1} f^{(\sigma)}(t)$ and completes the proof in accord with Lemma 5.38. ■

Dunkl [38] was the first who applied this result to design theory.

For some applications of Theorem 5.39 and Corollaries 5.40, 5.41 the optimal polynomial $f^{(\sigma)}(t)$ must have the additional property that all coefficients $f_i^{(\sigma)}(U)$, $i = 0, 1, ..., h(\sigma)$, of its expansion over the system U (not only $f_0^{(\sigma)}(U)$) are positive. Now we present a certain sufficient condition when this is true.

Similar to (5.77) for any polynomial $f(t) \in F[t]$ we define coefficients

$$f_i(U^{a,b}) = r_i^{a,b} \int_{-1}^{1} f(t) U_i^{a,b}(t) c^{a,b} (1-t)^a (1+t)^b d\nu(t), \ 0 \leq i < N^{a,b}. \tag{5.118}$$

Denote by $F_{\geq}(U^{a,b})$ (respectively, $F_{>}(U^{a,b})$) the set of polynomials $f(t) \in F[t]$ such that

$$f_i(U^{a,b}) \geq 0, \ 0 \leq i < N^{a,b}$$

(respectively, such that

$$f_i(U^{a,b}) > 0, \ i = 0, 1, ..., h-1, \text{ and } f_i(U^{a,b}) = 0, \ h \leq i < N^{a,b}$$

for some h, $1 \leq h < N^{a,b}$). In particular,

$$U_i^{1,0}(t) \in F_{>}(U) \tag{5.119}$$

by (5.65) and (5.14). We say that the system $U = \{U_i(t), \ 0 \leq i < N\}$ satisfies the *Krein condition* if

$$U_i(t) U_j(t) \in F_{\geq}(U) \text{ for any } i, j, \ 0 \leq i, j < N. \tag{5.120}$$

It is clear that if U satisfies the Krein condition, then

$$f(t) g(t) \in F_{>}(U) \text{ when } f(t) \in F_{>}(U) \text{ and } g(t) \in F_{>}(U). \tag{5.121}$$

We say that the system $U = \{U_i(t), \ 0 \leq i < N\}$ satisfies the *strengthened Krein condition* if together with (5.120) the following holds:

$$(t+1) U_i^{1,1}(t) U_j^{1,1}(t) \in F_{>}(U) \text{ for any } i, j, \ 0 \leq i, j < N^{1,1}. \tag{5.122}$$

We know (see Corollary 3.13) that in the case of polynomial spaces the system Q constructed in Section 2.2 satisfies the Krein condition. Moreover, by Lemmas 3.22 and 3.25 it satisfies the strengthened Krein condition for infinite distance-transitive polynomial spaces (in particular, for the Euclidean sphere S^{n-1}) and for finite decomposable polynomial spaces (in particular, for the Hamming space H_v^n and Johnson space J_w^n). This also holds [68] for all *antipodal* spaces defined as polynomial spaces X with a standard substitution $\sigma(d)$ for which for every $x \in X$ there exists a point $\overline{x} \in X$ such that $\sigma(d(x,y)) + \sigma(d(\overline{x},y)) = 0$ for any $y \in X$. For antipodal spaces (which include S^{n-1}, H_v^n for $v = 2$ and J_w^n for even n and $w = n/2$), the system Q has the following property: $Q_i(-t) = (-1)^i Q_i(t)$, $i = 0, ..., N-1$.

Theorem 5.42. *If a system $U = \{U_i(t), 0 \le i < N\}$ satisfies the strengthened Krein condition, then for any σ, $-1 \le \sigma < t_{N-2}^{1,1}$,*

$$f^{(\sigma)}(t) \in F_>(U). \tag{5.123}$$

Proof. By Theorem 5.4 and (5.14) (for the system $U^{1,\varepsilon}$) the polynomial $f^{(\sigma)}(t)$ defined by (5.93) can be represented in the form

$$(t+1)^{\varepsilon} r_{k-1}^{1,\varepsilon} b_{k-1}^{1,\varepsilon} \left(U_k^{1,\varepsilon}(t) U_{k-1}^{1,\varepsilon}(\sigma) - U_{k-1}^{1,\varepsilon}(t) U_k^{1,\varepsilon}(\sigma) \right) \sum_{i=0}^{k-1} r_i^{1,\varepsilon} U_i^{1,\varepsilon}(t) U_i^{1,\varepsilon}(\sigma) \tag{5.124}$$

where $\varepsilon = \varepsilon(\sigma)$ and $k = k(\sigma)$. Since $t_{k-1}^{1,0} < t_{k-1}^{1,1} < t_k^{1,0}$, according to Corollary 5.7 (for the system $U^{1,0}$) we have $U_k^{1,0}(\sigma) < 0$ and $U_i^{1,0}(\sigma) > 0$ for $i = 0, ..., k-1$ when $t_{k-1}^{1,1} \le \sigma < t_k^{1,0}$. Using the Krein condition, (5.119), and (5.121) we obtain that the polynomial (5.124) belongs to $F_>(U)$ when $\varepsilon = 0$ (note that simultaneously we proved that $f_{2k-1}^{(\sigma)}(t) \in F_>(U)$ when $t_{k-1}^{1,0} \le \sigma < t_k^{1,0}$). For $\varepsilon = 1$ and $t_k^{1,0} \le \sigma < t_k^{1,1}$ analogously we have $U_k^{1,1}(\sigma) < 0$ and $U_i^{1,1}(\sigma) > 0$ for $i = 0, ..., k-1$ by Corollary 5.7 (for the system $U^{1,1}$) and then we use the strengthened Krein condition. ■

From the proof it follows that (5.123) is valid if the systems U and $U^{1,1}$ satisfy the Krein condition. This was noted in [15]. However, (3.89), (3.91), and (3.93) show that this condition is stronger than the strengthened Krein condition.

Theorem 5.43. *If a system $U = \{U_i(t), 0 \le i < N\}$ satisfies the Krein condition, then $f_{2k-1}^{(\sigma)}(t) \in F_>(U)$ when $t_{k-1}^{1,0} \le \sigma < t_k^{1,0}$ and $f_{2k}^{(\sigma)}(t) \in F_>(U)$ when $\sigma = t_k^{1,0}$.*

Proof. The first statement follows from the last proof. For $\sigma = t_k^{1,0}$ by virtue of (5.99), (5.119), and (5.121) it is sufficient to prove that $(t+1)U_k^{1,0}(t)/(t-\sigma)$ can be expanded over the system U with positive coefficients. By (5.22) this polynomial coincides with

$$\frac{2}{(1-\sigma)T_k(1,1)} \left(T_k(t,1) - T_k(t,\sigma) \frac{T_k(-1,1)}{T_k(-1,\sigma)} \right)$$

because they have the same roots and are equal at $t = 1$. Therefore it remains to prove that

$$\xi_i = 1 - U_i(\sigma)\frac{T_k(-1,1)}{T_k(-1,\sigma)} > 0 \quad \text{for } i = 0, 1, ..., k.$$

Since $t^{0,1}_{k,k-1} < t^{1,0}_k < t^{0,1}_k$, $-T_k(-1,\sigma)/T_k(-1,1) = -U^{0,1}_k(\sigma) > 0$. In general, we do not know the sign of $U_i(\sigma)$; however, for any $i = 0, 1, ..., k-1$, $T_i(1,\sigma) > 0$ and hence $U_i(\sigma) > U_{i+1}(\sigma)$, and $\xi_i > \xi_k$. Using (5.24) and (5.25) we have

$$\xi_k = \frac{R_k(\sigma,-1,1)}{T_k(-1,\sigma)} = \frac{(1-\sigma)\, R_k(-1,\sigma,1)}{2T_k(-1,\sigma)} = \frac{1-\sigma}{2}.$$

■

Corollary 5.44. *Given that the system U satisfies the Krein condition, for any $l \in \{1, ..., N-1\}$ and $\theta \in \{0,1\}$,*

$$L_U(t^{1,\theta}_{l-\theta}) = \left(1 - \frac{1}{U_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r^{0,\theta}_i = \min \Omega_U(f) \tag{5.125}$$

where the minimum is taken over the class of polynomials $f(t) \in F_>(U)$ of degree at most $2l-\theta$ such that condition (5.79) is fulfilled for $\sigma = t^{1,\theta}_{l-\theta}$.

Corollary 5.45. *For any σ, $-1 \le \sigma < t^{1,0}_{N-2}$, the polynomial*

$$\widetilde{f}^{(\sigma)}(t) = (t-\sigma)\left(T^{1,0}_{m-1}(t,\sigma)\right)^2 \tag{5.126}$$

where $m = k(\sigma) + \varepsilon(\sigma)$ or, equivalently, $t^{1,0}_{m-1} \le \sigma < t^{1,0}_m$ $(t^{1,0}_0 = -1)$, has properties (5.79), $\widetilde{f}^{(\sigma)}_0 > 0$, and also $\widetilde{f}^{(\sigma)}(t) \in F_>(U)$ if the system U satisfies the Krein condition. The function

$$\widetilde{L}_U(\sigma) = \left(1 - \frac{U^{1,0}_{m-1}(\sigma)}{U_m(\sigma)}\right) \sum_{i=0}^{m-1} r_i \quad \text{with} \quad m = k(\sigma) + \varepsilon(\sigma)$$

being equal to $\Omega_U\left(\widetilde{f}^{(\sigma)}\right)$ is a positive-valued strictly increasing continuous function which takes the following values at the left ends of the half-open intervals:

$$\widetilde{L}_U(t^{1,\theta}_{l-\theta}) = L_U(t^{1,\theta}_{l-\theta}) = \left(1 - \frac{1}{U_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r^{0,\theta}_i \tag{5.127}$$

where l is an integer and $\theta \in \{0,1\}$.

Thus, $\widetilde{L}_U(\sigma) = L_U(\sigma)$ if $t_{k-1}^{1,1} \le \sigma \le t_k^{1,0}$ and it is possible to show [67] that $\widetilde{L}_U(\sigma) > L_U(\sigma)$ if $t_k^{1,0} < \sigma < t_k^{1,1}$ (this is not a consequence of Corollary 5.40 because the polynomial $\widetilde{f}^{(\sigma)}(t)$ has degree $2k+1 = h(\sigma)+1$ when σ lies inside the last interval).

One more extremum property of polynomials $f_{2k-1}^{(\sigma)}(t)$ and $f_{2k}^{(\sigma)}(t)$ was found by Sidelnikov [105] (see also [68]).

Corollary 5.46. *For any $\tau = 2l - 1 + \theta$ with an integer l and $\theta \in \{0, 1\}$,*

$$t_l^{0,\theta} = \sup \sigma$$

where the supremum is taken over all σ, $-1 \le \sigma < 1$, for which there exists a polynomial $f(t)$ of degree at most τ such that $f_0(U) > 0$ and $f(t) \le 0$ when $-1 \le t \le \sigma$.

This result was used in [39] to prove that the inequality of Theorem 4.14, which extends Tietäväinen's result [117] for the Hamming space to the case of any polynomial space (see Remark 4.16), cannot be improved by the method considered.

Note that by Corollary 5.40 $f^{(\sigma)}(t)$ also minimizes $\Omega_U(f) = f(1)/f_0$ over the class of polynomials $f(t) = \sum_{i=0}^{h(\sigma)} f_i U_i(t)$ such that $f_0 > 0$, $f_i \ge 0$, $i = 1, ..., h(\sigma)$, and condition (5.79) is fulfilled. Using Theorem 5.39 Boyvalenkov, Danev, and Bumova [20] (see also [21]) obtained necessary and sufficient conditions for the optimality of $f^{(\sigma)}(t)$ over the same class of polynomials but without the restriction that their degree does not exceed $h(\sigma)$. To describe this result, for any σ, $-1 \le \sigma < t_{N-2}^{1,1}$, we consider on $F[t]$ the following linear functional G_σ :

$$G_\sigma(f) = (L_U(\sigma))^{-1} f(1) + \sum_{j=0}^{k(\sigma)+\varepsilon(\sigma)-1} \rho_j^{(\sigma)} f(\alpha_j).$$

By Theorem 5.39, $G_\sigma(f) = f_0(U)$ for any polynomial $f(t)$ of degree at most $h(\sigma)$ and $G_\sigma(f) = (L_U(\sigma))^{-1} f(1)$ if $f(t)$ is divisible by $f^{(\sigma)}(t)$.

Theorem 5.47 ([20]). *Provided $f^{(\sigma)}(t) = \sum_{j=0}^{h(\sigma)} f_i^{(\sigma)} U_i(t) \in F_>(U)$ (Theorem 5.42 gives a sufficient condition for this property),*

$$L_U(\sigma) = \min \Omega_U(f),$$

where the minimum is taken over the class of polynomials $f(t) \in F[t]$ such that
(i) $f_0(U) > 0$, $f_i(U) \ge 0$, $i = 1, ..., \deg f$,
(ii) $f(t) \le 0$ for $-1 \le t \le \sigma$,
if and only if $G_\sigma(U_j) \ge 0$ for all j, $h(\sigma) < j < N$.

Proof. Let the condition of the theorem be fulfilled, where $f(t) = \sum_{i=0}^{l} f_i U_i(t)$ has properties (i) and (ii). By Corollary 5.40 we can assume that $l > h(\sigma)$. Applying G_σ to $\sum_{i=0}^{h(\sigma)} f_i U_i(t)$ we get

$$f_0 = (L_U(\sigma))^{-1} f(1) + \sum_{j=0}^{h(\sigma)} \rho_j^{(\sigma)} f(\alpha_j) - \sum_{j=h(\sigma)+1}^{l} f_j G_\sigma(U_j) \leq (L_U(\sigma))^{-1} f(1)$$

and, hence, $\Omega(f) \geq L_U(\sigma)$. Let, conversely, $G_\sigma(U_j) < 0$ for some j, $h(\sigma) < j < N$. There exist unique polynomials $a(t)$ of degree $j - h(\sigma)$ and $b(t) = \sum_{i=0}^{h(\sigma)-1} b_i U_i(t)$ such that $U_j(t) = f^{(\sigma)}(t)a(t) + b(t)$. Consider $f(t) = f^{(\sigma)}(t)(a(t) + c) = U_j(t) - b(t) + cf^{(\sigma)}(t)$, where

$$c = \max\left(- \min_{-1 \leq t \leq \sigma} a(t), \quad \max_{0 \leq i \leq h(\sigma)-1} \frac{b_i}{f_i^{(\sigma)}}, \ 0 \right).$$

This choice of c implies that $f_i(U) \geq 0$, $i = 0, ..., j$, and $f(1) > 0$ (we used $f_i^{(\sigma)} > 0$, $i = 0, ..., h(\sigma)$) and that $f(t) \leq 0$ for $-1 \leq t \leq \sigma$ because $a(t) + c \geq 0$ and $f^{(\sigma)}(t) \leq 0$ in this range. Since $G_\sigma(f) = (L_U(\sigma))^{-1} f(1) = G_\sigma(U_j) + f_0(U) < f_0(U)$ and $f(1) > 0$, we have $f_0(U) > 0$ and $\Omega(f) < L_U(\sigma)$. This completes the proof. ■

5.5. Applications to polynomial metric spaces and polynomial graphs

For a compact metric space $X = (X, d(x, y))$ we denoted by $A(X, d)$ the maximum size of a code $C \subseteq X$ with minimal distance $d(C) \geq d$ (see (2.1)). Considering on X a measure μ and fixing a continuous strictly monotone real function $\sigma(d)$ (substitution function), by means of (2.12) we defined a weighted τ-design $C = (C, m)$ where m is a certain positive-valued function on C, $\sum_{x \in C} m(x) = |C|$, and, in particular, a (simple) τ-design $C = (C, m)$ for which $m(x) \equiv 1$ for all $x \in C$ (see also (3.58) for polynomial spaces). The value $d'(C) = \tau(C) + 1$ was called the dual distance of C where $\tau(C)$ is the strength of C (see (2.13)). We denoted by $B(X, d)$ the minimum size of a (weighted) set $C \subseteq X$ with dual distance $d'(C) \geq d$ (see (2.14)). In Section 2.2 we assumed that in the infinite case the mean measure $\overline{\mu}(d)$ of closed metric balls of radius d and the substitution function $\sigma(d)$ have continuous derivatives $\overline{\mu}'(d)$ and $\sigma'(d)$ which are not zero when $0 < d < D(X)$ ($D(X)$ is the diameter of X). Furthermore, we remarked that without loss of generality it is possible to assume that $\sigma(d)$ is the standard substitution, i.e., a continuous strictly decreasing function on $[0, D(X)]$ such that $\sigma(D(X)) = -1 \leq \sigma(d) \leq \sigma(0) = 1$. (The definition of a weighted τ-design and the property of a metric space to be polynomial are invariant under any linear transformation of $\sigma(d)$.) Using these functions $\overline{\mu}(d)$ and $\sigma(d)$, and Theorem 2.7 we constructed on the interval $[-1, 1]$ a system Q of orthogonal polynomials $Q_i(t)$ of degree i and a system of positive constants r_i $(i = 0, 1, ..., s(X))$ such that

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)d\nu(t) = \delta_{i,j}, \tag{5.128}$$

$$Q_i(1) = 1, \quad i = 0, 1, ..., s(X),$$

where $\nu(t) = 1 - \overline{\mu}(\sigma^{-1}(t))$ and $r_0 = 1$. The system Q satisfies all conditions imposed on the system U considered in Sections 5.1-5.4. In particular, in the infinite case (5.128) takes the form

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)w(t)dt = \delta_{i,j},$$

where $w(t) = -\overline{\mu}'(d)/\sigma'(d)$ and $d = \sigma^{-1}(t)$, and in the finite case when $\Delta(X) = \{d_0, d_1, \ldots, d_s\}$, $d_0 = 0 < d_1 < \ldots < d_s = D(X)$, $s = s(X)$, $\nu(t)$ has $N = s + 1$ steps at the points $t_{N,l} = \sigma(d_{N-l})$ with sizes $w_l = \frac{1}{|X|^2} \sum_{x \in X} B_{d_{N-l}}(x, X) > 0$, $\sum_{l=1}^{N} w_l = 1$, and (5.128) takes the form

$$r_i \sum_{l=1}^{N} Q_i(t_{N,l})Q_j(t_{N,l})w_l = \delta_{i,j}.$$

(Note that in the latter case $Q_N(t) = (t - \sigma(d_0))...(t - \sigma(d_s))$, $N^{0,1} = N^{1,0} = N - 1$, $N^{1,1} = N - 2$, and $t^{1,1}_{N-2} = \sigma(d_1)$.) We proved that for polynomial metric spaces the system Q satisfies the Krein condition and for infinite distance transitive spaces and decomposable graphs it satisfies the strengthened Krein condition (Lemmas 3.22 and 3.25). Using the fact that for any polynomial $f(t)$ and for any (weighted) set $C \subseteq X$ the equality

$$\sum_{x,y \in C} f(\sigma(d(x,y)))m(x)m(y) = |C| \sum_{i=0}^{s(X)} \frac{f_i(Q)}{r_i} B'_i(C) \tag{5.129}$$

holds and all coefficients $B'_i(C)$ are nonnegative for polynomial spaces, we proved Theorems 3.14 and 3.15. According to these theorems, for any code C with minimal distance $d(C) \geq d$ in a polynomial space X and for any polynomial $f(t)$ such that $f_0(Q) > 0$, $f_i(Q) \geq 0$ for $i = 1, ..., s(X)$ and $f(t) \leq 0$ for $-1 \leq t \leq \sigma(d)$, the following holds

$$|C| \leq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{5.130}$$

with equality if and only if the polynomial $f(t)$ is annihilating for C and

$$f_i(Q)B'_i(C) = 0 \quad \text{for} \quad i = 1, ..., s(X). \tag{5.131}$$

Analogously, for any weighted τ-design $C = (C, m)$ in a polynomial space X and for any polynomial $f(t)$ such that $f_0(Q) > 0$, $f_i(Q) \leq 0$ for $i = \tau+1, ..., s(X)$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$, the following holds

$$|C| \geq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{5.132}$$

with equality if and only if C is a simple τ-design, the polynomial $f(t)$ is annihilating for C and

$$f_i(Q)B_i'(C) = 0 \quad \text{for} \quad i = \tau+1, ..., s(X). \tag{5.133}$$

This reduces the problem of obtaining universal bounds on $A(X,d)$ and $B(X,d)$ by the method considered to the σ-packing and τ-design problems for the system Q of which exploratory discussion is contained in Section 3.3. Considering the system Q (and the system P for polynomial graphs) as U we obtain the corresponding bounds as corollaries of the results on orthogonal polynomials given above. A code C in a polynomial space X is referred to as a *maximal code* if $A(X, d(C)) = |C|$ and a *minimal design* if $B(X, d'(C)) = |C|$.

Theorem 5.48. *Let X be a polynomial space with standard substitution $\sigma(d)$ and suppose the system Q defined by Theorem 2.7 satisfies the strengthened Krein condition. Then for any code $C \subset X$ (with $d(C) > d_1$ in the finite case) the following holds*

$$|C| \leq L_Q(\sigma(d(C))) \tag{5.134}$$

where by Lemma 5.37 $L_Q(\sigma)$ is a positive-valued strictly increasing continuous function defined by (5.96). The bound (5.134) is attained if and only if $\tau(C) = 2k-1+\varepsilon$ and

$$f_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\varepsilon\beta} \frac{T_{k-1}^{1,\varepsilon}(t,\sigma)}{T_{k-1}^{1,\varepsilon}(1,\sigma)} \tag{5.135}$$

where $\sigma = \sigma(d(C))$, $\beta = \beta(C)$, $k = k(\sigma)$, $\varepsilon = \varepsilon(\sigma)$.

Proof. By Corollary 5.40 and Theorem 5.42 the polynomial $f^{(\sigma)}(t) = (t-\sigma)(t+1)^{\varepsilon}\left(T_{k-1}^{1,\varepsilon}(t,\sigma)\right)^2$ of degree $h(\sigma) = 2k-1+\varepsilon$ has the desired properties and (5.130) implies (5.134) with equality if and only if $(t-\sigma)(t+1)^{\varepsilon}T_{k-1}^{1,\varepsilon}(t,\sigma)$ is annihilating for C and (5.131) holds. This proves (5.134) and the sufficiency of the stated conditions. Now we prove their necessity. Note that $\tau(C) < s(X)$ since by Corollary 3.20 $\tau(C) = s(X)$ implies that $C = X$ and $d(C) = d_1$. Since $f^{(\sigma)}(t) \in F_{>}(Q)$ and $\tau(C) < s(X)$, the condition (5.131) means that $h(\sigma) < s(X)$ and $\tau(C) \geq h(\sigma)$. Therefore, using Lemma 2.17 we have

$$2k-1+\varepsilon \leq \tau(C) \leq 2s(C) - \beta$$

and hence $s(C) > k-1+\varepsilon\beta$. On the other hand, the polynomial (5.135) of degree $k+\varepsilon\beta$ is also annihilating for C because -1 is not a root of the minimal polynomial for C if

$\beta = 0$. This means that $s(C) = k+\varepsilon\beta$ and (5.135) is minimal for C. By the definitions of $k = k(\sigma)$ and $\varepsilon = \varepsilon(\sigma)$, $\sigma < t_k^{1,\varepsilon}$ and hence by Lemma 5.37, $|C| = L_Q(\sigma) < L_Q(t_k^{1,\varepsilon})$. In the case $\tau(C) \geq 2k+\varepsilon$ by Theorem 2.16 and (5.97) we would have $|C| \geq L_Q(t_k^{1,\varepsilon})$. Hence $\tau(C) = 2k-1+\varepsilon$. ■

The polynomial (5.135) can have only simple roots $\alpha_0, \alpha_1, \ldots, \alpha_{k+\varepsilon-1}$ $(-1 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_{k+\varepsilon-1} = \sigma)$ and hence only d_i, $i = 1, ..., k+\varepsilon$, defined by $\sigma(d_i) = \alpha_{k+\varepsilon-i}$ can be nonzero distances of C. Therefore, (3.66) holds if we replace $f_C(t)$ by the polynomial $g(t)$ defined in Theorem 5.39 whose degree does not exceed $\tau(C)+1$. Applying (5.101) to all the Lagrange polynomials (5.109)-(5.110) we can describe the distance distribution of any code C, for which (5.134) is attained, in terms of the coefficients $\rho_j^{(\sigma)}$ of Theorem 5.39.

Corollary 5.49. *For any code C for which (5.134) is attained,*

$$B_{d_i}(C) = L_Q(\sigma) \cdot \rho_{k+\varepsilon-i}^{(\sigma)}, \quad i = 1, ..., k+\varepsilon, \tag{5.136}$$

where $\sigma = \sigma(d(C))$, $k = k(\sigma)$, $\varepsilon = \varepsilon(\sigma)$, and the coefficients $\rho_i^{(\sigma)}$ are the same as in Theorem 5.39.

Note that by Theorem 5.39 all $\rho_i^{(\sigma)}$ are positive except the cases $\sigma = t_k^{1,0}$ when $\rho_0^{(\sigma)} = 0$. Since by (5.135) $s(C) = k(\sigma) + \varepsilon(\sigma)\beta(C)$, then the condition $\tau(C) = 2k(\sigma) - 1 + \varepsilon(\sigma)$ can be rewritten in the form

$$\tau(C) = 2s(C) - \beta(C) - \delta_{\varepsilon(\sigma),\beta(C)} \tag{5.137}$$

and gives rise to tight designs if (and only if) $\varepsilon(\sigma) \neq \beta(C)$. We consider these significant special cases of the universal bound (5.134) and verify that they are valid for any polynomial space. We use the notation $d_i^{a,b} = \sigma^{-1}(t_i^{a,b})$ (see (2.71)).

Theorem 5.50. *For any code C in a polynomial space X such that $d(C) \geq d_{l-\theta}^{1,\theta}$ for some $l \in \{1, ..., s(X)\}$ and $\theta \in \{0, 1\}$,*

$$|C| \leq L_Q(t_{l-\theta}^{1,\theta}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r_i^{0,\theta} \tag{5.138}$$

with equality if and only if

$$\tau(C) = 2s(C) - \beta(C) \quad \text{and} \quad f_C(t) = \left(\frac{1+t}{2}\right)^{\theta} Q_{l-\theta}^{1,\theta}(t). \tag{5.139}$$

Proof. If $\sigma = t_{l-\theta}^{1,\theta}$, then $k(\sigma) = l$, $\varepsilon(\sigma) = 1-\theta$, and $h(\sigma) = 2l-\theta$. Taking into account that for any polynomial space $f_{2l-\theta}^{(\sigma)}(t) \in F_>(Q)$ when $\sigma = t_{l-\theta}^{1,\theta}$ by Theorem 5.43, we can repeat the proof of Theorem 5.48 for this case and use Corollary 5.44 to obtain

(5.138). If conditions (5.139) are satisfied, then $d(C) = d^{1,\theta}_{l-\theta}$, $s(C) = l$, $\beta(C) = \theta$, $\tau(C) = 2l - \theta$, and by Theorem 2.16 we have equality in (5.138). If equality in (5.138) holds, then $\tau(C) = 2k(\sigma) - 1 + \varepsilon(\sigma) = 2l - \theta$ and the same theorem proves the necessity of conditions (5.139). ■

Corollary 5.51. *Any tight τ-design C in a polynomial space X is a maximal code.*

The optimality of the choice $f^{(\sigma)}(t)$, for the method based on using the inequality (5.130), can be expressed in the form of the following statement which is a direct consequence of Corollaries 5.40 and 5.44.

Corollary 5.52. *For any σ, $-1 \le \sigma < t^{1,1}_{N-2}$, and any polynomial $f(t) \in F[t]$ of degree at most $h(\sigma)$ such that $f_0(U) > 0$ and $f(t) \le 0$ for $-1 \le t \le \sigma(d)$ which is not proportional to $f^{(\sigma)}(t)$ or to $f^{(\sigma)}(t)/(t+1)$ when $\sigma = t^{1,\theta}_{l-\theta}$ for some integer l and $\theta \in \{0, 1\}$, the following holds:*

$$\Omega_Q(f) = \frac{f(1)}{f_0(Q)} > L_Q(\sigma).$$

Since the question of whether the system Q corresponding to a polynomial metric space satisfies the strengthened Krein condition is still open (1997), we can also consider for any σ, $-1 \le \sigma < t^{1,0}_{s-1}$, the polynomial

$$\widetilde{f}^{(\sigma)}(t) = (t - \sigma)\left(T^{1,0}_{m-1}(t, \sigma)\right)^2 \tag{5.140}$$

of odd degree $2m - 1$ where $m = k(\sigma) + \varepsilon(\sigma)$ or, equivalently, $t^{1,0}_{m-1} \le \sigma < t^{1,0}_{m}$. By Theorem 5.43 and Corollary 5.45 this polynomial has the properties $\widetilde{f}^{(\sigma)}(t) \in F_>(Q)$ and $\widetilde{f}^{(\sigma)}(t) \le 0$ if $-1 \le t \le \sigma$, and the function

$$\widetilde{L}_Q(\sigma) = \Omega_Q(\widetilde{f}^{(\sigma)}) = \left(1 - \frac{Q^{1,0}_{m-1}(\sigma)}{Q_m(\sigma)}\right) \sum_{i=0}^{m-1} r_i.$$

is a positive-valued strictly increasing continuous function. Herewith,

$$\widetilde{L}_Q(\sigma) = L_Q(\sigma) \quad \text{if} \quad \varepsilon(\sigma) = 0 \tag{5.141}$$

and

$$\widetilde{L}_Q(t^{1,\theta}_{l-\theta}) = L_Q(t^{1,\theta}_{l-\theta}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r^{0,\theta}_i \tag{5.142}$$

where l is an integer and $\theta \in \{0, 1\}$.

Corollary 5.53. *For any code C in a polynomial space,*

$$|C| \le \widetilde{L}_Q(\sigma) = \left(1 - \frac{Q^{1,0}_{m-1}(\sigma)}{Q_m(\sigma)}\right) \sum_{i=0}^{m-1} r_i. \tag{5.143}$$

where $\sigma = \sigma(d(C))$ and $m = k(\sigma) + \varepsilon(\sigma)$.

According to Lemma 5.38 for any $l \in \{1, ..., s(X)\}$ and $\theta \in \{0, 1\}$ the polynomial

$$f(t) = \frac{t - \sigma}{t + 1} f^{(\sigma)}(t) \quad \text{where } \sigma = t_{l-\theta}^{1,\theta} \tag{5.144}$$

has degree $h(\sigma) = 2l - \theta$ and up to a constant multiple is equal to the polynomial $(t+1)^\theta \left(U_{l-\theta}^{1,\theta}(t)\right)^2$ and hence to the polynomial $g^{(2l-\theta)}(t)$ defined by (2.72). Moreover, by Theorem 5.39,

$$\Omega(f) = \Omega(f^{(\sigma)}) = L_Q(t_{l-\theta}^{1,\theta}).$$

Thus the polynomial (5.144) gives rise to the same solution of the restricted $(2l-\theta)$-design problem as in Theorem 2.16. However, Corollary 5.41 also finds the optimality and the uniqueness of the solution.

Corollary 5.54. *For any $(2l - \theta)$-design C in a polynomial space X where $l \in \{1, ..., s(X)\}$ and $\theta \in \{0, 1\}$,*

$$|C| \geq L_Q(t_{l-\theta}^{1,\theta}) = \left(1 - \frac{1}{Q_1(-1)}\right)^\theta \sum_{i=0}^{l-\theta} r_i^{0,\theta} \tag{5.145}$$

with equality in (5.145) if and only if $f_C(t) = (\frac{t+1}{2})^\theta \; Q_{l-\theta}^{1,\theta}(t)$. For any polynomial $f(t) \in F[t]$ of degree at most $\tau = 2l - \theta$ such that $f_0(Q) > 0$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$ which is not proportional to (5.144), the following holds:

$$\Omega_Q(f) = \frac{f(1)}{f_0(Q)} < L_Q(t_{l-\theta}^{1,\theta}).$$

Using the function $\gamma(d)$ (see (5.92)) we can express (5.145) as follows

$$|C| \geq L_Q(\gamma_Q(d)) \quad \text{if} \quad d'(C) \geq d. \tag{5.146}$$

Furthermore, we can rewrite (5.138) in the form

$$|C| \leq L_Q(\gamma_Q(d)) \quad \text{if} \quad d(C) \geq \sigma_Q^{-1}(\gamma_Q(d)). \tag{5.147}$$

We know that for any code C in a compact metric space X,

$$\tau(C) \leq 2s(C) - \beta(C) \tag{5.148}$$

and equality in (5.148) is necessary for C to be a tight design. For polynomial spaces the equality in (5.148) is already sufficient for attainability of the bounds (5.138) and (5.145), and the equality (5.137) is sufficient for attainability of the bound (5.134).

Theorem 5.55 ([68]). *Let C be a code in a polynomial space X, $\tau = \tau(C) \geq 1$, $s = s(C) \geq 1$, $\beta = \beta(C)$, $\sigma = \sigma_Q(d(C))$, $k = k_Q(\sigma)$, $\varepsilon = \varepsilon_Q(\sigma)$, and hence*

$$t_{k-1+\varepsilon}^{1,1-\varepsilon} \leq \sigma < t_k^{1,\varepsilon}.$$

Then $\tau = 2s - \beta$ if and only if $s = k$, $\beta = 1 - \varepsilon$ and

$$\sigma = t^{1,1-\varepsilon}_{k-1+\varepsilon}, \quad |C| = L_Q(\sigma), \quad f_C(t) = (\frac{t+1}{2})^{\beta} Q^{1,\beta}_{k-\beta}(t) \tag{5.149}$$

and $\tau = 2s - 1 - \beta$ if and only if $s = k + \varepsilon$, $\beta = \varepsilon$ and

$$\sigma \neq t^{1,1-\varepsilon}_{k-1+\varepsilon},\ |C| = L_Q(\sigma),\ f_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\beta} \frac{T^{1,\beta}_{k-1}(t,\sigma)}{T^{1,\beta}_{k-1}(1,\sigma)}. \tag{5.150}$$

In the both cases $f_C(t) \in F_>(Q)$.

The proof in the case $\tau = 2s$ follows from the bound for designs (Theorem 2.16) and the absolute bound of Delsarte (Theorem 3.18). A similar proof would be valid in the case $\tau = 2s - \beta$ with $\beta = 1$ if Conjecture 3.26 is true. In this case and in the general case Theorems 2.16, 3.17, 3.18, Corollary 5.53 monotonicity of $L_Q(\sigma)$ and properties of adjacent systems of orthogonal polynomials only were used. (The strengthened Krein condition is not needed.)

Theorem 5.55 describes some properties of the class of codes C in a polynomial space X defined by the condition $\tau(C) \geq 2s(C) - 1 - \beta(C) \geq 1$ (cf. (5.148)). By Theorem 3.21 this class consists of distance-regular codes. By Theorem 5.48 any code C, for which (5.134) is attained, belongs to this class. On the other hand, Theorems 5.48, 5.55 and (5.141)-(5.143) imply the following results on the optimality of codes of this class.

Corollary 5.56. *In a polynomial space X, any code C such that $\tau(C) \geq 2s(C) - 1$ is maximal.*

Corollary 5.57. *In a polynomial space X with the system Q satisfying the strengthened Krein condition, any code C such that $\tau(C) \geq 2s(C) - 2$ and $\beta(C) = 1$ is maximal.*

Remark 5.58. *It should be noted that the main parameters of a hypothetical code C, for which the inequality (5.134) is attained (i.e., $|C| = L_Q(d(C))$), can be uniquely defined by means of only the parameter $d = d(C)$ and some parameters of the system Q for the whole space X. Indeed, let $\sigma = \sigma_Q(d)$, $k = k_Q(\sigma)$, $\varepsilon = \varepsilon_Q(\sigma)$. Then by Theorems 5.48 and 5.55 $s(C) = k$, $\beta(C) = 1 - \varepsilon$ and $\tau(C) = 2s(C) - \beta(C)$ if $\sigma = t^{1,1-\varepsilon}_{k-1+\varepsilon}$, and $s(C) = k + \varepsilon$, $\beta(C) = \varepsilon$ and $\tau(C) = 2s(C) - \beta(C) - 1$ if $\sigma \neq t^{1,1-\varepsilon}_{k-1+\varepsilon}$. In the both cases the set $\Delta_0(C) = \{d_1, ..., d_s\}$ of $s = s(C)$ distances between different code points and the distance distribution of C can be found (see Corollary 5.49). Finally, C is a distance-regular code and its intersection numbers (2.9) are given by (3.67) and (3.68). Since the intersection numbers (and also the distances d_i in the case of polynomial graphs) are integers, these arguments can be used to prove nonexistence of a code C, for which (5.134) is attained. For example, it was observed [22] that in the case of antipodal spaces and $\varepsilon_Q(\sigma) = 1$, by Corollary 5.49 $B_{d_{k+1}}(C) < 1$. Hence,* *under these conditions, the inequality (5.134) might be attained only for tight $2k$-designs when $\sigma = t_{k-1}^{1,0}$ and $B_{d_{k+1}}(C) = 0$. On the other hand, this approach can be considered as a general part of a proof of the uniqueness (up to isometry) of codes with different minimal distances and in different polynomial spaces for which (5.134) is attained (see, for example, [78, Chapt. 20], [28, Chapt. 14]).*

Now we consider a polynomial graph X with standard substitutions $\sigma_Q(d)$ and $\sigma_P(d)$ (see (4.11)) and formulate statements which follow from the results given above and the duality in bounding the optimal sizes of codes and designs (Section 4.3). We shall write $k_U(\sigma)$, $\varepsilon_U(\sigma)$, $L_U(\sigma)$, $\gamma_U(d)$, $t_k^{a,b}(U) = \sigma_U(d_k^{a,b}(U))$ and $T_k^{a,b}(t, \sigma, U)$ for $U = Q$ and $U = P$ to explain for what system these values are considered.

Theorem 5.59. *For any code C in a polynomial graph X,*

$$L_Q(\gamma_Q(d'(C))) \leq |C| \leq \frac{|X|}{L_P(\gamma_P(d(C)))} \tag{5.151}$$

with equality in the left-hand side if and only if

$$d'(C) = 2s(C) - \beta(C) + 1 \quad \text{and} \quad f_C(t) = \left(\frac{1+t}{2}\right)^{\beta(C)} Q_{s(C)-\beta(C)}^{1,\beta(C)}(t) \tag{5.152}$$

and with equality in the right-left side if and only if

$$d(C) = 2s'(C) - \beta'(C) + 1 \quad \text{and} \quad \widetilde{f}_C(t) = \left(\frac{1+t}{2}\right)^{\beta'(C)} P_{s'(C)-\beta'(C)}^{1,\beta'(C)}(t). \tag{5.153}$$

Codes for which the upper bound in (5.151) is attained are called *perfect* codes. Thus we consider perfect codes with odd and even minimal distance (similar to tight designs defined as codes for which the lower bound in (5.151) is attained).

Theorem 5.60. *For any code C in a polynomial graph X such that $d(C) \geq d_{l-\theta}^{1,\theta}(Q)$ and $d'(C) \geq d_{m-\vartheta}^{1,\vartheta}(P)$ for some $l, m \in \{1, ..., s(X)\}$ and $\theta, \vartheta \in \{0, 1\}$,*

$$\frac{|X|}{L_P(\gamma_P(2m - \vartheta + 1))} \leq |C| \leq L_Q(\gamma_Q(2l - \theta + 1)) \tag{5.154}$$

with equality in the right-hand side if and only if

$$d'(C) = 2s(C) - \beta(C) + 1 \quad \text{and} \quad f_C(t) = \left(\frac{1+t}{2}\right)^{\theta} Q_{l-\theta}^{1,\theta}(t) \tag{5.155}$$

and with equality in the left-hand side if and only if

$$d(C) = 2s'(C) - \beta'(C) + 1 \quad \text{and} \quad \widetilde{f}_C(t) = \left(\frac{1+t}{2}\right)^{\vartheta} P_{m-\vartheta}^{1,\vartheta}(t). \tag{5.156}$$

Corollary 5.61. *In a polynomial graph X any perfect code is a minimal design and any tight design is a maximal code.*

Theorem 5.62. *Let X be a polynomial graph for which the systems Q and P satisfy the strengthened Krein condition. Then for any code $C \subset X$ with $d(C) > 1$ and $d'(C) > 1$ the following holds:*

$$\frac{|X|}{L_P(\sigma_P(d'(C)))} \leq |C| \leq L_Q(\sigma_Q(d(C))) \tag{5.157}$$

with equality in the right-hand side if and only if

$$d'(C) = 2k + \varepsilon \quad \text{and} \quad f_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\varepsilon\beta} \frac{T_{k-1}^{1,\varepsilon}(t,\sigma,Q)}{T_{k-1}^{1,\varepsilon}(1,\sigma,Q)} \tag{5.158}$$

where $\sigma = \sigma_Q(d(C))$, $\beta = \beta(C)$, $k = k_Q(\sigma)$, $\varepsilon = \varepsilon_Q(\sigma)$, and with equality in the left-hand side if and only if

$$d(C) = 2k + \varepsilon \quad \text{and} \quad \widetilde{f}_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\varepsilon\beta'} \frac{T_{k-1}^{1,\varepsilon}(t,\sigma,P)}{T_{k-1}^{1,\varepsilon}(1,\sigma,P)} \tag{5.159}$$

where $\sigma = \sigma_P(d'(C))$, $\beta' = \beta'(C)$, $k = k_P(\sigma)$, $\varepsilon = \varepsilon_P(\sigma)$.

For polynomial graphs for which Q and/or P do not satisfy the strengthened Krein condition one can use Corollary 5.53 and the corresponding dual analog. Similar to the case of polynomial spaces we can use Theorems 5.59, 4.13, Corollary 4.19, monotonicity of $L_P(\sigma)$ and properties of adjacent systems of orthogonal polynomials to obtain the following statements (cf. Theorem 5.55).

Theorem 5.63. *Let C be a code in a polynomial graph X, $d = d(C) \geq 2$, $s' = s'(C) \geq 1$, $\beta' = \beta'(C)$, $\sigma = \sigma_P(d'(C))$, $k = k_P(\sigma)$, $\varepsilon = \varepsilon_P(\sigma)$, and hence*

$$t_{k-1+\varepsilon}^{1,1-\varepsilon}(P) \leq \sigma < t_k^{1,\varepsilon}(P).$$

Then $d = 2s' - \beta' + 1$ if and only if $s' = k$, $\beta' = 1 - \varepsilon$ and

$$\sigma = t_{k-1+\varepsilon}^{1,1-\varepsilon}(P), \quad |C| = \frac{|X|}{L_P(\sigma)}, \quad \widetilde{f}_C(t) = (\frac{t+1}{2})^{\beta'} P_{k-\beta'}^{1,\beta'}(t), \tag{5.160}$$

and $d = 2s' - \beta'$ if and only if $s' = k + \varepsilon$, $\beta' = \varepsilon$ and

$$\sigma \neq t_{k-1+\varepsilon}^{1,1-\varepsilon}(P),\ |C| = \frac{|X|}{L_P(\sigma)},\ \widetilde{f}_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\beta'} \frac{T_{k-1}^{1,\beta'}(t,\sigma,P)}{T_{k-1}^{1,\beta'}(1,\sigma,P)}. \tag{5.161}$$

In the both cases $\widetilde{f}_C(t) \in F_{>}(P)$.

Thus the class of codes C in a polynomial graph X defined by the condition $d(C) \geq 2s'(C) - \beta'(C) \geq 2$ (cf. the statement 3 of Theorem 4.5) has the following property: each of the values $d'(C)$ or $|C|$ uniquely determines the other as well as the values $d(C)$, $s'(C)$, $\beta'(C)$, and the dual-annihilating polynomial $\widetilde{f}_C(t)$ (and hence the set of $s'(C)$ integers which are dual distances). Note that the condition $d(C) \geq 2s'(C) - \beta'(C)$ (as distinguished from the condition $d'(C) \geq 2s(C) - \beta(C)$; see Theorem 3.21 and Remark 5.58) does not imply that C is a distance-regular code. In particular, any perfect code $C \subset H_2^n$ for which $d(C) = 2s'(C) - \beta'(C) = 3$ is not distance-regular for $n > 7$ (appears in [6]). We also have the following results on the optimality of codes of this class.

Corollary 5.64. *In a polynomial graph X, any code C such that $d(C) \geq 2s'(C)$ is a minimal design.*

Corollary 5.65. *In a polynomial graph X with the system P satisfying the strengthened Krein condition, any code C such that $d(C) \geq 2s'(C) - 1$ and $\beta'(C) = 1$ is a minimal design.*

One more pair of universal bounds for designs and codes in polynomial graphs was presented in Theorem 4.27.

6. Summary for the basic polynomial spaces

6.1. The unit Euclidean sphere and the projective spaces

The unit Euclidean sphere in $\mathbf{R}^n$ (see Examples 2.3, 2.10, 3.11)

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, ..., x_n) \in \mathbf{R^n},\ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

with Euclidean distance $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a compact metric space of essential interest in coding theory and information theory. We can also measure the distance between $\mathbf{x}, \mathbf{y} \in S^{n-1}$ by the *angular distance* $\varphi(\mathbf{x}, \mathbf{y})$ where

$$\cos \varphi(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i = 1 - \frac{1}{2} d^2(\mathbf{x}, \mathbf{y}). \tag{6.1}$$

Denoting the normalized Lebesgue measure on S^{n-1} (the normalized surface area) by μ, we state that S^{n-1} is a distance invariant space and

$$\overline{\mu}(d) = \frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} \text{ if } \cos \varphi = 1 - d^2/2, \tag{6.2}$$

where $\sigma_{n-1}(\varphi)$ is the surface area of a spherical cap on S^{n-1} of radial angle φ and $\sigma_{n-1} = 2\sigma_{n-1}(\frac{\pi}{2})$ is the surface area of S^{n-1}. The function

$$\sigma(d) = 1 - \frac{d^2}{2} \tag{6.3}$$

is the standard substitution for the Euclidean sphere $X = S^{n-1}$. Using (2.34), (6.2), (2.20), (6.3) we can see that the corresponding system Q consists of polynomials orthogonal on the interval [-1,1] with respect to the weight function

$$w(t) = \nu'(t) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}(1-t^2)^{\frac{n-3}{2}}.$$

As we remarked, the properties (2.49) and (2.50) of the normalized Jacobi polynomials $P_i^{\alpha,\beta}(t)$ defined by (2.48) imply that for the case S^{n-1},

$$Q_i(t) = P_i^{\frac{n-3}{2},\frac{n-3}{2}}(t), \quad i = 0, 1, \dots , \tag{6.4}$$

and $r_i = \frac{2i+n-2}{i+n-2}\binom{i+n-2}{i}$. (The Jacobi polynomials $P_i^{\alpha,\beta}(t)$ with $\alpha = \beta$ are called Gegenbauer polynomials.) By Lemma 3.22 the system (6.4) satisfies the strengthened Krein condition.

The fact that S^{n-1} is a polynomial space with standard substitution (6.3) is well known (see, for example, [14]) and can be proved using the approach described in the end of Section 3.2. The isometry group of S^{n-1} is the group $O(n)$ of orthogonal matrices of order n and S^{n-1} is distance-transitive with respect to $O(n)$. The subspaces V_i , $i = 0, 1, \dots$, on which the components of representation $L(g)$ defined by (3.75) act, consist of all functions on S^{n-1} which are represented by harmonic homogeneous polynomials $h(x_1, ..., x_n)$ of total degree i in variables $x_1, ..., x_n$. For the zonal spherical function $\Phi_i(d)$ (see (3.76)) there exists a polynomial $U_i(t)$ of degree i such that $\Phi_i(d(\mathbf{x}, \mathbf{y})) = U_i((\mathbf{x}, \mathbf{y}))$. Hence S^{n-1} is polynomial with respect to (6.3) and $U_i(t)$ coincides with $Q_i(t) = P_i^{\frac{n-3}{2},\frac{n-3}{2}}(t)$.

In the case of $X = S^{n-1}$ the concept of a (weighted) τ-design $C = (C, m)$ is connected [35] (see also (3.58)) with the approximation formula for the evaluation of multi-dimensional integrals over S^{n-1} of the following sort

$$\int\limits_{S^{n-1}} u(\mathbf{x}) d\mu(\mathbf{x}) \approx \frac{1}{\sum\limits_{x \in C} m(\mathbf{x})} \sum_{\mathbf{x} \in C} u(\mathbf{x}) m(\mathbf{x}). \tag{6.5}$$

The (weighted) set $C = (C, m)$ is a (weighted) τ-design in S^{n-1} with respect to the substitution $\sigma(d) = 1 - d^2/2$ (such τ-designs are called *spherical)* if and only if the approximation formula (6.5) becomes equality for all functions $u(\mathbf{x})$ which are polynomials in coordinates of $\mathbf{x} = (x_1, ..., x_n) \in S^{n-1}$ of total degree at most τ. Thus $B(S^{n-1}, \tau + 1)$ is the minimum number of nodes in the approximation formula of this type.

The relationship (6.1) allows us to consider $A(S^{n-1},d)$ as the maximum size $M(n,\varphi)$ of a *spherical* code $C\subset S^{n-1}$ with the *minimal angular distance* $\varphi(C)=\varphi$ between its distinct points where $\cos\varphi=1-d^2/2$ and $d=2\sin\frac{\varphi}{2}$. There exists a close connection between the value $M(n,\varphi)$ and some classical packing problems in $\mathbf{R}^n$. We consider the compact (but not distance invariant) metric space

$$R_\rho^n=\left\{\mathbf{x}=(x_1,...,x_n)\in\mathbf{R^n},\ \sum_{i=1}^n x_i^2\le\rho\right\}$$

with the Euclidean distance. For the measure ν equal to n-dimensional volume, it is known (see, for example, [40]) that $\nu(R_\rho^n)=\rho^n\nu(R_1^n)$. Therefore the ratio

$$\frac{A(R_\rho^n,2)\nu(R_1^n)}{\nu(R_\rho^n)}=\frac{A(R_\rho^n,2)}{\rho^n}$$

characterizes the density of packing the ball R_ρ^n of radius ρ by balls of radius 1. The *highest density of a sphere packing in* $\mathbf{R}^n$ is defined as follows:

$$\delta_n=\lim_{\rho\to\infty}\frac{A(R_\rho^n,2)}{\rho^n}. \tag{6.6}$$

This limit really exists and does not change if we replace $A(R_\rho^n,2)$ in (6.6) by $A(R_{\rho-1}^n,2)$. Another characteristic of the space $\mathbf{R}^n$ is the *kissing number* M_n equal to the maximum number of unit spheres in $\mathbf{R}^n$ that touch one such sphere and do not intersect. The following statements are valid:

$$M_n=M(n,\frac{\pi}{3}), \tag{6.7}$$

$$\delta_{n-1}=\lim_{\varphi\to 0}\frac{M(n,2\varphi)\sigma_{n-1}(\varphi)}{\sigma_{n-1}}, \tag{6.8}$$

$$\delta_n\le M(n+\varepsilon,2\varphi)\sin^n\varphi\quad \text{for any } \varphi,\ 0<\varphi\le\frac{\pi}{2}, \tag{6.9}$$

where $\varepsilon=0$ if $\frac{\pi}{6}\le\varphi\le\frac{\pi}{2}$, and $\varepsilon=1$ if $\varphi<\frac{\pi}{6}$, and, in particular,

$$\delta_n\le M_n2^{-n} \tag{6.10}$$

((6.7) is evident and proofs of (6.8)-(6.9) can be found in [67]).

Now we apply the results of Section 5.5 to bound $M(n,\varphi)=A(S^{n-1},2\sin\frac{\varphi}{2})$ and $B(S^{n-1},\tau+1)$. Let $Q_i(t)=Q_i^{(n)}(t)=P_i^{\frac{n-3}{2},\frac{n-3}{2}}(t),\quad i=0,1,...$, and let $p_i^{\alpha,\beta}$ be the largest root of the Jacobi polynomial $P_i^{\alpha,\beta}(t)$. Thus for the system Q we have

$$t_i=p_i^{\frac{n-3}{2},\frac{n-3}{2}},\ t_i^{1,0}=p_i^{\frac{n-1}{2},\frac{n-3}{2}},\ t_i^{1,1}=p_i^{\frac{n-1}{2},\frac{n-1}{2}}.$$

Using the known properties of the Gegenbauer polynomials we have

$$r_i = \frac{2i+n-2}{i+n-2}\binom{i+n-2}{i},\ b_i = \frac{i+n-2}{2i+n-2},\ a_i = 0$$

and hence (see Theorem 5.3 and (5.9)) the following recurrence

$$(i+n-2)Q_{i+1}(t) = (2i+n-2)tQ_i(t) - iQ_{i-1}(t). \tag{6.11}$$

The first seven polynomials $Q_i(t) = Q_i^{(n)}(t)$ have the following form:

$$Q_0(t) = 1,\ Q_1(t) = t,\ Q_2(t) = \frac{nt^2-1}{n-1},\ Q_3(t) = \frac{(n+2)t^3-3t}{n-1},$$

$$Q_4(t) = \frac{(n+2)(n+4)t^4 - 6(n+2)t^2 + 3}{n^2-1},$$

$$Q_5(t) = \frac{(n+4)(n+6)t^5 - 10(n+4)t^3 + 15t}{n^2-1},$$

$$Q_6(t) = \frac{(n+4)(n+6)(n+8)t^6 - 15(n+4)(n+6)t^4 + 45(n+4)t^2 - 15}{(n+3)(n^2-1)}.$$

Using (2.77) and (6.4) we can find that

$$\sum_{i=0}^{k} r_i^{0,\theta} = \binom{k+n-1}{k} + (1-\theta)\binom{k+n-2}{k-1},\ \text{where } \theta \in \{0,1\}.$$

By (5.10) and Lemma 5.24 we can express $Q_k^{1,0}(t)$, $Q_k^{0,1}(t)$, and $Q_k^{1,1}(t)$ via polynomials of the system Q as follows

$$Q_k^{1,0}(t) = \frac{(n-1)\left(Q_k(t) - Q_{k+1}(t)\right)}{(2k+n-1)(1-t)},\ Q_k^{0,1}(t) = \frac{Q_k(t)+Q_{k+1}(t)}{1+t},$$

$$Q_k^{1,1}(t) = \frac{2k(Q_{k-1}(t) - Q_{k+1}(t))}{(2k+n-2)(1-t^2)}.$$

In particular, this gives

$$t_0^{1,1} = -1,\ t_1^{1,1} = 0,\ t_2^{1,1} = \frac{1}{\sqrt{n+2}},\ t_3^{1,1} = \frac{\sqrt{3}}{\sqrt{n+4}},$$

$$t_4^{1,1} = \sqrt{\frac{3(n+4)+\sqrt{6(n+3)(n+4)}}{(n+4)(n+6)}},\ t_5^{1,1} = \sqrt{\frac{5(n+6)+\sqrt{10(n+3)(n+6)}}{(n+6)(n+8)}},$$

$$t_1^{1,0} = -\frac{1}{n},\ t_2^{1,0} = \frac{\sqrt{n+3}-1}{n+2}.$$

These calculations show that the function $L_Q(\sigma) = L_Q^{(n)}(\sigma)$ for the case of the Euclidean sphere S^{n-1} can be expressed as follows:

$$L_Q^{(n)}(\sigma) = \begin{cases} \binom{k+n-3}{k-1}\left(\frac{2k+n-3}{n-1} - \frac{Q_{k-1}(\sigma)-Q_k(\sigma)}{(1-\sigma)Q_k(\sigma)}\right) & \text{if } t_{k-1}^{1,1} \le \sigma < t_k^{1,0}, \\ \binom{k+n-2}{k}\left(\frac{2k+n-1}{n-1} - \frac{(1+\sigma)(Q_k(\sigma)-Q_{k+1}(\sigma))}{(1-\sigma)(Q_k(\sigma)+Q_{k+1}(\sigma))}\right) & \text{if } t_k^{1,0} \le \sigma < t_k^{1,1}, \end{cases} \tag{6.12}$$

and takes the following values at the ends of the half-open intervals

$$L_Q^{(n)}(t_{k-1}^{1,1}) = 2\binom{k+n-2}{k-1}, \quad L_Q^{(n)}(t_k^{1,0}) = \binom{k+n-1}{k} + \binom{k+n-2}{k-1}. \tag{6.13}$$

Using Theorem 2.16, Corollary 5.54 and (3.58) one can extend the known Delsarte-Goethals-Seidel bound [35] to the case of weighted spherical designs.

Theorem 6.1 (DGS bound). *For any (weighted) $(2l-\theta)$-design C in S^{n-1} where $l = 1, 2, \dots$, and $\theta \in \{0, 1\}$,*

$$|C| \ge L_Q^{(n)}(t_{l-\theta}^{1,\theta}) = \binom{l+n-2}{n-1} + \binom{l+n-1-\theta}{n-1} \tag{6.14}$$

with equality in (6.14) if and only if C is a simple design and $f_C(t) = (\frac{t+1}{2})^\theta\ Q_{l-\theta}^{1,\theta}(t)$.

Spherical codes C for which (6.14) is attained are called tight spherical designs and characterized by the condition $\tau(C) = 2s(C) - \beta(C)$ according to Theorem 5.55.

Theorems 5.48, 5.50 and 5.55 and Lemma 3.22 imply the following statement.

Theorem 6.2 ([64], [65], [68]). *Let C be a code in S^{n-1} with the minimal angular distance $\varphi = \varphi(C)$ or with the minimal distance $d(C) = 2\sin\frac{\varphi}{2}$, $\sigma = \sigma_Q(d(C)) = \cos\varphi$, $k = k_Q(\sigma)$, $\varepsilon = \varepsilon_Q(\sigma)$ and hence $t_{k-1+\varepsilon}^{1,1-\varepsilon} \le \sigma < t_k^{1,\varepsilon}$. Then*

$$|C| \le L_Q^{(n)}(\cos\varphi) \tag{6.15}$$

where $L_Q(\sigma)$ is defined by (6.12) and, in particular,

$$|C| \le \binom{l+n-2}{n-1} + \binom{l+n-1-\theta}{n-1} \quad \text{if } \cos\varphi \le t_{l-\theta}^{1,\theta} \tag{6.16}$$

for some $l = 1, 2, \dots$, and $\theta \in \{0, 1\}$. The bound (6.15) is attained if and only if

$$\tau(C) \ge 2s(C) - \beta(C) - 1 \ge 1.$$

If $|C| = L_Q^{(n)}(\cos\varphi)$ and $\cos\varphi = t_{k-1+\varepsilon}^{1,1-\varepsilon}$, then $s(C) = k$, $\beta(C) = 1-\varepsilon$, $\tau(C) = 2s(C) - \beta(C)$ (and hence C is a tight design), $f_C(t) = (\frac{t+1}{2})^{1-\varepsilon}\ Q_{k-1+\varepsilon}^{1,1-\varepsilon}(t)$; if $|C| = L_Q^{(n)}(\cos\varphi)$ and $\cos\varphi \ne t_{k-1+\varepsilon}^{1,1-\varepsilon}$, then $\varepsilon = 0$ (see Remark 5.58), $s(C) = k$, $\beta(C) = 0$, $\tau(C) = 2s(C) - 1$, $f_C(t) = \frac{t-\sigma}{1-\sigma}\frac{T_{k-1}^{1,0}(t,\sigma)}{T_{k-1}^{1,0}(1,\sigma)}$. In both cases C is a distance-regular code and $f_C(t) \in F_>(Q)$.

Degree of $f^{(\sigma)}(t)$	$L_Q^{(n)}(\sigma)$	Interval for $\sigma = \cos\varphi$
1	$\frac{1-\sigma}{-\sigma}$	$[-1, -\frac{1}{n}]$
2	$\frac{2n(1-\sigma)}{1-n\sigma}$	$[-\frac{1}{n}, 0]$
3	$\frac{n(2+(n+1)\sigma)(1-\sigma)}{1-n\sigma^2}$	$[0, \frac{1}{\sqrt{n+3}+1}]$
4	$\frac{2n(1+(n+2)\sigma)(1-\sigma)}{1+2\sigma-(n+2)\sigma^2}$	$[\frac{1}{\sqrt{n+3}+1}, \frac{1}{\sqrt{n+2}}]$
5	$\frac{n((n+2)(n+3)\sigma^2+4(n+2)\sigma-n+1)(1-\sigma)}{2\sigma(3-(n+2)\sigma^2)}$	$[\frac{1}{\sqrt{n+2}}, t_3^{1,0}]$

Table 6.1: The First Five Bounds on the Size of a Spherical Code with Minimal Angular Distance φ ($\sigma = \cos\varphi$)

The function $L_Q^{(n)}(\sigma)$ is continuous and is expressed by the explicit formula in every interval considered. The first five formulas are given in Table 6.1. They are obtained using the optimal (for the restricted σ-packing problem) polynomials $f^{(\sigma)}(t)$ of degree one, two, three, four, and five, respectively.

Example 6.3. *Find parameters of a hypothetical spherical code $C \subset S^{n-1}$ with $n = 24$ and $d(C) = 1$ (or with the minimal angular distance $\varphi(C) = 60°$) for which (6.15) is attained. Using Remark 5.58 and the calculations given above we have consecutively $\sigma = \sigma_Q(C) = \frac{1}{2}$, $k = k_Q(\sigma) = 6$, $\varepsilon = \varepsilon_Q(\sigma) = 1$, $\sigma = t_5^{1,1}$, $|C| = L_Q^{(24)}(t_5^{1,1}) = 2\binom{28}{7} = 196560$, $s = s(C) = 6$, $\beta = \beta(C) = 1$, $\tau = \tau(C) = 11$ and C must be a tight 11-design. Moreover,*

$$f_C(t) = (t+1)Q_5^{1,1}(t) = (t+1)Q_5^{(26)}(t) = \frac{32}{45}(t-\frac{1}{2})(t-\frac{1}{4})t(t+\frac{1}{4})(t+\frac{1}{2})(t+1)$$

and hence $\sigma(d_j)$, $j = 1, ..., 6$, are equal to $\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, -\frac{1}{2}, -1$, respectively. The coefficients of polynomials $f_C^{(j)}(t)$, $j = 1, ..., 6$, over the system $\{Q_i^{(24)}(t)\}$ (see (3.64) and

(3.65)) and the values $f_C^{(j)}(1)$ are given in the following table:

j	$f_{j,0}$	$f_{j,1}$	$f_{j,2}$	$f_{j,3}$	$f_{j,4}$	$f_{j,5}$	$f_C^{(j)}(1)$	$f_{j,0}\|C\| - f_C^{(j)}(1)$
1	$\frac{11}{468}$	$\frac{232}{819}$	$\frac{391}{252}$	$\frac{1702}{351}$	$\frac{2300}{273}$	$\frac{920}{189}$	20	4600
2	$\frac{28}{117}$	$\frac{1952}{1365}$	$\frac{92}{63}$	$-\frac{5888}{585}$	$-\frac{9200}{273}$	$-\frac{1472}{63}$	−64	47104
3	$\frac{37}{78}$	$\frac{1}{91}$	$-\frac{253}{42}$	$\frac{46}{39}$	$\frac{4600}{91}$	$\frac{920}{21}$	90	93150
4	$\frac{28}{117}$	$-\frac{1184}{819}$	$\frac{92}{63}$	$\frac{2944}{351}$	$-\frac{9200}{273}$	$-\frac{7360}{189}$	−64	47104
5	$\frac{11}{468}$	$-\frac{76}{273}$	$\frac{391}{252}$	$-\frac{506}{117}$	$\frac{2300}{273}$	$\frac{920}{63}$	20	4600
6	0	$-\frac{1}{4095}$	0	$-\frac{46}{1755}$	0	$-\frac{184}{189}$	−1	1

By Theorem 3.21 C is distance-regular with distance distribution $B_{d_j}(C) = p_{j,j}^0$, $j = 1, ..., 5$, given in the last column of this table (see (3.66), (3.67), and Corollary 5.49) and with other intersection numbers $p_{i,j}^k$ defined by (3.68). For example,

$$p_{2,3}^3 = |C| \sum_{l=0}^{5} \frac{(l+22) f_{2,l} f_{3,l}}{(2l+22)\binom{l+22}{l}} Q_l(0) - f_C^{(3)}(1) = 22374.$$

The sought code really exists. It consists of vectors of minimal norm in the Leech lattice and is unique up to isometry (see [28, Chapter 14]).

The parameters of the known (to the author) codes for which the bound (6.15) is attained are presented in Table 6.2. All these codes are described by Delsarte, Goethals, and Seidel in [35]. An exception is the infinite sequence of codes $C \subset S^{n-1}$ of length $n = q(q^2 - q + 1)$ with $\cos\varphi = q^{-2}$ where $d(C) = 2\sin\frac{\varphi}{2}$ and q is a power of a prime. These codes are obtained with the help of arranging on S^{n-1} the point graph of the partial geometry $(q^2+1, q+1, 1)$ and were considered by Cameron, Goethals, and Seidel in [27] as spherical 3-designs. The author noticed that these codes are maximal and is not aware of any other infinite family of maximal codes on S^{n-1} with a minimal angular distance less than $\pi/2$. Table 6.2 also contains parameters of all known tight designs because, for them, both the bounds (6.14) and (6.16) are attained. Detailed information on the existence of tight spherical t-designs can be found in [10].

The two most interesting cases are where equality in the bound (6.15) allow the author to find two kissing numbers

$$M_8 = 240 \quad \text{and} \quad M_{24} = 196560.$$

n	$\tau(C)$	$s(C)$	$\beta(C)$	Tight or not	$\cos\varphi(x,y)$ $x,y \in C, x \neq y$	$\lvert C\rvert = L_Q^{(n)}(d(C))$	Comments
n	1	1	1	T	-1	2	
n	2	1	0	T	$-\frac{1}{n}$	$n+1$	Regular simplex
n	3	2	1	T	$-1, 0$	$2n$	Generalized octahedron
$q\frac{q^3+1}{q+1}$	3	2	0		$-\frac{1}{q}, \frac{1}{q^2}$	$(q+1)(q^3+1)$	q is a power of a prime, $q \geq 3$
5	3	2	0		$-\frac{3}{5}, \frac{1}{5}$	16	
21	3	2	0		$-\frac{2}{7}, \frac{1}{7}$	162	
22	3	2	0		$-\frac{4}{11}, \frac{1}{11}$	100	
6	4	2	0	T	$-\frac{1}{2}, \frac{1}{4}$	27	
22	4	2	0	T	$-\frac{1}{4}, \frac{1}{6}$	275	
3	5	3	1	T	$-1, \pm\frac{1}{\sqrt{5}}$	12	Regular icosahedron
7	5	3	1	T	$-1, \pm\frac{1}{3}$	56	
23	5	3	1	T	$-1, \pm\frac{1}{5}$	552	
22	5	3	0		$-\frac{1}{2}, -\frac{1}{8}, +\frac{1}{4}$	891	
8	7	4	1	T	$-1, 0, \pm\frac{1}{2}$	240	
23	7	4	1	T	$-1, 0, \pm\frac{1}{3}$	4600	
24	11	6	1	T	$-1, 0, \pm\frac{1}{4}, \pm\frac{1}{2}$	196560	
2	$r-1$	$\left[\frac{r}{2}\right]$	$\frac{1+(-1)^r}{2}$	T	$\cos\frac{2\pi j}{r}$ $j = 1, ..., \left[\frac{r}{2}\right]$	r	Regular r-polygon

Table 6.2: The Parameters of the Known Spherical Codes $C \subset S^{n-1}$ for Which the Upper Bound (6.15) Is Attained

This result was obtained independently by Odlyzko and Sloane [87] who used a computer to find a solution of the σ-packing problem for the system $\{Q_i^{(n)}(t)\}$ when $\sigma = \frac{1}{2} = \cos\frac{\pi}{3}$ and $3 \leq n \leq 24$. Their study shows that, despite the presence of many cases of attainability, in some cases there exist polynomials of higher degree which improve (6.15). Boyvalenkov, Danev and Bumova [20] found necessary and sufficient conditions for the existence of such polynomials (see Theorem 5.47). This situation is not surprising because the asymptotic bounds of Theorem 6.7 and Corollary 6.8 below show that (6.15) can be improved when φ is sufficiently small. It should also be noted that results of Rankin [91] and Astola [5], obtained with the help of other methods, improve, respectively, the first and the second, and the third bounds in Table 6.1 for some φ.

Note that (6.15), (6.7), and (6.9) give the following bounds

$$M_n \leq L_Q^{(n)}(\frac{1}{2}), \quad n = 2, 3, ... \tag{6.17}$$

$$\delta_n \leq \min_{\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{2}} \left(\sin^n\varphi \; L_Q^{(n)}(\cos 2\varphi)\right), \; n = 2, 3, ... \tag{6.18}$$

which are better than the known bounds (see [94], [28]) for sufficiently large n.

Theorems 6.1 and 6.2 imply that for any spherical $(2l-\theta)$-design $C \subset S^{n-1}$, where $l \in \{1, 2, ...\}$ and $\theta \in \{0, 1\}$,

$$\sigma(d(C)) = \cos\varphi(C) \geq t_{l-\theta}^{1,\theta} = p_{l-\theta}^{\frac{n-1}{2}, \frac{n-3}{2}+\theta} \tag{6.19}$$

with equality if and only if C is a tight $(2l-\theta)$-design. Let $\sigma(\rho(C)) = \cos\psi(C)$ ($\psi(C)$ can be considered as the *angular covering radius* of C), and let, for any $x \in S^{n-1}$, $f_{x,C}(t)$ be the polynomial of minimal degree such that $f_{x,C}(\cos\varphi(x,y)) = 0$ for all $y \in C$ and $f_{x,C}(1) = 1$.

Theorem 6.4. *[39]For any spherical $(2l-\theta)$-design $C \subset S^{n-1}$, where $l \in \{1, 2, ...\}$ and $\theta \in \{0, 1\}$,*

$$\sigma(\rho(C)) = \cos\psi(C) \geq t_l^{0,1-\theta} = p_l^{\frac{n-3}{2}, \frac{n-1}{2}-\theta} \tag{6.20}$$

with equality if and only if there exists a point $x \in S^{n-1} \backslash C$ such that $f_{x,C}(t) = \left(\frac{t+1}{2}\right)^{1-\theta} Q_l^{0,1-\theta}(t)$. In particular, the bound (6.20) is attained for all tight $2l$-designs.

Thus, the spherical caps of radial angle $\psi(C) \geq \arccos p_l^{\frac{n-3}{2}, \frac{n-1}{2}-\theta}$ with centers at the points of any spherical $(2l-\theta)$-design $C \subset S^{n-1}$ cover S^{n-1}.

Corollary 6.5. *For any spherical $(2l-\theta)$-design $C \subset S^{n-1}$, where $l \in \{1, 2, ...\}$ and $\theta \in \{0, 1\}$,*

$$|C|\sigma_{n-1}(\arccos p_l^{\frac{n-3}{2}, \frac{n-1}{2}-\theta}) \geq \sigma_{n-1}. \tag{6.21}$$

Recently Yudin [119] obtained the following stronger result.

Theorem 6.6. *[119]For any spherical τ-design $C \subset S^{n-1}$,*

$$|C|\sigma_{n-1}(\arccos p_{\tau}^{\frac{n-1}{2},\frac{n-1}{2}}) \geq \sigma_{n-1}. \qquad (6.22)$$

The bound (6.22) is attained for $n = 2$ and any τ, similarly to bounds (6.14) and (6.21), and improves both these bounds for sufficiently large τ. Combining Theorems 6.2 and 6.6, for any spherical τ-design $C \subset S^{n-1}$ we have

$$L_Q^{(n)}(\cos\varphi(C)) \cdot \sigma_{n-1}(\arccos p_{\tau}^{\frac{n-1}{2},\frac{n-1}{2}}) \geq \sigma_{n-1}. \qquad (6.23)$$

This gives one more bound (compare with (6.19)) on the minimal angular distance of spherical τ-designs.

Some known asymptotic results for spherical codes are consequences of the following special case of the bound (6.16):

$$|C| \leq 2\binom{k+n-1}{k} \quad \text{if} \quad 1 - \frac{(d(C))^2}{2} = \cos\varphi(C) \leq t_k^{1,1}. \qquad (6.24)$$

Since $t_k^{1,1} = p_k^{\frac{n-1}{2},\frac{n-1}{2}}$, by Corollary 5.17 we have

$$\sqrt{2}h_k \leq t_k^{1,1}\sqrt{\frac{(2k+n-4)(2k+n-2)}{k+n-2}} \leq 2\sqrt{k-1} \qquad (6.25)$$

where $n \geq 2$ and h_k is defined by (5.35) (take into account (5.36)).

Theorem 6.7 (KL bound [57]). *For any fixed φ, $0 < \varphi < \frac{\pi}{2}$, and $n \to \infty$*

$$\frac{1}{n}\log M(n,\varphi) \lesssim \frac{1+\sin\varphi}{2\sin\varphi}\log\frac{1+\sin\varphi}{2\sin\varphi} - \frac{1-\sin\varphi}{2\sin\varphi}\log\frac{1-\sin\varphi}{2\sin\varphi}. \qquad (6.26)$$

It is natural to compare this bound with the trivial sphere packing bound (2.17) which in the case of S^{n-1} takes the form

$$M(n,\varphi) \leq \frac{\sigma_{n-1}}{\sigma_{n-1}(\varphi/2)}$$

and gives rise (see (2.20)) to the following asymptotic bound for any fixed φ, $0 < \varphi < \frac{\pi}{2}$, as $n \to \infty$

$$\frac{1}{n}\log M(n,\varphi) \lesssim -\frac{1}{2}\log(1-\cos\varphi). \qquad (6.27)$$

The difference between the right-hand sides of (6.27) and (6.26) (hereafter we use log to the base 2) is a positive valued convex function which has a maximum equal to 0.0990 (with a deficiency) at the point φ^* being the unique root of the equation $\cos\varphi \ln\frac{1+\sin\varphi}{1-\sin\varphi} + (1+\cos\varphi)\sin\varphi = 0$. ($\varphi^* = 63°$ to within 10 angular sec.) Using the known inequality (see, for example, [103])

$$\left(\sin\frac{\varphi}{2}\right)^{n-1} M(n,\varphi) \leq \sqrt{2\pi n}\left(\sin\frac{\psi}{2}\right)^{n-1} M(n+1,\psi) \quad \text{if} \quad \varphi < \psi \qquad (6.28)$$

for $\psi = \varphi^*$ one can improve (6.27) by 0.0990 for smaller angles.

Corollary 6.8 ([57]). *For any fixed* φ, $0 < \varphi \leq \varphi^*$, *and* $n \to \infty$

$$\frac{1}{n} \log M(n, \varphi) \lesssim -\frac{1}{2} \log (1 - \cos \varphi) - 0.0990 \tag{6.29}$$

and, in particular,

$$M_n = M(n, \frac{\pi}{3}) \leq 2^{-n(0.401+o(1))} \quad \text{as } n \to \infty.$$

Corollary 6.9 ([57]).

$$\delta_n \leq 2^{-n(0.599+o(1))} \quad \text{as } n \to \infty. \tag{6.30}$$

Note that the first asymptotic bound $\delta_n \leq 2^{-n(c+o(1))}$ with $c > \frac{1}{2}$ was obtained by Sidelnikov [103].

One can use (6.8) and the asymptotic behavior of $L_Q^{(n)}(\cos 2\varphi)$ as $\varphi \to 0$ to obtain an upper bound for δ_n which is valid for all $n = 1, 2, \ldots$ (similar to the bound (6.18)).

Corollary 6.10 ([65]). *For any* n, $n = 1, 2, \ldots$,

$$\delta_n \leq \Delta(n) = \frac{\left(j(\frac{n}{2})\right)^n}{\left(\Gamma\left(\frac{n}{2}+1\right)\right)^2 4^n}, \tag{6.31}$$

where $j(\nu)$ *is the smallest positive zero of the Bessel function* $J_\nu(z)$ *(see [14]).*

In proving Corollary 6.10 it was also used that

$$\frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} \sim \frac{\Gamma(\frac{n}{2})}{2\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})} \sin^{n-1} \varphi \quad \text{as} \quad \varphi \to 0 \tag{6.32}$$

and that, for any fixed α and β (α , $\beta > -1$),

$$\arccos p_k^{\alpha, \beta} \sim \frac{j(\alpha)}{k} \quad \text{as} \quad k \to \infty \tag{6.33}$$

(see [113]). Since

$$j(v) = \nu + c\nu^{1/3} + O(\nu^{-1/3}) \quad \text{as} \quad \nu \to \infty, \text{where } c = 1{,}855757, \tag{6.34}$$

(see [122]), we have

$$\Delta(n) \sim \frac{e^{c(4n)^{1/3}}}{\pi n} \cdot \left(\frac{e}{4}\right)^n \quad \text{as} \quad n \to \infty. \tag{6.35}$$

Therefore, (6.31) gives the asymptotic bound

$$\delta_n \leq 2^{-n(\log_2 \frac{4}{e} + o(1))} \quad \text{as} \quad n \to \infty,$$

which is weaker in comparison with (6.30).

Since the function $L_Q^{(n)}(\cos\varphi)$ is monotone, we can use it to obtain upper bounds on the minimal angular distance $\varphi(C)$ or lower bounds on the maximal inner product $\cos\varphi(C)$ of spherical codes $C \subset S^{n-1}$ of a given size. Let

$$s(n, M) = \min_{C\subset S^{n-1}, |C|=M} \cos\varphi(C).$$

From (6.24) and (6.25) we have the following.

Corollary 6.11. *For any $n \geq 2$ and $k \geq 1$,*

$$s(n, 2\binom{k+n-1}{k}) \geq \sqrt{\frac{2(n+k-2)}{(n+2k-4)(n+2k-2)}} h_k.$$

Corollary 6.12 ([67]). *Let $n \to \infty$, then*

$$s(n, M) \gtrsim \begin{cases} \sqrt{\frac{4\log M}{n(\log n - \log\log M)}} & \textit{if } \frac{\log M}{\log n} \to \infty,\ \frac{\log M}{n} \to 0, \\ \frac{2\sqrt{\gamma(1+\gamma)}}{1+2\gamma} & \textit{if } \frac{\log M}{n} \to G(\gamma),\ 0 < \gamma < \infty, \end{cases}$$

where

$$G(x) = (1+x)H(\frac{1}{1+x}) \text{ and } H(x) = -x\log x - (1-x)\log(1-x).$$

Using (6.24), (6.25), (6.32)-(6.34) one can also find the asymptotic behavior of the bounds (6.14), (6.22), (6.19), (6.20), and (6.23) for spherical τ-designs $C \subset S^{n-1}$ in different processes: when τ is fixed and $n \to \infty$, when τ grows with n, and when n is fixed and $\tau \to \infty$ (see [67], [39]). In particular, in the latter process the both bounds (6.14) and (6.22) for spherical τ-designs $C \subset S^{n-1}$ have the following form:

$$|C| \gtrsim l(n)\tau^{n-1} \quad \text{as} \quad \tau \to \infty$$

with

$$l(n) = \frac{1}{2^{n-2}\Gamma(n)} \quad \text{and} \quad l(n) = \frac{2\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})\left(j(\frac{n-1}{2})\right)^{n-1}},$$

respectively. It is surprising that the ratio of the function $l(n)$ for DGS bound (6.14) to that for Yudin's bound (6.22) is exactly equal to $\Delta(n-1)$ (The definition and asymptotic behavior of $\Delta(n)$ are given in (6.31) and (6.35).)

Now we calculate bounds for the projective spaces $\mathbf{T}_m P^{n-1}$ (see Examples 2.4, 2.11 and (3.77)), which can be used to estimate the size of codes with given maximum modulus of the inner product (2.22) over distinct code points. Let $\mathbf{T}_1 = \mathbf{R}$, $\mathbf{T}_2 = \mathbf{C}$, and $\mathbf{T}_4 = \mathbf{H}$, the quaternionic algebra, and

$$\mathbf{T}_m S^{n-1} = \left\{ \mathbf{u} = (u_1, ..., u_n);\ u_i \in \mathbf{T}_m,\ \sum_{i=1}^{n} |u_i|^2 = 1 \right\}; \tag{6.36}$$

in particular, $\mathbf{R}S^{n-1} = S^{n-1}$. For any code $C \subset \mathbf{T}_m S^{n-1}$ we define the angle $\varsigma(C)$, $0 \le \varsigma(C) \le \frac{\pi}{2}$, as follows:

$$\cos \varsigma(C) = \max_{\mathbf{u}, \mathbf{v} \in C,\ \mathbf{u} \neq \mathbf{v}} |(\mathbf{u}, \mathbf{v})|. \tag{6.37}$$

If $\dot{\varsigma(C)} > 0$, then any pair of different points $\mathbf{u}$ and $\mathbf{v}$ of $C \subset \mathbf{T}_m S^{n-1}$ lies on different lines U and V of $\mathbf{T}_m P^{n-1}$ and (see (2.24))

$$d(U, V) = \sqrt{1 - |(\mathbf{u}, \mathbf{v})|} \ge \sqrt{1 - \cos \varsigma(C)} = \sqrt{2} \sin \frac{\varsigma(C)}{2}.$$

Hence, for any code $C \subset \mathbf{T}_m S^{n-1}$,

$$|C| \le A(\mathbf{T}_m P^{n-1}, \sqrt{2} \sin \frac{\varsigma(C)}{2}) \quad \text{if} \quad \dot{\varsigma(C)} > 0.$$

The compact metric spaces $\mathbf{T}_m P^{n-1}$ with the standard substitution $\sigma(d) = 2\left(1 - d^2\right)^2 - 1$ (or $\sigma(d) = \cos 2\varphi$ if $d = \sqrt{2} \sin \frac{\varphi}{2}$) are polynomial and $Q_i(t) = P_i^{\alpha,\beta}(t)$, where

$$\alpha = \frac{m}{2}(n-1) - 1, \quad \beta = \frac{m}{2} - 1. \tag{6.38}$$

Let $L_{\alpha,\beta}(\sigma) = L_Q(\sigma)$ if $Q = \left\{ P_i^{\alpha,\beta}(t), i = 0, 1, ... \right\}$. From (5.90), (5.96), (5.97), and (2.79) it follows that

$$L_{\alpha,\beta}(\sigma) = \binom{k+\alpha}{k-1} \frac{\binom{k+\alpha+\beta+\varepsilon}{k-1+\varepsilon}}{\binom{k+\beta-1+\varepsilon}{k-1+\varepsilon}} \left(1 - \frac{P_{k-1}^{\alpha+1,\beta+\varepsilon}(\sigma)}{P_k^{\alpha,\beta+\varepsilon}(\sigma)} \right), \tag{6.39}$$

where $k = k(\sigma)$ and $\varepsilon = \varepsilon(\sigma)$ (or, equivalently, $p_{k-1+\varepsilon}^{\alpha+1,\beta+1-\varepsilon} \le \sigma < p_k^{\alpha+1,\beta+\varepsilon}$); in particular, for any $l = 1, 2, ...$ and $\theta \in \{0, 1\}$, $L_{\alpha,\beta}(p_{l-\theta}^{\alpha+1,\beta+\theta})$ equals (2.79). By Lemma 3.22 the system Q for $\mathbf{T}_m P^{n-1}$ satisfies the strengthened Krein condition and we can use Theorem 5.48.

Theorem 6.13 ([66], [68]). *For any code $C \subset \mathbf{T}_m S^{n-1}$ with $\dot{\varsigma(C)} > 0$,*

$$|C| \le L_{\alpha,\beta}(\cos 2\varsigma(C)), \tag{6.40}$$

and, in particular, for any $l = 1, 2, ...$ and $\theta \in \{0, 1\}$,

$$|C| \le \frac{\binom{l+\alpha+1-\theta}{l-\theta}\binom{l+\alpha+\beta+1}{l}}{\binom{l+\beta}{l}} \text{ if } \cos 2\varsigma(C) \le p_{l-\theta}^{\alpha+1,\beta+\theta}, \tag{6.41}$$

where α, β, and $L_{\alpha,\beta}(\sigma)$ are defined in (6.38) and (6.39).

In the first interval, where $-1 \le \cos 2\varsigma(C) \le -1 + \frac{2m}{mn+2}$ or $0 \le \cos\varsigma(C) \le \sqrt{\frac{m}{mn+2}}$, (6.40) coincides with Welch bound $|C| \le \frac{n(1-\cos^2\varsigma(C))}{1-n\cos^2\varsigma(C)}$. For larger $\cos\varsigma(C)$ (6.40) improves the Sidelnikov [102] and Welch [123] bounds for real and complex codes C with both the given $\cos\varsigma(C)$ and the given periodic and aperiodic crosscorrelation (see Chapter (Helleseth, Kumar), where (6.40) for the first four intervals is also given). By Corollary 5.54 the right-hand side of the first inequality in (6.41) is the lower bound (5.145) on the size of $(2l-\theta)$-designs in $\mathbf{T}_m P^{n-1}$. It was also obtained in Section 2.2 and included in Table 2.1. Numerous examples of codes with equality in (6.40) and tight designs in $\mathbf{T}_m P^{n-1}$ can be found in [34], [35], [66], [53], [54], [68]. They all are distance-regular (see Corollary 5.51 and Remark 5.58).

6.2. The Hamming space

The Hamming space $X = H_v^n$ $(n, v = 2, 3, \ldots)$ (see Examples 2.1, 2.8, 4.1) consists of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ where $x_i \in \{0, 1, \ldots, v-1\}$ with the distance $d(\mathbf{x}, \mathbf{y})$ being equal to the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ. This space is distance-transitive and hence distance invariant with the measure $w_i = v^{-n}\binom{n}{i}(v-1)^i$, $i = 0, 1, \ldots, n$. H_v^n is a self-dual (systems Q and P coincide) polynomial graph [30] with respect to the linear standard substitution $\sigma(d) = 1 - \frac{2d}{n}$. The relationship (3.6) in fact proves this fact and shows that the system $\{Q_k(t), k = 0, 1, \ldots, n\}$ may be defined by means of

$$Q_k(\sigma(d)) = (r_k)^{-1} K_k^{n,v}(d) \tag{6.42}$$

where

$$K_k^n(z) = K_k^{n,v}(z) = \sum_{j=0}^{k} (-1)^j (v-1)^{k-j} \binom{z}{j} \binom{n-z}{k-j} \tag{6.43}$$

is the Krawtchouk polynomial of degree k and $r_k = \binom{n}{k}(v-1)^k$. For the polynomials $K_k^{n-a-b}(z-a)$ where $a, b \in \{0, 1\}$ the following orthogonality conditions hold:

$$\sum_{d=0}^{n} K_i^{n-a-b}(d-a) K_j^{n-a-b}(d-a)(v-1)^d d^a (n-d)^b \binom{n}{d} = 0$$

if $i, j \in \{0, 1, \ldots, n-a-b\}$ and $i \ne j$ (see [68] or [71]). Considering (4.33)-(4.38) for this case one can find

$$c^{0,1} = \frac{v}{2}, \quad c^{1,0} = \frac{v}{2\,(v-1)}, \quad c^{1,1} = \frac{nv^2}{4(n-1)(v-1)},$$

$$r_k^{0,1} = \binom{n-1}{k}(v-1)^k, \; r_k^{1,0} = \frac{\left(\sum\limits_{i=0}^{k} \binom{n}{i}(v-1)^i\right)^2}{\binom{n-1}{k}(v-1)^k}, \; r_k^{1,1} = \frac{\left(\sum\limits_{i=0}^{k} \binom{n-1}{i}(v-1)^i\right)^2}{\binom{n-2}{k}(v-1)^k},$$

$$Q_k^{0,1}(\sigma(d)) = \frac{K_k^{n-1}(d)}{\binom{n-1}{k}(v-1)^k}, \; Q_k^{1,0}(\sigma(d)) = \frac{K_k^{n-1}(d-1)}{\sum\limits_{i=0}^{k}\binom{n}{i}(v-1)^i},$$

$$Q_k^{1,1}(\sigma(d)) = \frac{K_k^{n-2}(d-1)}{\sum\limits_{i=0}^{k}\binom{n-1}{i}(v-1)^i},$$

and, in particular, prove that

$$d_k^{a,b}(Q) = d_k^{a,b}(P) = d_k(n-a-b)+a, \tag{6.44}$$

where $d_k(n) = d_k(n,v)$ is the smallest root[6] of (6.43). By Lemma 3.25 (see also Example 3.23) the system Q for the Hamming space satisfies the strengthened Krein condition.

Let

$$L^{n,v}(d) = L_Q(\sigma(d)) \text{ and } M_{2k-1}^{n,v}(d) = M_{2k-1}(\sigma(d))$$

(see (5.83)). Then using (5.84) and (5.96) we have

$$M_{2k-1}^{n,v}(d) = \sum_{i=0}^{k-1}\binom{n}{i}(v-1)^i - \binom{n}{k}(v-1)^k\frac{K_{k-1}^{n-1}(d-1)}{K_k^n(d)} \tag{6.45}$$

and

$$L^{n,v}(d) = \begin{cases} M_{2k-1}^{n,v}(d) & \text{if} \quad d_k(n-1) < d-1 \le d_{k-1}(n-2) \\ vM_{2k-1}^{n-1,v}(d) & \text{if} \quad d_k(n-2) < d-1 \le d_k(n-1) \end{cases} \tag{6.46}$$

and, in particular (see (5.97)), for any $\theta \in \{0,1\}$ and any integer l, $1 \le l+\theta \le n$,

$$L^{n,v}(d_l(n-1-\theta)+1) = L_Q(t_l^{1,\theta}) = v^\theta \sum_{i=0}^{l}\binom{n-\theta}{i}(v-1)^i. \tag{6.47}$$

It is useful to introduce the following notation for the right-hand side of (6.47):

$$H^{n,v}(2l+1+\theta) = v^\theta \sum_{i=0}^{l}\binom{n-\theta}{i}(v-1)^i. \tag{6.48}$$

For $\theta = 0$ it equals the size of a metric ball of radius l around any point of H_v^n. Note that if $f(t)$ and $\widetilde{f}(t)$ are annihilating and dual-annihilating polynomials for a code $C \subset H_v^n$, then $f(\sigma(z))$ and $\widetilde{f}(\sigma(z))$ are polynomials in z of the same degree ($\sigma(d)$ is a linear function) such that all nonzero distances or dual distances are their

[6] Since $t_0^{1,1} = -1$ and $t_{n+1} = 1$ (see (5.89) and (5.2)), and $\sigma(d) = 1 - 2\frac{d}{n}$, we should put $d_0(n) = n+1$ and $d_{n+1}(n) = 0$.

roots. These polynomials in z are referred to as *annihilating for distances or for dual distances of* C, respectively. Such nonzero polynomials of the minimal degree are called *minimal*.

The universal bounds for designs and codes in Hamming space are complete because Conjectures 3.26 and 4.20 are true and the strengthened Krein condition is satisfied. Their symmetry is explained by the coinciding systems Q and P (compare with Johnson space). The following statements are consequences of Theorems 4.27, 5.59, 5.55, 5.63, 5.60 and 5.62.

Theorem 6.14 ([107]). *For any code* $C \subseteq H_v^n$,

$$v^{d'(C)-1} \leq |C| \leq v^{n-d(C)+1}. \tag{6.49}$$

Each of the bounds is attained if and only if $d(C) + d'(C) = n + 2$. *In this case* $\prod_{i=d(C)}^{n} (i-d)$ *and* $\prod_{i=d'(C)}^{n} (i-d)$ *are annihilating polynomials for distances and dual distances of* C *respectively.*

Codes for which these bounds are attained are called MDS-codes. The Reed-Solomon codes belong to the class. Detailed information about the existence of MDS-codes is contained in [78].

Theorem 6.15 ([92], [50]). *For any code* $C \subset H_v^n$,

$$H^{n,v}(d'(C)) \leq |C| \leq \frac{v^n}{H^{n,v}(d(C))}. \tag{6.50}$$

Equality in the left-hand side holds if and only if $d'(C) = 2s(C) + 1 - \beta(C)$; *in this case the polynomial*

$$(n-z)^{\beta} K_{s-\beta}^{n-1-\beta}(z-1) \tag{6.51}$$

with $s = s(C)$, $\beta = \beta(C)$ *is minimal for distances of* C, *and* C *is a distance-regular code. Equality in the right-hand side holds if and only if* $d(C) = 2s'(C) + 1 - \beta'(C)$; *in this case the polynomial (6.51) with* $s = s'(C)$, $\beta = \beta'(C)$ *is minimal for dual distances of* C.

Codes of size equal to the left-hand side or the right-hand side of (6.50) are called *tight* designs or *perfect* codes respectively (in particular, this means that we consider perfect codes with even minimal distance as well). The problem of existence of perfect codes is considered in [76], [78] and in Chapter (Honkala, Tietavainen). To simplify some future formulations we assume that $d(C) \geq 2$ and $d'(C) \geq 2$.

Theorem 6.16 ([64], [71]). *Let* C *be a code in* H_v^n, $\sigma = \sigma(d(C))$, $k = k(\sigma)$, $\varepsilon = \varepsilon(\sigma)$, $\sigma' = \sigma(d'(C))$, $k' = k(\sigma')$, $\varepsilon' = \varepsilon(\sigma')$ *and hence* $d_k(n-1-\varepsilon) < d(C) - 1 \leq d_{k-1+\varepsilon}(n-2+\varepsilon)$, $d_{k'}(n-1-\varepsilon') < d'(C) - 1 \leq d_{k'-1+\varepsilon'}(n-2+\varepsilon')$. *Then*

$$\frac{v^n}{L^{n,v}(d'(C))} \leq |C| \leq L^{n,v}(d(C)) \tag{6.52}$$

where $L^{n,v}(d)$ is defined by (6.46) and, in particular,

$$|C| \le H^{n,v}(2l+1+\theta) \text{ if } d(C) \ge d_l(n-1-\theta)+1 \tag{6.53}$$

$$|C| \ge \frac{v^n}{H^{n,v}(2l'+1+\theta')} \text{ if } d'(C) \ge d_{l'}(n-1-\theta')+1. \tag{6.54}$$

The upper bound in (6.52) is attained if and only if

$$d'(C) \ge 2s(C) - \beta(C).$$

Herewith, if $|C| = L^{n,v}(d(C))$ and $d(C) = d_{k-1+\varepsilon}(n-2+\varepsilon)+1$, then $s(C) = k$, $\beta(C) = 1-\varepsilon$, $\tau(C) = 2s(C)-\beta(C)$ (and hence C is a tight design), and the polynomial

$$(n-z)^{\beta} K_{s-\beta}^{n-1-\beta}(z-1) \tag{6.55}$$

with $s = s(C)$, $\beta = \beta(C)$ is minimal for distances of C; if $|C| = L^{n,v}(d(C))$ and $d(C) \ne d_{k-1+\varepsilon}(n-2+\varepsilon)+1$, then $s(C) = k+\varepsilon$, $\beta(C) = \varepsilon$, $\tau(C) = 2s(C)-\beta(C)-1$ and the polynomial

$$(z-d)(z-n)^{\beta} \sum_{i=0}^{s-1} \frac{K_i^{n-1-\beta}(d-1) K_i^{n-1-\beta}(z-1)}{\binom{n-1-\beta}{i}(v-1)^i} \tag{6.56}$$

with $s = s(C)$, $\beta = \beta(C)$, $d = d(C)$ is minimal for distances of C. In the both cases C is a distance-regular code. The lower bound in (6.52) is attained if and only if

$$d(C) \ge 2s'(C) - \beta'(C).$$

If $|C|L^{n,v}(d'(C)) = v^n$ and $d'(C) = d_{k'-1+\varepsilon'}(n-2+\varepsilon')+1$, then $s'(C) = k'$, $\beta'(C) = 1-\varepsilon'$, $d(C) = 2s'(C)+1-\beta'(C)$ (and hence C is a perfect code), and (6.55) with $s = s'(C)$, $\beta = \beta'(C)$ is minimal for dual distances of C; if $|C|L^{n,v}(d'(C)) = v^n$ and $d'(C) \ne d_{k'-1+\varepsilon'}(n-2+\varepsilon')+1$, then $s'(C) = k'+\varepsilon'$, $\beta'(C) = \varepsilon'$, $d(C) = 2s'(C)-\beta'(C)$ and the polynomial (6.56) with $s = s'(C)$, $\beta = \beta'(C)$, $d = d'(C)$ is minimal for dual distances of C.

The values $L^{n,2}(d)$, that give rise (see (6.52)) to the first five bounds for binary codes and designs, are given in Table 6.3. The first three bounds for codes are the same as in the case of the Euclidean sphere. This is explained by the coincidence of polynomials $Q_i(t)$, $i = 0, 1, 2$. The first bounds for codes and designs belong to Plotkin [89] and Friedman [41], respectively. The third bound for binary codes, in some range, was improved by Tietäväinen [116] with the help of a special method (see also [60]). At the same time, by Theorem 6.16 the general third bound

$$|C| \le \frac{vd\left((n(v-1)+1)(n(v-1)-vd+2)-v\right)}{vd\left(2n(v-1)-v+2-vd\right)-n(n-1)(v-1)^2},$$

Degree of $f^{(\sigma)}(t)$	$L_Q(\sigma) = L^{n,2}(d)$ for $\sigma = 1 - 2\frac{d}{n}$	Interval for $\sigma = 1 - 2\frac{d}{n}$
1	$\frac{1-\sigma}{-\sigma}$	$[-1, -\frac{1}{n}]$
2	$\frac{2n(1-\sigma)}{1-n\sigma}$	$[-\frac{1}{n}, 0]$
3	$\frac{n(2+(n+1)\sigma)(1-\sigma)}{1-n\sigma^2}$	$[0, \frac{\sqrt{n-1}-1}{n}]$
4	$\frac{2n(n-2+\sigma n^2)(1-\sigma)}{n-1-(n\sigma-1)^2}$	$[\frac{\sqrt{n-1}-1}{n}, \frac{\sqrt{n-2}}{n}]$
5	$\frac{(4n^3\sigma+n^2(n^2+n+2)\sigma^2-(n-1)(n-2)^2)(1-\sigma)}{2\sigma(3n-2-n^2\sigma^2)}$	$[\frac{\sqrt{n-2}}{n}, \frac{\sqrt{3n-5}-1}{n}]$

Table 6.3: Values $L_Q(\sigma) = L^{n,2}(d)$, $\sigma = 1 - 2\frac{d}{n}$, for the First Five Intervals

where $d_2(n-1)+1 \leq d = d(C) < d_1(n-2)+1$, can be attained only for distance-regular 3-designs with two distances d and $\frac{(v-1)(n-1)}{v}\left(1 + \frac{1}{(v-1)(n-1)-v(d-1)}\right) + 1$; there exist infinite families of such codes (see [24] and Table 6.4).

The parameters of the known (to the author) codes for which the upper bound in (6.52) is attained are presented in Table 6.4. This class of codes consists of distance-regular codes and includes all tight designs. The lower bound in (6.52) is attained for codes with dual parameters (in particular, for all perfect codes), if they exist.

For the Hamming space Conjectures 3.26 and 4.20 are completely proved.

Theorem 6.17 ([29], [71]). *For any code $C \subset H_v^n$,*

$$\frac{v^n}{H^{n,v}(2s'(C)) + 1 - \beta'C))} \leq |C| \leq H^{n,v}(2s(C) + 1 - \beta(C)). \tag{6.57}$$

The conditions for attainability of the lower and upper bound in (6.57) coincide with those for the upper and lower bounds in (6.50) respectively.

All statements of Theorem 4.5 are consequences of universal bounds (6.49), (6.50), (6.52), (6.57). Moreover, together with the inequality $d(C)+d'(C) \leq n+2$ they imply two other inequalities

$$H^{n,v}(d'(C))H^{n,v}(d(C)) \leq v^n, \quad L^{n,v}(d'(C))L^{n,v}(d(C)) \leq v^n$$

which show that $d(C)$ and $d'(C)$ cannot be too large simultaneously. The last inequality is stronger and gives rise to an improvement of some bounds on minimal distance of self-dual codes [71].

The following result of Tietäväinen [117], [118] can be considered as a special case of Theorem 4.14.

Theorem 6.18 (Tietäväinen bound). *If for a code $C \subset H_v^n$, $d'(C) = 2l + 1 - \theta$ for some $l \in \{1, 2, ...\}$ and $\theta \in \{0, 1\}$, then*

$$\rho(C) \leq d_l^{n,v}(n - 1 + \theta). \tag{6.58}$$

n	v	$s(C)$	$\beta(C)$	$d'(C)$	$d(C) = d$	$\lvert C\rvert = L^{n,v}(d)$	Comments
n	v	1	1	2	n	v	
n	v	1	0	2	$n > d > \frac{(v-1)n+1}{v}$	$\frac{vd}{vd-n(v-1)}$	coexistence [99] with resolvable block-designs $2-(L, L/v, n-d)$
n	v	1	0	3	$\frac{(v-1)n+1}{v}$	$(v-1)n+1$	coexistence [100] with affine resolvable block-designs $2-(L, L/v, n-d)$
$p^l q$	$q = p^m$	2	1	3	$n - p^l$	nq	$l, m = 1, 2,$ Semakov, Zinovjev, Zaitzev [100]
$qh+$ $h-q$	q	2	1	3	$n - h$	q^3	$2\lvert q, h\rvert q, 2 < h < q$ Denniston [37]
$q^2 + 1$	q	2	0	4	$n - q - 1$	q^4	ovoid in $PG(3, q)$ Bose [19], Qvist [90]
56	3	2	0	4	36	3^6	projective cap Hill [51]
78	4	2	0	4	56	4^6	projective cap Hill [52]
$4l$	2	2	1	4	$n/2$	$2n$	Hadamard codes
$q + 2$	q	2	1	4	$n - 2$	q^3	$2\lvert q$ hyperoval in $PG(2, q)$ Bose [19]
11	3	2	0	5	6	3^5	projection of Golay code
12	3	3	1	6	6	3^6	Golay code [49]
22	2	3	0	6	8	2^{10}	projection of Golay code
23	2	3	0	7	8	2^{11}	projection of Golay code
24	2	4	1	8	8	2^{12}	Golay code [49]
n	2	$\lfloor n/2 \rfloor$	$\frac{1+(-1)^n}{2}$	n	2	2^{n-1}	even weight code

Table 6.4: The Parameters of the Known Codes $C \subset H_v^n$ for Which the Upper Bound (6.52) Is Attained

In [39] it was proved that equality in (6.58) holds if and only if there exists a point $x \in H_v^n \backslash C$ such that the polynomial $(n-z)^{1-\theta} K_l^{n-1-\theta,v}(z)$ is minimal for all distances $d(x,y)$, $y \in C$, and that in the binary case the bound (6.58) is attained for all tight $2l$-designs.

Some asymptotic results are consequences of the following special case of the bound (6.53)

$$|C| \le \sum_{i=0}^{k} \binom{n}{i} (v-1)^i \quad \text{if} \quad d(C) \ge d_k(n-1)+1 \tag{6.59}$$

and estimates for $d_k(n)$ (see Corollaries 5.20 and 5.21). In particular, we have the following result of McEliece, Rodemich, Rumsey Jr., and Welch [79].

Theorem 6.19 (The first form of the MRRW bound). *If v is fixed and $\lim\limits_{n\to\infty} \frac{d}{n} = \delta$ where $0 \le \delta \le \frac{v-1}{v}$, then*

$$\frac{1}{n} \log_v A(H_v^n, d) \lesssim H_v(\gamma_v(\delta)) \tag{6.60}$$

where

$$H_v(x) = -x \log_v x - (1-x) \log_v (1-x) + \log_v (v-1), \tag{6.61}$$

$$\gamma_v(x) = \frac{1}{v} \left(v - 1 - (v-2)x - 2\sqrt{(v-1)x(1-x)} \right). \tag{6.62}$$

Notice that $\gamma_v(x)$, $0 \le x \le (v-1)/v$, is a decreasing continuous function which, as one can check, coincides with its inverse function, that is

$$\gamma_v(\gamma_v(x)) = x \text{ when } 0 \le x \le \frac{v-1}{v}.$$

Using (6.54) one can obtain the following result.

Theorem 6.20 ([71]). *If v is fixed and $\lim\limits_{n\to\infty} \frac{d}{n} = \delta$ where $0 \le \delta \le \frac{v-1}{v}$, then*

$$\frac{1}{n} \log_v B(H_v^n, d) \gtrsim 1 - H_v(\gamma_v(\delta)). \tag{6.63}$$

Similar to the Euclidean case the asymptotic bound (6.60) (and also (6.63)) is not good for small δ. In this case it becomes even worse than the trivial sphere packing bound (the upper bound in (6.50)) which in the stated asymptotic process takes the form

$$\frac{1}{n} \log_v A(H_v^n, d) \lesssim 1 - H_v(\delta). \tag{6.64}$$

In the binary case the following inequality (Bassalygo-Elias Lemma [11])

$$\binom{n}{w} A(H_2^n, 2d) \le 2^n A(J_w^n, d) \tag{6.65}$$

is valid. This plays the same role as (6.28) and allows us to improve (6.60) for small δ using the universal bound for codes in Johnson space. In the non-binary case the situation is more difficult because a natural generalization of (6.65) connects H_v^n with a metric space which is not polynomial (see [106], [1], [114]).

Similar difficulties arise for studying codes in $\widetilde{H}_v^n = \left\{ \mathbf{u} = \frac{1}{\sqrt{n}}(u_1, ..., u_n);\ u_i \in \widetilde{H}_v \right\}$, where $\widetilde{H}_v$ is the set of all v-th roots of unity. Since $\widetilde{H}_v^n \subset \mathbf{C}S^{n-1}$ (see (6.36)), one can consider the problem of finding the maximum size of a code $C \subset \widetilde{H}_v^n$ with a given value $\varsigma = \varsigma(C)$ (see (6.37)), which is of important interest for codes with given cross-correlation properties. However, the problem to describe all polynomials $f(t)$ such that $f(|(\mathbf{u}, \mathbf{v})|^2)$ is an FDNDF on $\widetilde{H}_v^n$, $v \geq 3$, is solved now (1997) only for polynomials of small degrees (the corresponding first three bounds can be found in [66] and Chapter (Helleseth, Kumar). At the same time, for $v = 2$, 2^{n-1} pairs $U = \{\mathbf{u}, -\mathbf{u}\}$ of complementary points of $\widetilde{H}_2^n$ (lines of "projective Hamming space") with the distance $d(U, V) = \min_{\mathbf{u} \in \mathbf{U}, \mathbf{v} \in \mathbf{V}} d(\mathbf{u}, \mathbf{v})$ form a polynomial graph (called a *folded cube* [23]) with $D = \lfloor \frac{n}{2} \rfloor$ and the (standard for even n) substitution $\sigma(d) = 2(1 - \frac{2d}{n})^2 - 1$. The corresponding inequality $|C| \leq L_Q(\cos 2\dot{\varsigma}(C))$ if $\dot{\varsigma}(C) > 0$ (cf. (6.40)) was proved and calculated in [69]. This improves the Sidelnikov bound [102] for binary codes when $\cos \dot{\varsigma}(C) > \frac{\sqrt{3n-8}}{n}$. The first four bounds are also given in [69] and Chapter (Helleseth, Kumar).

6.3. The Johnson space

We consider Johnson space J_w^n ($n = 2, 3, ...\ ; w = 1, ..., \lfloor n/2 \rfloor$) as the set X of all w-subsets of the n-set $\{1, ..., n\}$, where the distance between two elements $x, y \in J_w^n$ is defined to be $w - |x \cap y|$ (see Examples 2.2, 2.9). J_w^n can also be considered as the subset of the binary Hamming space H_2^n consisting of all vectors which have exactly w non-zero coordinates, with distance being equal to half of the Hamming distance. For this reason codes in J_w^n are often called *constant weight* codes. In this case $\Delta(X) = \{0, 1, ..., w\}$ and hence $s(X) = w$, $D(X) = w$. As shown by Delsarte [30], Johnson space is a polynomial graph with

$$v_i = \binom{w}{i}\binom{n-w}{i}, \quad i = 0, 1, \ldots, w, \tag{6.66}$$

$$r_i = \binom{n}{i} - \binom{n}{i-1}, \quad i = 1, \ldots, w \quad (r_0 = 1), \tag{6.67}$$

with the standard substitutions (see Example 4.2)

$$\sigma_Q(d) = 1 - 2\frac{d}{w}, \quad \sigma_P(d) = 1 - 2\frac{d(n+1-d)}{w(n+1-w)}. \tag{6.68}$$

The corresponding systems $\{Q_i(t),\ i = 0, 1, ..., w\}$ and $\{P_i(t),\ i = 0, 1, ..., w\}$ may be defined by

$$Q_i(\sigma_Q(d)) = J_i(d), \quad i = 0, 1, ..., w, \tag{6.69}$$

where

$$J_i(z) = J_i^{n,w}(z) = \sum_{j=0}^{i} (-1)^j \frac{\binom{i}{j}\binom{n+1-i}{j}}{\binom{w}{j}\binom{n-w}{j}} \binom{z}{j} \tag{6.70}$$

are Hahn polynomials, and by

$$P_i(\sigma_P(d)) = (v_i)^{-1} E_i(d), \; i = 0, 1, ..., w, \tag{6.71}$$

where

$$E_i(z) = E_i^{n,w}(z) = \sum_{j=0}^{i} (-1)^{i-j} \binom{w-j}{i-j} \binom{w-z}{j} \binom{n-w+j-z}{j}. \tag{6.72}$$

(We shall verify that (6.72) is really a polynomial of degree i in $\sigma_P(z)$.) By Lemma 3.25 (see also Example 3.23) the system Q for the Johnson space satisfies the strengthened Krein condition. Unfortunately this question with respect to the system P is still open (1997).

A set C is a simple τ-design in J_w^n with respect to a linear substitution $\sigma_Q(d)$ if and only if (see [30]) it forms a *block τ-design* or *tactical configuration.* This object is defined as a set C of w-subsets called *blocks* a of a n-set such that, for some λ, each τ-subset of the n-set belongs to exactly λ w-subsets of C and denoted by $S_\lambda(\tau, w, n)$. In other words, $S_\lambda(\tau, w, n)$ is a binary code consisting of words of length n with w ones such that in any τ positions the word of all ones occurs the same number λ times. The systems $S_\lambda(\tau, w, n)$ with $\lambda = 1$ are called *Steiner systems* and denoted by $S(\tau, w, n)$. Weighted τ-designs in J_w^n were also considered when C is a multiset and λ is the total multiplicity of the w-subsets of C containing a fixed τ-subset. Since a τ-design is an i-design for any i, $0 \leq i \leq \tau$, we have the following necessary conditions for the existence of a τ-design:

$$\lambda \binom{n-i}{\tau-i} \equiv 0 \pmod{\binom{w-i}{\tau-i}}, \; i = 0, 1, ..., \tau. \tag{6.73}$$

By Wilson's theorem [124], for any integers n, w, and τ, $0 \leq \tau \leq w \leq n$, there exists a (in general, weighted) system $S_\lambda(\tau, w, n)$ for all sufficiently large integers λ satisfying the congruencies (6.73). We emphasize once more that by Theorem 2.15 the lower bounds given below are already valid for the number of different blocks of a τ-design.

The following statement follows from Theorem 4.27 if one takes into account that by (4.68), (4.70), (6.66), and (6.68),

$$N_Q(d) = \sum_{j=0}^{w} \binom{n-w}{j} \binom{d-1}{j} = \binom{n-w+d-1}{n-w}.$$

Theorem 6.21.

$$\frac{\binom{n}{d'(C)-1}}{\binom{w}{d'(C)-1}} \leq |C| \leq \frac{\binom{n}{w+1-d(C)}}{\binom{w}{w+1-d(C)}}. \tag{6.74}$$

Each of the bounds is attained if and only if $d(C)+d'(C)=w+2$. In this case C is a Steiner system, and $\prod\limits_{i=d(C)}^{n}(i-d)$ and $\prod\limits_{i=d'(C)}^{n}(i-d)$ are annihilating polynomials for distances and dual distances of C, respectively.

There exists a simple combinatorial proof of (6.74). Complete information about the existence of Steiner systems can be found in [16].

Hahn polynomials (6.70) and the corresponding polynomials $Q_i(t)$ defined by (6.69) satisfy the following conditions of orthogonality and normalization:

$$r_i \sum_{d=0}^{w} J_i(d) J_j(d) v_d = \delta_{i,j} \binom{n}{w},$$

$$Q_k(1) = J_k(0) = 1, \quad k = 0, 1, ..., w.$$

In particular, $J_0(d) = 1$,

$$J_1(d) = 1 - \frac{nd}{w(n-w)}, \; J_2(d) = 1 - \frac{2(n-1)d}{w(n-w)} + \frac{(n-1)(n-2)d(d-1)}{w(w-1)(n-w)(n-w-1)}.$$

Using (6.67) - (6.70) and Theorem 5.3 one can obtain the recurrence (5.49) for polynomials $Q_i(t)$. This recurrence implies by induction that

$$Q_k(-1) = J_k(w) = (-1)^k \frac{\binom{w}{k}}{\binom{n-w}{k}}, \quad k = 0, 1, ..., w. \tag{6.75}$$

Moreover, for $k = 0, 1, ..., w$, we have the following equalities (see (5.14) and (5.24)):

$$T_k(1,1) = \sum_{i=0}^{k} r_i = \binom{n}{k}, \tag{6.76}$$

$$T_k(1,-1) = \sum_{i=0}^{k} r_i Q_i(-1) = Q_k(-1) \binom{n}{k} \frac{w-k}{w}, \tag{6.77}$$

$$R_k(1,1,-1) = Q_k(-1) \binom{n}{k} \frac{k}{w}. \tag{6.78}$$

Using Lemma 5.24, (6.67), (6.76)-(6.78) we find that in this case the polynomials $Q_k^{1,0}(t)$ and $Q_k^{0,1}(t)$ for $k \leq w-1$, and $Q_k^{1,1}(t)$ for $k \leq w-2$ are defined as follows:

$$Q_k^{1,0}(\sigma_Q(d)) = \frac{(w-k)(n-w-k)}{n-2k} \cdot \frac{J_k(d) - J_{k+1}(d)}{d}, \tag{6.79}$$

$$Q_k^{0,1}(\sigma_Q(d)) = \frac{w}{n-2k} \cdot \frac{(w-k)J_k(d) + (n-w-k)J_{k+1}(d)}{w-d}, \tag{6.80}$$

$$Q_k^{1,1}(\sigma_Q(d)) = \frac{w(n-w-k)}{(n-k)(n-2k)} \cdot \frac{w(w-k)J_k(d) - (w(w-k) - d(n-2k))J_{k+1}(d)}{d(w-d)} \tag{6.81}$$

(the values $Q_k^{1,0}(1) = Q_k^{1,1}(1) = 1$ and the values $Q_k^{0,1}(-1)$ and $Q_k^{1,1}(-1)$ coincide with the corresponding limits of the right-hand sides when d tends to 0 or w.) At the same time Example 3.23 and Theorem 3.24 show that

$$Q_k^{0,1}(\sigma_Q(d)) = J_k^{n-1,w-1}(d) \tag{6.82}$$

and, in particular,

$$\sum_{i=0}^{k} r_i^{0,1} = \binom{n-1}{k}, \ k = 0, 1, ..., w-1. \tag{6.83}$$

Let $d_k^{a,b}(U) = d_k^{a,b}(U, n, w)$ for $U = Q$ and $U = P$. Since in the general case (see (5.2) and (5.89)) $t_{N-1}^{1,0} = t_{N-2}^{1,1} = t_{N,N-1}$ and $t_0^{1,1} = -1$, we have $d_w^{1,0}(U) = d_{w-1}^{1,1}(U) = 1$ and $d_0^{1,1}(U) = w$. From (6.82) it follows that

$$d_k^{0,1}(Q, n, w) = d_k(Q, n-1, w-1).$$

We shall use that $d_1^{1,0}(Q, n, w) = \frac{w(n-w)}{n-1}$, $d_1^{1,1}(Q, n, w) = \frac{(w-1)(n-w)}{n-2}$, and
$d_2^{1,0}(Q, n, w) = \frac{(n-2)(2w(n-w)-n-1) - \sqrt{(n-2)(4w(n-w)(w(n-w)-3n+5)+(n-2)(n+1)^2)}}{2(n-2)(n-3)}$.

Assuming for the Johnson space $L_Q(\sigma_Q(d)) = L^{n,w,2}(d)$ and using (5.83), (5.85), (5.95), (6.76), (6.79), and (6.82) one can find that for $1 < d \leq w$,

$$L^{n,w,2}(d) = \begin{cases} M_{2k-1}^{(d)} & \text{if } d_k^{1,0}(Q, n, w) < d \leq d_{k-1}^{1,1}(Q, n, w), \\ M_{2k}^{(d)} & \text{if } d_k^{1,1}(Q, n, w) < d \leq d_k^{1,0}(Q, n, w), \end{cases} \tag{6.84}$$

where

$$M_{2k-1}^{(d)} = \left(1 - \frac{(w-k+1)(n-w-k+1)}{n-2k+2} \frac{J_{k-1}^{n,w}(d) - J_k^{n,w}(d)}{dJ_k^{n,w}(d)}\right) \binom{n}{k-1},$$

$$M_{2k}^{(d)} = \left(1 - \frac{(w-k)(n-w-k)}{n-2k} \frac{J_k^{n,w}(d) - J_{k+1}^{n,w}(d)}{dJ_k^{n-1,w-1}(d)}\right) \binom{n}{k}.$$

Using (5.97), (6.75), (6.76), and (6.83) one can also find the following special values of (6.84) for $d = d_k^{1,\theta}(Q, n, w)$ where $\theta \in \{0, 1\}$ and $1 \leq k + \theta \leq w$:

$$L^{n,w,2}(d_k^{1,\theta}(Q, n, w)) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{k} r_i^{0,\theta} = \left(\frac{n}{w}\right)^{\theta} \binom{n-\theta}{k}. \tag{6.85}$$

The function (6.72) is a polynomial of degree i in $\sigma_P(z)$ since

$$(w-z-l)(n+1-z-w+l) = -z(n+1-z) + (w-l)(n+1-w+l) \qquad (6.86)$$

for any $l = 0, 1, ..., j$. The polynomials $P_i(t)$ defined by (6.71) satisfy the following conditions of orthogonality and normalization:

$$v_i \sum_{d=0}^{w} P_i(\sigma_P(d) P_j(\sigma_P(d)) r_d = \delta_{i,j} \binom{n}{w}, \qquad (6.87)$$

$$P_i(1) = (v_i)^{-1} E_i(0) = 1, \quad i = 0, 1, ..., w.$$

From (6.66), (6.71), (6.72), (6.86), and Theorem 5.3 it follows that for these polynomials $P_i(t)$ the recurrence (5.7) holds where

$$b_i = \frac{Co(P_i(t))}{Co(P_{i+1}(t))} = \frac{2(w-i)(n-w-i)}{w(n+1-w)}, \qquad (6.88)$$

$$c_i = \frac{v_{i-1}}{v_i} b_{i-1} = \frac{2i^2}{w(n+1-w)}, \; a_i = \frac{2i\,(n-2i) - w(n-w-1)}{w(n+1-w)}.$$

Now we calculate

$$L_P(t_k^{1,\theta}(P)) = \left(1 - \frac{1}{P_1(-1)}\right)^{\theta} \sum_{i=0}^{k} v_i^{0,\theta}. \qquad (6.89)$$

By (5.64) and (6.87)

$$v_i^{0,1} = \frac{(T_i(1,-1,P))^2}{-c^{0,1}(P) v_i b_i P_i(-1) P_{i+1}(-1)}. \qquad (6.90)$$

From (6.71) and (6.72) it follows that

$$P_i(-1) = (v_i)^{-1} E_i(w) = \frac{(-1)^i}{\binom{n-w}{i}}, \; i = 0, 1, ..., w, \qquad (6.91)$$

and, hence,

$$T_i(1,-1,P) = \sum_{j=0}^{i} v_j P_j(-1) = \sum_{j=0}^{i} \binom{w}{j} (-1)^j = (-1)^i \binom{w-1}{i}. \qquad (6.92)$$

Using (4.34), (6.68), and the fact that

$$d(n+1-d)\left(\binom{n}{d} - \binom{n}{d-1}\right) = n(n-1)\left(\binom{n-2}{d-1} - \binom{n-2}{d-2}\right) \qquad (6.93)$$

one can find that

$$c^{0,1}(P) = \frac{n+1-w}{2}\ . \qquad (6.94)$$

From (6.88)-(6.92) and (6.94) it follows that

$$1-\frac{1}{P_1(-1)}=n-w+1, \quad v_i^{0,1}=\frac{1}{i+1}\binom{w-1}{i}\binom{n-w}{i}$$

and, hence,

$$L_P(t_k^{1,\theta}(P))=\sum_{i=0}^{k}\binom{w-\theta}{i}\binom{n-w+\theta}{i+\theta}. \tag{6.95}$$

Note that from (4.36) and (6.93) it also follows that for $0\le i<j\le w-1$ the equality

$$\sum_{d=0}^{w}P_i^{1,0}(\sigma_P(d))P_j^{1,0}(\sigma_P(d))(1-\sigma_P(d))r_d=0$$

implies that

$$\sum_{d=0}^{w-1}P_i^{1,0}(\sigma_P(d+1))P_j^{1,0}(\sigma_P(d+1))\left(\binom{n-2}{d}-\binom{n-2}{d-1}\right)=0\ .$$

Therefore, $P_i^{1,0}(\sigma_P(d+1))$ is equal to $E_i^{n-2,w-1}(d)$ up to a constant factor and hence

$$d_k^{1,0}(P,n,w)=d_k(P,n-2,w-1)+1. \tag{6.96}$$

Consider codes $C\subset J_w^n$ with $d(C)>1$ and $d'(C)>1$. It is clear that for any d, $d=2,...,w$, there exist unique integers l and $\theta\in\{0,1\}$ such that $d=2l+1-\theta$. Moreover, for any d, $d=2,...,w$, and U ($U=Q$ or $U=P$) there exist minimal possible integers l and $\theta\in\{0,1\}$ such that $d\ge d_{l-\theta}^{1,\theta}(U)$ (since, by (5.90), $d_k^{1,\varepsilon}(U)<d\le d_{k-1+\varepsilon}^{1,1-\varepsilon}(U)$, where $k=k_U(\sigma_U(d))$ and $\varepsilon=\varepsilon_U(\sigma_U(d))$; indeed, we have $l=k$, $\theta=1-\varepsilon$ if $d=d_{k-1+\varepsilon}^{1,1-\varepsilon}(U)$ and $l=k+\varepsilon$, $\theta=\varepsilon$ if $d\ne d_{k-1+\varepsilon}^{1,1-\varepsilon}(U)$).

The following universal bounds follow from Theorems 5.59, 5.55, 5.63. For odd $d'(C)$ and $d(C)$ they coincide with the Ray-Chaudhuri-Wilson bound [93] for block designs and the sphere packing bound for codes in J_w^n, which are also proved by a combinatorial method.

Theorem 6.22. *For a code $C\subset J_w^n$, let $d(C)=2l+1-\theta$ and $d'(C)=2k+1-\vartheta$ for some $l,k\in\{1,...,w\}$ and $\theta,\vartheta\in\{0,1\}$. Then*

$$\left(\frac{n}{w}\right)^{\vartheta}\binom{n-\vartheta}{k-\vartheta}\le|C|\le\frac{\binom{n}{w}}{\sum_{i=0}^{l-\theta}\binom{w-\theta}{i}\binom{n-w+\theta}{i+\theta}}. \tag{6.97}$$

Equality in the left-hand side holds if and only if $s(C)=k$ and $\beta(C)=\vartheta$; in this case $f_C(t)=\left(\frac{1+t}{2}\right)^{\beta(C)}Q_{s(C)-\beta(C)}^{1,\beta(C)}(t)$. Equality in the right-hand side holds if and only if $s'(C)=l$ and $\beta'(C)=\theta$; in this case $\widetilde{f}_C(t)=\left(\frac{1+t}{2}\right)^{\beta'(C)}P_{s'(C)-\beta'(C)}^{1,\beta'(C)}(t)$.

Theorems 5.60, 5.55, 5.63 give rise to the following result.

Theorem 6.23. *For a code $C \subset J_w^n$, let $d(C) \geq d_{l-\theta}^{1,\vartheta}(Q)$ and $d'(C) \geq d_{k-\vartheta}^{1,\vartheta}(P)$ for some $l, k \in \{1, ..., w\}$ and $\theta, \vartheta \in \{0,1\}$. Then*

$$\frac{\binom{n}{w}}{\sum_{i=0}^{k-\vartheta} \binom{w-\vartheta}{i}\binom{n-w+\vartheta}{i+\vartheta}} \leq |C| \leq \binom{n}{w}^{\theta} \binom{n-\theta}{l-\theta}. \tag{6.98}$$

Equality in the right-hand side holds if and only if $d'(C) = 2s(C) - \beta(C) + 1$; in this case $f_C(t) = \left(\frac{1+t}{2}\right)^{\beta(C)} Q_{s(C)-\beta(C)}^{1,\beta(C)}(t)$. Equality in the left-hand side holds if and only if $d(C) = 2s'(C) - \beta'(C) + 1$; in this case $\widetilde{f}_C(t) = \left(\frac{1+t}{2}\right)^{\beta'(C)} P_{s'(C)-\beta'(C)}^{1,\beta'(C)}(t)$.

Example 6.24. *In the case of the known constant weight code C with parameters $n = 24, w = 8, |C| = 759, d'(C) = 6, d(C) = 4$ we have $d_2^{1,1}(Q) = 4, d_1^{1,1}(P) = 6$ and all six bounds (6.74), (6.97), and (6.98) are attained!*

The upper bounds in (6.98) are special cases of the following statement (see Theorems 5.55 and 5.62).

Theorem 6.25 ([64], [67]). *For any code $C \subset J_w^n$,*

$$|C| \leq L^{n,w,2}(d(C)) \tag{6.99}$$

where $L^{n,w,2}(d)$ is defined by (6.84). Equality holds if and only if

$$d'(C) \geq 2s(C) - \beta(C) > 1.$$

If $d'(C) = 2s(C)+1-\beta(C)$ (and hence C is a tight design), then $f_C(t) = \left(\frac{1+t}{2}\right)^{\beta} Q_{s-\beta}^{1,\beta}(t)$; if $d'(C) = 2s(C)-\beta(C)$, then $f_C(t) = \frac{t-\sigma}{1-\sigma}\left(\frac{t+1}{2}\right)^{\beta} \frac{T_{s-1}^{1,\beta}(t,\sigma)}{T_{s-1}^{1,\beta}(1,\sigma)}$ where $s = s(C), \beta = \beta(C)$ in both cases and $\sigma = \sigma_Q(d(C))$.

Note that a code $C \subset J_w^n$ for which $|C|$ equals the lower bound in (6.97) or the upper bound in (6.99) (in particular, in (6.98)) is distance-regular, and the polynomial $f_C(\sigma_Q(d))$ in d of degree $s(C)$ is minimal for distances of C. Theorem 6.25 gives rise to the following bounds for a code $C \subset J_w^n$ with large minimal distance $d(C) = d$:

$$|C| \leq \frac{dn}{dn - w(n-w)} \quad \text{if} \quad \frac{w(n-w)}{n-1} < d \leq w, \tag{6.100}$$

$$|C| \leq \frac{dn(n-1)}{w(d(n-1)-(w-1)(n-w))} \quad \text{if} \quad \frac{(w-1)(n-w)}{n-2} < d \leq \frac{w(n-w)}{n-1},$$

$$|C| \leq \frac{dn(n-1)(w(n-w)-1-d(n-2))}{d(n-1)(2w(n-w)-n-d(n-2))-w(n-w)(w-1)(n-w-1)}$$

if $\quad d_2^{1,0}(Q,n,w) < d \leq \frac{(w-1)(n-w)}{n-2}$.

The first of the bounds is the well known Johnson bound [55]. It should be noted that the bounds of Theorems 6.23 and 6.25 become stronger than those of Theorems 6.21 and 6.22 when the parameters w, $d(C)$ and $d'(C)$ are sufficiently large.

To derive asymptotic results one can use the following special cases, $\theta = 0$ and $\vartheta = 0$ in Theorem 6.23:

$$|C| \leq \binom{n}{l} \quad \text{if} \quad d(C) \geq d_l^{1,0}(Q), \tag{6.101}$$

$$|C| \geq \frac{\binom{n}{w}}{\sum_{i=0}^{k} \binom{w}{i}\binom{n-w}{i}} \quad \text{if} \quad d'(C) \geq d_k^{1,0}(P). \tag{6.102}$$

Consider for any η, $0 \leq \eta \leq \frac{1}{2}$, the decreasing continuous function

$$\xi_\eta(x) = \frac{\eta(1-\eta) - x(1-x)}{1 + 2\sqrt{x(1-x)}}, \tag{6.103}$$

which maps the interval $[0,\eta]$ onto $[0,\eta(1-\eta)]$. The inverse function $\xi_\eta^{-1}(x)$ can be expressed in the following explicit form:

$$\xi_\eta^{-1}(x) = \frac{1}{2}\left(1 - \sqrt{1 - 4\left(\sqrt{\eta(1-\eta) - x(1-x)} - x\right)^2}\right). \tag{6.104}$$

This fact is essentially used to obtained asymptotic results for the Johnson space (compare with the function $\gamma_v(x)$ for the Hamming space that coincides with its inverse, see (6.62)). First we note that by Corollary 5.26, Lemma 5.29, Theorem 5.12, and (5.54), $t_k(Q) - \frac{2n}{w(n-2k+1)} < t_k^{1,0}(Q) < t_k(Q)$ and, hence,

$$d_k(Q) < d_k^{1,0}(Q) < d_k(Q) - \frac{n}{(n-2k+1)}, \; k = 1, ..., w-1. \tag{6.105}$$

Corollary 5.22, (6.101), and (6.103)-(6.105) imply the following known result.

Theorem 6.26 ([79]). *If* $\lim_{n\to\infty} \frac{w}{n} = \eta$ *and* $\lim_{n\to\infty} \frac{d}{n} = \beta$, *where* $0 < \eta \leq \frac{1}{2}$ *and* $0 \leq \beta \leq \eta(1-\eta)$, *then*

$$\frac{1}{n} \log A(J_w^n, d) \lesssim H_2\left(\xi_\eta^{-1}(\beta)\right), \tag{6.106}$$

where $H_2(x)$ *is the binary Shannon entropy (see (6.61)).*

On the other hand, (6.102), (6.96), (6.103), (6.104), and Corollary 5.23 give rise to a new result.

Theorem 6.27. *If* $\lim_{n\to\infty} \frac{w}{n} = \eta$ *and* $\lim_{n\to\infty} \frac{d}{n} = \beta$, *where* $0 < \eta \le \frac{1}{2}$ *and* $0 \le \beta \le \eta$, *then*

$$\frac{1}{n}\log B(J_w^n, d) \gtrsim H_2(\eta) - \eta H_2(\frac{\xi_\eta(\beta)}{\eta}) - (1-\eta)H_2(\frac{\xi_\eta(\beta)}{1-\eta}). \tag{6.107}$$

We note for comparison that in the same asymptotic processes Theorem 6.22 implies the following bounds:

$$\frac{1}{n}\log A(J_w^n, d) \lesssim H_2(\eta) - \eta H_2(\frac{\beta}{2\eta}) - (1-\eta)H_2(\frac{\beta}{2(1-\eta)}),$$

$$\frac{1}{n}\log B(J_w^n, d) \gtrsim H_2(\frac{\beta}{2}).$$

From Theorem 6.26 and inequality (6.65) one can derive the following asymptotic bound for codes in binary Hamming space.

Theorem 6.28 (The second form of the MRRW bound [79]). *If* $\lim_{n\to\infty} \frac{d}{n} = \delta$ *, where* $0 \le \delta \le 2\eta(1-\eta)$, *then*

$$\frac{1}{n}\log A(H_2^n, d) \lesssim 1 - \max_{\frac{1}{2}(1-\sqrt{1-2\delta}) \le \eta \le \frac{1}{2}} \left(H_2(\eta) - H_2\left(\xi_\eta^{-1}(\frac{\delta}{2})\right)\right), \tag{6.108}$$

where $\xi_\eta^{-1}(x)$ *is defined by (6.104).*

This asymptotic bound is better than (6.60) for $v = 2$ when $\delta < 0.272$. Note that in the proof of Theorem 6.28 instead of (6.65) the following significant relationship between linear programming bounds for codes in Hamming and Johnson spaces

$$\binom{n}{w} A_Q(H_2^n, 2d) \le 2^n A_Q(J_w^n, d) \tag{6.109}$$

may be used. This was found by Rodemich (see [33]). New lower bounds on $B(H_2^n, d)$ and, in particular, an analog of the asymptotic inequality (6.108) were obtained in [72] with the help of (6.109) and Corollary 4.26 applied to the both spaces H_2^n and J_w^n.

References

[1] M. J. Aaltonen, "A new bound on nonbinary block codes", *Discrete Mathematics,* **83**, pp. 139-160, 1990.

[2] N. Alon, O. Goldreich, J. Hastad, R. Peralta, "Simple construction of almost k-wise independent random variables," *Random Structures and Algorithms*, **3**, no. 3 (1992), 289-304.

[3] R. Askey and G. Gasper, "Jacobipolynomial expansions of Jacobi polynomials with non-negative coefficients", *Proc. Camb. Phil. Soc.* **70** (1971), 243-255.

[4] R. Askey and J. Wilson, "A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols", *SIAM J. of Math. Anal.*, **10** (1979), 1008-1016.

[5] J. T. Astola, "The Tietäväinen bound for spherical codes", *Discr. Appl. Math.*, **7** (1984), 17-21.

[6] S. Augustinovich and F. Solovjeva, "On distance-regularity of perfect binary codes", to appear in *Problems of Information Transmission.*

[7] L. Babai, H. Snevily, and R. M. Wilson, "A new proof of several inequalities on codes and sets", *J. Combin. Th. (A)* **71** (1995), n.1, 146-153.

[8] E. Bannai, T. Ito, "Algebraic combinatorics. I. Association schemes", *Benjamin/Cummings*, London, 1984.

[9] E. Bannai and S. G. Hoggar, "On tight t-designs in compact symmetric spaces of rank one", *Proc. Japan Acad.,* **61**, Ser. A (1985), 78-82.

[10] E. Bannai, "Orthogonal polynomials in coding theory and algebraic combinatorics", in "Orthogonal Polynomials" , P.Nevai (ed), *Kluwer Acad.Publ.*, 1990, 25-53.

[11] L. A. Bassalygo, "New upper bounds for error correcting codes", *Problems of Inform. Transmission*, **1** (1965), n. 1, pp. 32–35.

[12] L. A. Bassalygo, G. V. Zaitsev, V. A. Zinoviev, "Uniformly packed codes", *Problems of Inform. Transmission*, **10** (1974), n. 1, pp. 6-9.

[13] L. A. Bassalygo, V. A. Zinoviev, "Remark on uniformly packed codes", *Problems of Inform. Transmission*, **13** (1977), n. 3, pp. 178–180.

[14] H. Batemann, A. Erdelyi, "Higher transcendental functions", Vol. **2** *McGraw-Hill Book C., Inc.*, New York, 1953.

[15] B. I. Belov, "Methods of solutions of mathematical programming and optimal control problems", Part IV, Novosibirsk, Nauka, 1984, pp. 76-116 (in Russian).

[16] Th. Beth, D. Jungnickel, and H. Lenz, Design Theory, *Bibl. Institut-Wissenschaftsverlag*, Manheim, 1985.

[17] H. F. Blichfeldt, "A new principle in the geometry of numbers with some applications", *Tran. Am. Math. Soc.,* **15**, (1914), 227-235.

[18] A. Blokhuis, "Polynomials in finite geometries and Combinatorics", in *Surveys in Combinatorics, 1993* (K. Walker, ed.), *Cambridge University Press*, 1993, pp. 35-52.

[19] R. C. Bose, "Mathematical theory of the symmetrical factorial design", *Sankhya* **8** (1947), 107–166.

[20] P. G. Boyvalenkov, D. P. Danev, and S. P. Bumova, "Upper bounds on the minimum distance of spherical codes", *IEEE Trans. Inform. Theory* **IT-42** (1996), 1576-1581.

[21] P. Boyvalenkov, D. Danev, "On linear programming bounds for codes in polynomial metric spaces", to appear in *Problems of Information Transmission.*

[22] P. Boyvalenkov, D. Danev, "On maximal codes in polynomial metric spaces", Lectures Notes in Computer Sciences vol. 1255, Springer, Berlin-Heidelberg (1997), 29-38.

[23] A. E. Brouwer, A. M. Cohen, A. Neumaier, "Distance-regular graphs", *Springer-Verlag*, Berlin, 1989.

[24] R. Calderbank, W. M. Kantor, "The geometry of two-weight codes", *Bull. London Math. Soc.* **18** (1986), 97–122.

[25] R. Calderbank, P. Delsarte, "Extending the t-design concept", *Trans. of the Amer. Math. Soc.* **338** (1993), 941–952.

[26] E. Cartan, "Sur la détermination d'un systéme orthogonal complet dans un espace de Rieman symmetrique clos", *Rend. Circ. Math. Palermo*, **53** (1929), 217-252.

[27] P. J. Cameron, J. M. Goethals, J. J. Seidel, "Strongly regular graphs having strongly regular subconstituents", *J. Algebra* **55** (1978), 257-280.

[28] J. H. Conway and N. J. A. Sloane, "Sphere Packings, Lattices and Groups". New York: Springer-Verlag, 2nd ed., 1993.

[29] Ph. Delsarte, "Four fundamental parameters of a code and their combinatorial significance", *Info. and Control* **23** (1973), 407–438.

[30] Ph. Delsarte, "An algebraic approach to the association schemes of coding theory", *Philips Res. Reports, Suppl.* **10** (1973).

[31] Ph. Delsarte, "Association schemes and t-design in regular semilattices", *J. Combin. Th. (A)* **20** (1976), 230–273.

[32] Ph. Delsarte, "Hahn polynomials, discrete harmonics and t-designs", *SIAM J. Appl. Math.* **34** (1978), 157–166.

[33] Ph. Delsarte, "Application and generalization of the MacWilliams transform in coding theory", in *Proc. of the Fifteenth Symposium on Inform. Theory in the Benelux,* Louvain-la-Neuve, Belgium (1994), 9-44.

[34] Ph. Delsarte, J. M. Goethals, J. J. Seidel, "Bounds for systems of lines, and Jacobi polynomials", *Philips Res. Reports,* **30** (1975), 91*-105*.

[35] Ph. Delsarte, J. M. Goethals, J. J. Seidel, "Spherical codes and designs", *Geometriae Dedicata*, **6** (1977), 363-388.

[36] Ph. Delsarte and J. J. Seidel, "Fisher-type inequalities for Euclidean t-designs", *Lin. Algebra Appl.* **114/115** (1989), 213-230.

[37] R. H. F. Denniston, "Some maximal arcs in finite projective planes", *J. Combin. Th.* **6** (1969), 317–319.

[38] C. F. Dunkl, "Discrete quadrature and bounds on t-design", *Mich. Math. J.* **26** (1979), 81-102.

[39] G. Fazekas, V.I. Levenshtein, "On upper bounds for code distance and covering radius of designs in polynomial metric spaces", *J. Comb. Th. (A)* **70** no. 2 (1995), 267-288.

[40] G. M. Fichtengoltz, Course of differential and integral calculation, vol. 3, *GIFML*, (1960).

[41] J. Friedman, "On the bit extraction problem",Proc. 33 IEEE Symp. on Foundations of Computer Science (1992), 314-319.

[42] R. Gangolli, "Positive definite kernels on homogeneous spaces and certain stochastic processes related to Levy's Brownian motion of several parameters", *Ann. Inst. Henri Poincare*, **3** (1967), 121-226.

[43] G. Gasper, "Linearization of the product of Jacobi polynomials, II", *Canad. J. Math.* **22** (1970), 582–593.

[44] I. M. Gelfand, "Spherical functions on symmetrical Riemann spaces", *Dokl. Akad. Nauk USSR*, **70**, no.1, (1950), 5-8.

[45] E. N. Gilbert, F. J. MacWilliams and N. J. A. Sloane, "Codes which detect deception", *Bell Syst. Techn. J.* **53** (1974), 405-424.

[46] C. D.Godsil, "Polynomial spaces", *Discrete Mathematics* **73** (1988-89), 71-88.

[47] J. M. Goethals, H. C. A. van Tilborg, "Uniformly packed codes", *Philips Res. Reports*, **30,** no.1 (1975), 9–36.

[48] J. M. Goethals and J. J. Seidel, "Cubature formulae, polytopes, and spherical designs", in *The Geometric Vein*, The Coxeter Festschrift (C. Davis, B. Grünbaum, and F.A. Sherk, Eds.), Springer, Berlin, 1982, pp. 203-218.

[49] M. J. E. Golay, "Notes on digital coding", *Proc. IEEE* **37** (1949), 657.

[50] R. W. Hamming, "Error detecting and error correcting codes", *Bell Syst. Techn. J.* **29** (1950), 147–160.

[51] R. Hill, "On the largest size cap in $S_{5,3}$", *Rend. Acad. Naz. Lincei* **54 : (8)** (1973), 378-384.

[52] R. Hill, "Caps and groups", in *Atti dei Covegni Lincei, Colloquio Intern. sulle Theorie Combinatorie (Roma, 1973)*, **17** *(Acad. Naz. Lincei)*, 1976, 384–394.

[53] S. G. Hoggar, "t-designs in projective spaces", *Europ. J. Comb.*, **3** (1982), 233-254.

[54] S. G. Hoggar, "t-designs with general angle set", *Europ. J. Comb.*, **13** (1992), 257-271.

[55] S. M. Johnson, "A new upper bound for error-correcting codes", *IRE Trans. Inform. Theory*, **8** (1962), 208-224.

[56] M. E. H. Ismail and X. Li, "Bound on the extreme zeros of orthogonal polynomials", *Proc. of AMS*, **115** (1992), 131-140.

[57] G. A. Kabatyanskii, V. I. Levenshtein, "Bounds for packings on a sphere and in space", *Problems of Information Transmission* **14**: (1) (1978), 1–17.

[58] A. N. Kolmogorov, S. V. Fomin, "Elements of theory of functions and functional analysis", Moscow, *Nauka*, 1972.

[59] T. Koornwinder, "The addition formula for Jacobi polynomials and spherical harmonics", *SIAM J. Appl. Math.* **25** (1973), 236–246.

[60] I. Krasikov, S. Litsyn, "Linear programming bounds for codes of small size", *Europ. J. Comb.*, **18** (1997), 647–654.

[61] M. Krawtchouk, "Sur une généralisation des polynômes d'Hermite", *Comptes Rendus* **189** (1929), 620–622.

[62] V. I. Levenshtein, "Methods for obtaining bounds in metric problems of coding theory", in *Proc. of the 1975 IEEE-USSR Joint Workshop on Information Theory,* New York (1976), pp. 126-143.

[63] V. I. Levenshtein, "Bounds on the probability of undetected error", *Problems of Information Transmission* **13**: (1) (1977), 1–12.

[64] V. I. Levenshtein, "On choosing polynomials to obtain bounds in packing problems", *In : Proc. Seventh All-Union Conf. on Coding Theory and Information Transmission*, Part II, Moscow, Vilnius, 1978, pp. 103–108 (in Russian).

[65] V. I. Levenshtein, "On bounds for packings in n-dimensional Euclidean space", *Soviet Math. Dokl.* **20 : (2)** (1979), 417–421.

[66] V. I. Levenshtein, "Bounds on the maximal cardinality of a code with bounded modulus of the inner product", *Soviet Math. Dokl.* **25 : (2)** (1982), 526–531.

[67] V. I. Levenshtein, "Bounds for packings of metric spaces and some their applications", in *Probl. Cybern.* **40**, Nauka, Moscow, 1983, pp. 43-110 (in Russian).

[68] V. I. Levenshtein, "Designs as maximum codes in polynomial metric spaces", *Acta Applicandae Mathematicae* **29** (1992), 1–82.

[69] V. I. Levenshtein, "Bounds for self-complementary codes and their applications", in *Eurocode-92. CISM Courses and Lectures,* vol. **339**. *Springer-Verlag*, Wien-New-York, 1993, 159-171.

[70] V. I. Levenshtein, "Packing and decomposition problems for polynomial association schemes", *Europ. J. Combinatorics* **14** (1993), 461–477.

[71] V. I. Levenshtein, "Krawtchouk polynomials and universal bounds for codes and designs in Hamming spaces", *IEEE Trans. Inform. Theory* **IT-41** (1995), 1303-1321.

[72] V. I. Levenshtein, "Equivalence of Delsarte's bounds for codes and designs in symmetric association schemes, and some applications", to appear in *Discrete Mathematics.*

[73] D. Leonard, "Orthogonal polynomials, duality, and association schemes", *SIAM J. Math. Anal.*, **13** (1982), 656–663.

[74] D. Leonard, "Parameters of association schemes that are both P- and Q-polynomial", *J. of Combin. Theory* (A), **36** (1984), 355-363.

[75] V. K. Leontjev, "Coding with error detection", *Problems of Information Transmission* **8**: (2) (1972)

[76] J. H. van Lint, "Coding Theory", *Springer-Verlag*, Berlin, 1992.

[77] L. Lovász, "On the Shannon capacity of a graph", *IEEE Trans. Inform. Theory* **IT-25**, 1 (1979), 1–7.

[78] F. J. MacWilliams, N. J. A. Sloane, "The theory of error-correcting codes", *North Holland*, Amsterdam, 1977.

[79] R. J. McEliece, E. R. Rodemich, H. C. Rumsey Jr., L. R. Welch, "New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities", *IEEE Trans. Inform. Theory* **IT-23** (1977), 157–166.

[80] R. J. McEliece, E. R. Rodemich, H. C. Rumsey Jr., "The Lovász bound and some generalizations", *Journal of Combinatorics, Information and System Sciences*, **3**, 3 (1978), 134-152.

[81] H. L. Montgomery, "Topics in multiplicative number theory", *Springer-Verlag,* 1971.

[82] A. Neumaier, "Combinatorial configurations in terms of distances", *Memorandum* **81-09** (Wiskunde), Eindhoven Univ. Technol., 1981.

[83] A. Neumaier and J. J. Seidel, "Discrete measures for spherical designs, eutactic stars, and lattices", *Indag. Math.*, **50** (1988), 321-334.

[84] A. Neumaier and J. J. Seidel, "Measures of strength $2e$ and optimal designs of degree e", *Sankhyã* **54** (1992), 299-309.

[85] A. F. Nikiforov and V. B. Uvarov, "Special functions of mathematical physics", *Birkhäuser*, 1988.

[86] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, "Classical orthogonal polynomials of a discrete variable", *Springer-Verlag*, 1991.

[87] A. M. Odlyzko, N. J. A. Sloane, "New bounds on the number of unit spheres that can touch a unit sphere in n dinensions", *J. of Combin. Theory* (A), **26** (1979), 210-214.

[88] V. Pless, "Introduction to the theory of error-correcting codes", 2nd ed., New York : *Wiley*, 1989.

[89] M. Plotkin, "Binary codes with specified minimum distance", *IEEE Trans. Inform. Theory,* **6** (1960), 445–450.

[90] B. Quist, "Some remarks concerning curves of the second degree in a finite plane", *Ann. Acad. Fenn. Sci. Ser. A*, **134** (1952).

[91] R. A. Rankin, "The closest packing of spherical caps in n dimensions", *Proc. Glasgow Math. Assoc.,* **2** (1955), 139-144.

[92] C. R. Rao, "Factorial experiments derivable from combinatorial arrangements of arrays", *J. R. Stat. Soc.* **89** (1947), 128–139.

[93] D. K. Ray-Chaudhuri and R. M. Wilson, "On t-designs", *Osaka J. Math.*, **12** (1975), 737-744.

[94] C. A. Rogers, Packing and Covering, *Cambridge University Press*, 1964.

[95] I. J. Schoenberg, "Positive definite functions on spheres", *Duke Math. J.*, **9** (1942), 96-107.

[96] I. Schoenberg and G. Szego, "An extremum problem for polynomials", *Composito Math.* **14** (1960), 260–268.

[97] A. A. Schrijver, "A comparison of the bounds of Delsarte and Lovász", *IEEE Trans. Inform. Theory* **25**, 4 (1979), 425–429.

[98] J. J. Seidel, "Isometric embeddings and geometric designs", *Discrete Mathematics*, **136** (1994), 281-293.

[99] N. V. Semakov, V. A. Zinoviev, "Equidistant q-ary codes and resolved balanced incomplete block designs", *Problems of Information Transmission* **4 : (2)** (1968).

[100] N. V. Semakov, V. A. Zinoviev, G. V. Zaitsev, "Class of maximal equidistant codes", *Problems of Information Transmission* **5 : (2)** (1969).

[101] C. Shannon, "The zero error capacity of a noisy channel", *IRE Trans. Inform. Theory* **IT-2** (1956), 8–19.

[102] V. M. Sidelnikov, "On mutual correlation of sequences", *Soviet Math. Doklady* **12,** no.1 (1971), 197–201.

[103] V. M. Sidelnikov, "New bounds for the densest packing of spheres in n-dimensional Euclidean space", *Math. Sbornik,* **95** (1974), 148-158.

[104] V. M. Sidelnikov, "Upper bounds on the number of points of binary code with specified code distance", *Probl. of Inform. Transm.,* **10** (1974), 43-51.

[105] V. M. Sidelnikov, "On extremal polynomials used to estimate the size of codes", *Problems of Information Transmission* **16 : (3)** (1980), 174–186.

[106] V. R. Sidorenko, "Upper bound of the size of q-ary codes", *Problems of Information Transmission* **11: (3)** (1975), 14-20.

[107] R. C. Singleton, "Maximum distance q-ary codes", *IEEE Trans. Inform. Theory* **10** (1964), 116–118.

[108] N. J. A. Sloane, "Recent bounds for codes, sphere packing and related problems obtained by linear programming and other methods", *Contemp. Math.*, **9** (1982), 153-185.

[109] D. Stanton, "Some q-Krawtchouk polynomials on Chevalley groups", *Am. J. Math.*, **102** (1980), 625-662.

[110] D. Stanton, "A partially ordered set and q-Krawtchouk polynomials", *J. Combin. Theory*, Ser. A, **30** (1981), 276-284.

[111] D. Stanton, "An introduction to group representations and orthogonal polynomials", in Orthogonal Polynomials, *Kluwer Acad. Publ.*, 1990, 419-433.

[112] D. R. Stinson, "Combinatorial designs and cryptography". In *Surveys in Combinatorics, 1993* (K. Walker, ed.), *Cambridge University Press*, 1993, 257-287.

[113] G. Szego, "Orthogonal polynomials", *AMS Col. Publ.* **Vol. 23**, Providence, RI, 1939.

[114] H. Tarnanen, M. J. Aaltonen and J. M. Goethals, "On the nonbinary Johnson scheme", *Europ. J. Combin.*, (1985), 279-285.

[115] P. Terwilliger, "A characterization of P- and Q-polynomial association schemes, *J. of Combin. Theory* (A), **45** (1987), 8-26.

[116] A. Tietäväinen, "Bounds for binary codes just outside the Plotkin range", *Inform. and Control*, **47** (1980), 85–93.

[117] A. Tietäväinen, "An upper bound on the covering radius of codes as a function of the dual distance", *IEEE Trans. Inform. Theory* **IT-36 (6)** (1990), 1472–1474.

[118] A. Tietäväinen, "Covering radius and dual distance", *Designs, Codes and Cryptography* **1** (1991), 31–46.

[119] V.A. Yudin, "Lower bounds for spherical designs", *Izvestiya: Mathematics* **61**:3 (1997), 673-683.

[120] N. J. Vilenkin, "Special functions and the theory of group representations "(in Russian), Moscow, "Nauka", 1965. English translation published by *Amer. Math. Soc.*, Providence RI, 1968.

[121] H.-C. Wang, "Two point homogeneous spaces", *Ann. Math.* **55** (1952), 177-191.

[122] G. N.Watson, "A treatise of theory of Bessel functions", *Cambridge Univ. Press*, London, 1962.

[123] L. R. Welch, "Lower bounds on the maximum correlation of signals", *IEEE Trans. Inform. Theory* **20** (1974), 397–399.

[124] R. M. Wilson, "The necessary conditions for t-designs are sufficient for something", *Utilitas Math.*, **4** (1973), 207-215.