On designs in compact metric spaces and a universal bound on their size

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Abstract

For finite and compact infinite metric spaces, a concept of a (weighted) τ-design is introduced which depends on a choice of a substitution function. To estimate the minimum size of a τ-design a system of orthogonal polynomials is defined using the average measure of metric balls and the substitution function. A universal lower bound on the size of τ-designs is obtained with the help of the solution of the known extremum problem for systems of orthogonal polynomials. The concept of a τ-design and the bound considered coincide with those in the case of polynomial association schemes of Delsarte and the Euclidean sphere for the proper choices of the substitution functions. This bound is also calculated for some other spaces.

1. Introduction

Compact metric spaces $X = (X, d(x, y), \mu)$ with a measure μ defined on metric balls are considered. All finite metric spaces (in particular, all graphs with the path metric) with the ordinary counting measure are included in the consideration. The compactness condition implies that, for any $d > 0$, any set $A \subseteq X$, for which the open metric balls of (*packing*) radius $d/2$ with centers at points of A do not intersect, is finite and, for any $\rho > 0$, there exists a finite set $C \subseteq X$ such that the closed metric balls of (*covering*) radius ρ with centers at all points of C cover the whole space X. This gives rise to the classical *packing and covering problems* for compact metric spaces: for any $d > 0$, find a finite set (code) $C \subseteq X$ of the maximum size with the packing radius $d/2$ (or with distance at least d between its distinct points) and, for any $\rho > 0$, find a finite set (code) $C \subseteq X$ of the minimum size with the covering radius ρ. The covering problem can be treated as an approximation of the whole space X by a code

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$C \subseteq X$ of the minimum size, and the covering radius ρ of C characterizes a degree of the approximation.

One of the most important and fruitful results of Delsarte theory [11] is a concept of *a design C of strength τ* (or τ-design) in an association scheme X with s classes. A chain of certain subspaces $W_0 \subset W_1 \subset ... \subset W_s$ of functions $f(z)$, $z \in X$, is considered and a code $C \subseteq X$ is called a τ-design ($\tau = 1, 2, \ldots, s$) if for any $f(z) \in W_\tau$ the mean values of $f(z)$ over X and over C are the same. Thus, any code $C \subseteq X$ can be also considered as an approximation of the whole space X in another sense, and the strength τ of the design C characterizes a degree of the approximation. This gave rise to a new *design problem*: for any τ, $\tau = 1, 2, \ldots, s$, find a τ-design $C \subseteq X$ of the minimum size. Later the definition of a τ-design based on subspaces of functions of one variable was extended to the Euclidean sphere, the projective spaces and some other spaces (see [13], [12], [14], [26], [27], [28], [18], [38], [21], [9]). The concept of design allowed to understand the unified nature of such different notions as the tactical configurations, orthogonal arrays, sets of nodes of cubature formulas, and to apply general methods for their investigation and for obtaining universal bounds on their size. Furthermore, it turns out to be that a number of extremum problems of significant interest in factorial experiments, cryptology, complexity theory, and calculation theory are reduced to the design problem for some spaces (see [7], [30], [17], [40], [1], [19], [34], [10]).

The purpose of the paper is to extend the notion of a τ-design to an arbitrary compact metric space $X = (X, d(x,y), \mu)$ and obtain an universal bound on the size of τ-designs. In Section 2 a concept of a (weighted) τ-design in X is presented, which depends on a fixed monotone substitution function $\sigma(d)$ and is defined as follows: a (weighted) finite set $C \subseteq X$ is called τ-design ($\tau = 1, 2, \ldots$) if for any polynomial $f(z)$ of degree at most τ, the mean value of $f(\sigma(d(x,y)))$ over $X \times X$ with the given measure and over $C \times C$ coincide. For certain choices of $\sigma(d)$ this concept includes τ-designs of Delsarte in P- and Q-polynomial association schemes and their generalizations mentioned above. To study (weighted) τ-designs and obtain a universal (valid for all $X, \sigma(d)$ and τ) lower bound on their size in Section 3, for an arbitrary compact metric space $X = (X, d(x,y), \mu)$, a system Q of orthogonal polynomials on interval $[-1, 1]$ is introduced. This system is defined using the substitution function $\sigma(d)$ and the function $\overline{\mu}(d)$ being equal to the mean measure of metric balls in X of the radius d. One can expect that the system Q might be useful not only for the design and packing problems. In Section 3 we also find the systems Q for different spaces as examples. The desirable universal bound on the size of (weighted) τ-designs is given in Section 4. As in the case of Delsarte designs it is obtained with the help of the solution of the known extremum problem for systems Q of orthogonal polynomials. This bound includes the known bounds on the size of τ-designs in P- and Q-polynomial association schemes and of spherical τ-designs. It was calculated for some other metric spaces, for example, for the n-dimensional torus and cube with natural substitution functions. Necessary and sufficient conditions for attainability of this bound are found. In particular, the necessary conditions show that although this bound is valid for weighted τ-designs, it may be attained only for simple designs (i.e., when all weights are equal). In Section 5 we introduce the classes of so-called FDNDF-polynomial and polynomial compact metric spaces. For such spaces the definition of a τ-design given above is equivalent to that of Delsarte. Moreover, some additional possibilities to improve the universal bound for τ-designs are appeared and an extremum problem for the system Q can be used for obtaining universal (upper) bounds on the size of codes with given packing radius. An universal bound on the size of codes in FDNDF-polynomial spaces, which is, in a sense, dual to the universal bound for τ-designs, is given. More general results on the packing problem can be found in [24], [25].

It should be noted that in the paper we put some restrictions which are useful for the consideration of the packing and design problems simultaneously. Some of them, in particular, the triangle axiom for $d(x,y)$ and the monotony of $\sigma(d)$, can be omitted for the consideration of the design problem.

2. Designs in compact metric spaces

We consider compact metric spaces X with distance $d(x,y)$ including all finite metric spaces. In particular, we consider arbitrary connected undirected graphs Γ, since the set $X = X(\Gamma)$ of vertices of Γ may be considered as a finite metric space with the *path metric* $d(x,y)$ equal to the minimum number of edges in a path connecting the vertices x and y. Any finite set (code) $C, C \subseteq X$, is characterized by its *minimal distance*

$$d(C) = \min_{x,y \in C, x \neq y} d(x,y)$$

when $|C| \geq 2$ and by its *covering radius*

$$\rho(C) = \max_{x \in X} d(x,C),$$

where $d(x,C)$ denotes the minimal distance between x and points of C. The classical packing and covering problems consist in finding a code $C \subseteq X$ of the maximum size with given minimum distance and a code $C \subseteq X$ of the minimum size with given covering radius.

We assume that X is endowed with a normalized measure μ, $\mu(X) = 1$, such that for any ρ ($\rho \geq 0$) and any $x \in X$ the *metric ball*

$$S_\rho(x,X) = \{y : y \in X, \quad d(x,y) \leq \rho\}$$

is measurable and

$$\mu(x,\rho) = \mu(S_\rho(x,X)) > 0 \quad \text{if } \rho > 0. \tag{2.1}$$

In the infinite case for simplicity we also will assume that $\mu'_\rho(x,\rho) = \frac{\partial \mu(x,\rho)}{\partial \rho}$ is continuous as a function in two variables. In the case of finite X we assume that μ is the normalized *counting measure*, that is,

$$\mu(A) = \frac{|A|}{|X|} \quad \text{for any } A \subseteq X.$$

The space X is called *distance invariant* if $\mu(x, \rho)$ does not depend on $x \in X$ for any ρ. In the general case the function

$$\bar{\mu}(\rho) = \int_X \mu(x, \rho) d\mu(x) \tag{2.2}$$

characterizes the *mean measure* of a metric ball of radius ρ. Thus, for distance invariant spaces $\bar{\mu}(\rho) = \mu(x, \rho)$ for any $x \in X$.

We also consider some other parameters of codes connected with their metric properties. For any $W, W \subseteq X$, let $\Delta(W)$ denote the set of values of $d(x, y)$ when $x, y \in W$ and let $\Delta_0(W)$ be $\Delta(W) \backslash \{0\}$. For any finite set (code) $C, C \subseteq X$, the parameter

$$s(C) = |\Delta_0(C)| \tag{2.3}$$

characterizes the *number of* (distinct) *distances* between distinct points of C. In particular, $s(X)$ is defined in the case of a finite metric space X. In the case of compact infinite metric spaces X, we put $s(X) = \infty$ since then the set $\Delta(X)$ is infinite. Clearly, for any code C,

$$d(C) = \min \Delta_0(C). \tag{2.4}$$

Together with the *diameter* $D(X) = \max \Delta_0(X)$ of the whole space X, we define the *diameter* of a code C by

$$D(C) = \max \Delta_0(C).$$

We also consider the following auxiliary parameter of a code C:

$$\beta(C) = \begin{cases} 1 \text{ if } D(C) = D(X), \\ 0 \text{ otherwise.} \end{cases} \tag{2.5}$$

Note that for any code $C \subseteq X$, $x \in X$ and ρ, the set

$$\overline{S}_\rho(x, C) = \{y : y \in C, \quad d(x, y) = \rho\}$$

is finite and one can define

$$B_\rho(x, C) = |\overline{S}_\rho(x, C)|. \tag{2.6}$$

Now we introduce one more fundamental parameter of a code C and the concept of a τ-design ($\tau = 1, 2, \ldots$). For some applications it is useful to consider a finite set C as a weighted set $C = (C, m)$ where m is a certain positive-valued function on C (for example, multiplicity or probability). It is convenient to normalize *weights* $m(x)$ of elements $x \in C$ such that

$$\sum_{x\in C} m\,(x) = |C|\,, \tag{2.7}$$

where $|C|$ as always is the number of distinct elements (the *size*) of a weighted set $C = (C, m)$. Hereafter we consider a *code* C as a special case of a weighted set when $m\,(x) \equiv 1$ for all $x \in C$. The concept of a (weighted) τ-design depends on a continuous strictly monotone real function (*substitution)* $\sigma\,(d)$ defined on the interval $[0, D(X)]$ (for example, $\sigma(d) = d$, d^2 or e^d). A weighted set $C = (C, m)$ will be referred to as a *weighted τ-design (in X with respect to the substitution $\sigma\,(d)$)* if for any polynomial $f\,(t)$ in a real t of degree at most τ,

$$\int_X \int_X f(\sigma(d(x,y)))d\mu\,(x)\,d\mu(y) = \frac{1}{|C|^2} \sum_{x,y\in C} f(\sigma(d(x,y)))m\,(x)\,m(y). \tag{2.8}$$

A *τ-design* is a special case of a weighted τ-design when $m\,(x) \equiv 1$ for all $x \in C$. Such τ-designs are also called *simple.* The maximum integer τ ($\tau \le s(X)$) such that a (weighted) set C is a (weighted) τ-design is called the *strength of C* and denoted by $\tau(C)$. The value

$$d'(C) = \tau(C) + 1 \tag{2.9}$$

is referred to as the *dual distance* of C. These definitions of a τ-design and of the dual distance are natural extensions to compact metric spaces of the corresponding definitions of Delsarte for association schemes [11] and Delsarte, Goethals and Seidel for the Euclidean sphere [13]. We shall see in Section 5 that they coincide with the classical definitions for corresponding substitutions (linear for Hamming and Johnson spaces and quadratic for the Euclidean sphere).

The equality (2.8) shows that in a definite sense a (weighted) τ-design C is a good approximation to the whole space X, and the parameter (2.9) characterizes the degree of such an approximation. The design problem consists in finding a (weighted) τ-design $C \subseteq X$ of the minimum size.

3. A system Q of orthogonal polynomials for a compact metric space. Examples

We endow a compact metric space $X = (X, d(x,y), \mu)$ and a given substitution $\sigma(d)$ with a system of orthogonal polynomials $Q_i(t)$ of degree i, $i = 0, 1, ..., s(X)$. This system $Q = \{Q_i(t)\}$ is in fact defined by the substitution $\sigma(d)$ and the mean measure $\overline{\mu}(d)$ of closed metric balls of radius d. Considering some properties of the system Q we obtain universal bounds on the sizes of designs and codes in compact metric spaces. Notice that the definition of a (weighted) τ-design (with respect to substitution $\sigma(d)$) is invariant under any linear transformation of $\sigma(d)$. This allows us to assume without additional loss of generality that $\sigma(d)$ is a continuous strictly decreasing function on $[0, D(X)]$ such that

$$\sigma(D(X)) = -1 \le \sigma(d) \le \sigma(0) = 1. \tag{3.1}$$

Such a substitution function $\sigma(d)$ is referred to as *standard*. The inverse function of the standard function $\sigma(d)$ is denoted by $\sigma^{-1}(\ )$, i.e., for any t, $-1 \leq t \leq 1$, $\sigma^{-1}(t) = d$ when $\sigma(d) = t$.

Consider the following inner product for continuous functions $f(t)$ and $g(t)$ in a real t on the interval $[-1, 1]$:

$$f \cdot g = \int\limits_X \int\limits_X f(\sigma(d(x,y)))g(\sigma(d(x,y)))d\mu(x)d\mu(y). \tag{3.2}$$

We verify that the integral on the right-hand side of (3.2) can be rewritten as the Rimann-Stieltjes integral on $[-1, 1]$. To this end we consider the function $\nu(t)$ in a real t on the interval $[-1, 1]$ defined by

$$\nu(t) = 1 - \overline{\mu}(\sigma^{-1}(t)), \tag{3.3}$$

which increases with t from $\nu(-1) = 0$ up to $\nu(1) = 1 - \overline{\mu}(0)$. When X is finite and $\Delta(X) = \{d_0 = 0, d_1, ..., d_s\}$ where $s = s(X)$, $\nu(t)$ is left continuous (i.e., $\nu(t - 0) = \nu(t)$) and has $s + 1$ steps at points $t_i = \sigma(d_i)$ with positive step sizes

$$w_i = \nu(t_i + 0) - \nu(t_i) = \overline{\mu}(d_i) - \overline{\mu}(d_i - 0) = \frac{1}{|X|^2} \sum_{x \in X} B_{d_i}(x, X), \tag{3.4}$$

$i = 0, 1, ..., s$, $\sum\limits_{i=0}^{s} w_i = 1$ (see (2.6)). In the infinite case we assumed that $\mu'_\rho(x, \rho)$ is continuous and hence $\overline{\mu}'(\rho) = \int\limits_X \mu'_\rho(x, \rho)d\mu(x)$. Therefore, for some natural additional assumptions on $\sigma(d)$, $\nu(t)$ is differentiable and $w(t) = \nu'(t) = -\frac{\overline{\mu}'(d)}{\sigma'(d)}$ is continuous on $[-1, 1]$ and positive inside the interval. An arbitrary left continuous function $\nu(t)$ on $[-1, 1]$ generates the Lebesgue-Stieltjes measure ν on the same interval by

$$\nu[a, b] = \nu(b + 0) - \nu(a), \quad \nu(a, b) = \nu(b) - \nu(a + 0), \tag{3.5}$$

$$\nu(a, b] = \nu(b + 0) - \nu(a + 0), \quad \nu[a, b) = \nu(b) - \nu(a). \tag{3.6}$$

In our case the measure ν is normalized ($\nu[-1, 1] = 1$),

$$\nu[a, b] = \overline{\mu}(d_a) - \overline{\mu}(d_b - 0) \quad \text{where } a = \sigma(d_a),\ b = \sigma(d_b),$$

and we can rewrite (3.2) as

$$f \cdot g = \int\limits_{-1}^{1} f(t)g(t)d\nu(t). \tag{3.7}$$

Since $|\Delta_0(X)| = s(X)$, the polynomials t^i, $i = 0, ..., s(X)$, for $t = \sigma(d)$, $d \in \Delta(X)$, are linearly independent, and using them in the orthogonalization process with respect to (3.7) we obtain the following statement.

Theorem 3.1. *For a compact metric space $X = (X, d(x,y), \mu)$ with a fixed standard substitution $\sigma(d)$ there exists a unique system Q of orthogonal polynomials $Q_i(t)$ of degree i on the interval $[-1,1]$ and a unique system of positive constants r_i $(i = 0,1,...,s(X))$ such that*

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)d\nu(t) = \delta_{i,j}, \tag{3.8}$$

$$Q_i(1) = 1, \quad i = 0,1,...,s(X), \tag{3.9}$$

where $\nu(t)$ is defined by (3.3).

Notice that

$$Q_0(t) \equiv 1 \quad \text{and} \quad r_0 = 1 \tag{3.10}$$

since the measure is normalized. Under our assumption, the orthogonality condition (3.8) for a finite and infinite X takes respectively the following forms:

$$r_i \sum_{k=0}^{s} Q_i(\sigma(d_k))Q_j(\sigma(d_k))w_k = \delta_{i,j}, \quad i,j = 0,1,...,s = s(X), \tag{3.11}$$

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)w(t)dt = \delta_{i,j}, \quad i,j = 0,1,... \quad . \tag{3.12}$$

Example 3.2. *The Hamming space $X = H_v^n$ $(n, v = 2,3,...)$ consists of vectors $\mathbf{x} = (x_1,...,x_n)$ where $x_i \in \{0,1,...,v-1\}$ with the distance $d(\mathbf{x},\mathbf{y})$ being equal to the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ. In this case $\Delta(X) = \{0,1,...,n\}$ and hence $s(X) = n$, $D(X) = n$. The Hamming space is distance invariant and*

$$\overline{\mu}(d) = v^{-n} \sum_{0 \le i \le d} \binom{n}{i} (v-1)^i.$$

For $X = H_v^n$ with the standard substitution $\sigma(d) = 1 - \frac{2d}{n}$ we have $s = n$, $d_i = i$, $w_i = v^{-n}\binom{n}{i}(v-1)^i$, $i = 0,1,...,n$, and conditions (3.11) and (3.9) uniquely determine constants $r_i = w_i v^n$ and polynomials $Q_i(t)$ which are connected with the Krawtchouk polynomials

$$K_i^{n,v}(z) = \sum_{j=0}^{i} (-1)^j (v-1)^{i-j} \binom{z}{j} \binom{n-z}{i-j}$$

as follows

$$Q_i(t) = (r_i)^{-1} K_i^{n,v}(\frac{n(1-t)}{2}).$$

Example 3.3. *The Johnson space J_w^n ($n = 2, 3, \ldots ; w = 1, \ldots, \lfloor \frac{n}{2} \rfloor$) is the set of all w-subsets of the n-set $\{1, \ldots, n\}$, where the distance between two elements $x, y \in J_w^n$ is defined by $w - |x \cap y|$. In this case $\Delta(X) = \{0, 1, \ldots, w\}$ and hence $s(X) = w$, $D(X) = w$. The Johnson space is distance invariant and*

$$\overline{\mu}(d) = \binom{n}{w}^{-1} \sum_{0 \leq i \leq d} \binom{w}{i} \binom{n-w}{i}.$$

J_w^n can also be considered as the subset of the binary Hamming space H_2^n consisting of vectors which have w non-zero coordinates, with distance being equal to half of the Hamming distance. For the Johnson space J_w^n ($w \leq n/2$) with the standard substitution $\sigma(d) = 1 - \frac{2d}{w}$ we have $s = w$, $d_i = i$, $w_i = \binom{n}{w}^{-1} \binom{w}{i} \binom{n-w}{i}$, $i = 0, 1, \ldots, w$, and conditions (3.11) and (3.9) uniquely determine constants $r_i = \binom{n}{i} - \binom{n}{i-1}$ and polynomials $Q_i(t)$ which are connected with the Hahn polynomials

$$J_i(z) = \sum_{j=0}^{i} (-1)^j \frac{\binom{i}{j} \binom{n+1-i}{j}}{\binom{w}{j} \binom{n-w}{j}} \binom{z}{j}$$

as follows

$$Q_i(t) = J_i(\frac{w(1-t)}{2}).$$

Example 3.4. *The unit Euclidean sphere in $\mathbf{R}^n$*

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R^n}, \ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

with the Euclidean distance $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a compact metric space. In this case $\Delta(X) = [0, 2]$ and, hence, $D(S^{n-1}) = 2$, $s(X) = \infty$. We can also measure the distance between $\mathbf{x}, \mathbf{y} \in S^{n-1}$ by the angle $\varphi(\mathbf{x}, \mathbf{y})$ where

$$\cos \varphi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i = 1 - \frac{1}{2} d^2(\mathbf{x}, \mathbf{y}).$$

The following facts are well-known [32]. The isometry group G of S^{n-1} consists of all orthogonal matrices of order n and acts transitively on S^{n-1}. There exists the unique normalized measure μ on S^{n-1} which is invariant (i.e., $\mu(gA) = \mu(A)$ for any measurable $A \subseteq S^{n-1}$ and any $g \in G$). This measure μ coincides with the normalized Lebesgue measure on S^{n-1} (the normalized surface area). The metric space S^{n-1} is distance invariant and for any d, $0 \leq d \leq 2$,

$$\overline{\mu}(d) = \frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} \text{ if } \cos \varphi = 1 - d^2/2, \tag{3.13}$$

where $\sigma_{n-1}(\varphi)$ is the surface area of a spherical cap on S^{n-1} of radial radius φ (i.e., of the set $\{\mathbf{y} : \mathbf{y} \in S^{n-1}, \varphi(\mathbf{x},\mathbf{y}) \le \varphi\}$ with $\mathbf{x} \in S^{n-1}$) and $\sigma_{n-1} = 2\sigma_{n-1}(\frac{\pi}{2})$ is the surface area of S^{n-1}. It is also known that

$$\frac{\sigma_{n-1}(\varphi)}{\sigma_{n-1}} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{\cos\varphi}^{1} (1-z^2)^{\frac{n-3}{2}} dz \tag{3.14}$$

and $\sigma_{n-1} = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$, where $\Gamma(x)$ is the gamma function ($\Gamma(x) = \int_0^\infty v^{x-1}e^{-v}dv$). For the Euclidean sphere $X = S^{n-1}$ with the standard substitution $\sigma(d) = 1 - \frac{d^2}{2}$ we have $s = \infty$ and $w(t) = \nu'(t) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}(1-t^2)^{\frac{n-3}{2}}$ (see (3.3), (3.13) and (3.14)). It is known [5] that the Jacobi polynomials

$$P_i^{\alpha,\beta}(t) = \frac{1}{2^i\binom{i+\alpha}{\alpha}} \sum_{j=0}^{i} \binom{i+\alpha}{j}\binom{i+\beta}{i-j}(t-1)^{i-j}(t+1)^j, \tag{3.15}$$

($\alpha \ge -\frac{1}{2}$, $\beta \ge -\frac{1}{2}$), normalized by $P_i^{\alpha,\beta}(1) = 1$, satisfy the following orthogonality condition:

$$\frac{2i+\alpha+\beta+1}{i+\alpha+\beta+1}\binom{i+\alpha+\beta+1}{i}\frac{\binom{i+\alpha}{i}}{\binom{i+\beta}{i}}c^{\alpha,\beta}\int_{-1}^{1} P_i^{\alpha,\beta}(t)P_j^{\alpha,\beta}(t)(1-t)^\alpha(1+t)^\beta dt = \delta_{i,j}, \tag{3.16}$$

where the constant

$$c^{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$$

normalizes the measure, i.e., $c^{\alpha,\beta}\int_{-1}^{1}(1-t)^\alpha(1+t)^\beta d(t) = 1$. Therefore, in the case S^{n-1}, $n = 2,3,\dots$, we have $r_i = \frac{2i+n-2}{i+n-2}\binom{i+n-2}{i}$ and

$$Q_i(t) = P_i^{\frac{n-3}{2},\frac{n-3}{2}}(t).$$

Example 3.5. *The real, complex, and quaternionic projective spaces. For $m = 1,2$, and 4, consider*

$$\mathbf{T}_m^n = \{\mathbf{u} = (u_1, \dots, u_n);\ u_i \in \mathbf{T}_m\},$$

where $\mathbf{T}_1 = \mathbf{R}$ is the real number field, $\mathbf{T}_2 = \mathbf{C}$ is the complex number field, and $\mathbf{T}_4 = \mathbf{H}$ is the associative (but noncommutative) quaternionic algebra. In $\mathbf{H}$ there is the basis 1 (the unity), i, j, k, such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and any $u \in \mathbf{H}$ can be represented in the form $u = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbf{R}$. Thus $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$ and m is the dimension of $\mathbf{T}_m$ over $\mathbf{R}$. For any $u \in \mathbf{H}$ there is defined the conjugate element $u^* = x_0 - x_1 i - x_2 j - x_3 k$. Since $uu^* = u^* u = x_0^2 + x_1^2 + x_2^2 + x_3^2$, one can define a norm $|u|$ of $u \in \mathbf{T}_m$ by means of $|u| = \sqrt{uu^*}$ and verify that $(uv)^* = v^* u^*$ and $|uv| = |u| \cdot |v|$ for any $u, v \in \mathbf{T}_m$. For any vectors (or points) $\mathbf{u} = (u_1, ..., u_n) \in \mathbf{T}_m^n$ and $\mathbf{v} = (v_1, ..., v_n) \in \mathbf{T}_m^n$, define the inner product as follows:

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} u_i v_i^*.$$

Points $\mathbf{u} = (u_1, ..., u_n) \in \mathbf{T}_m^n$ and $\mathbf{v} = (v_1, ..., v_n) \in \mathbf{T}_m^n$ are said to be equivalent if there exists a non-vanishing $\lambda \in \mathbf{T}_m$ such that $v_i = \lambda u_i$ for each $i = 1...,n$. Elements of projective spaces $\mathbf{T}_m P^{n-1}$ are defined as equivalence classes of non-zero points of $\mathbf{T}_m^n$ and called lines (through the origin in $\mathbf{T}_m^n$). For any $U, V \in \mathbf{T}_m P^{n-1}$, one can define the angle $\varphi = \varphi(U, V)$, $0 \leq \varphi \leq \frac{\pi}{2}$, between lines U and V by means of

$$\cos \varphi(U, V) = \frac{|(\mathbf{u}, \mathbf{v})|}{\sqrt{(\mathbf{u}, \mathbf{u})(\mathbf{v}, \mathbf{v})}} \text{ for any } \mathbf{u} \in U, \ \mathbf{v} \in V, \tag{3.17}$$

since the right-hand side of (3.17) does not depend on a choice of points $\mathbf{u}, \mathbf{v}$ on these lines, and the distance

$$d(U, V) = \sqrt{1 - \cos \varphi(U, V)} = \sqrt{2} \sin \frac{\varphi(U, V)}{2}.$$

For the metric space $X = \mathbf{T}_m P^{n-1}$ we have $\Delta(X) = [0, 1]$ and $D(X) = 1$. The following facts are well-known (see [16], [20]). The isometry group of $\mathbf{T}_m P^{n-1}$ for $m = 1, 2$, and 4, respectively, consists of all orthogonal, unitary, and symplectic matrices of order n and acts transitively. As in the case of S^{n-1}, on $\mathbf{T}_m P^{n-1}$ there exists the unique normalized invariant measure μ. The metric space $\mathbf{T}_m P^{n-1}$ is distance invariant and for any $d = \sqrt{2} \sin \frac{\varphi}{2}$, $0 \leq \varphi \leq \frac{\pi}{2}$,

$$\overline{\mu}(d) = \frac{\Gamma(\frac{m}{2} n)}{\Gamma(\frac{m}{2}(n-1)) \Gamma(\frac{m}{2})} \int_{\cos^2 \varphi}^{1} (1 - z)^{\frac{m}{2}(n-1)-1} z^{\frac{m}{2}-1} dz.$$

Therefore, for the projective spaces $X = \mathbf{T}_m P^{n-1}$ with the standard substitution

$$\sigma(d) = 2\left(1 - d^2\right)^2 - 1 \text{ (or } \sigma(d) = \cos 2\varphi \text{ if } d = \sqrt{2} \sin \frac{\varphi}{2}),$$

using (3.3) and the change of variable $2z = t + 1$ we have $s = \infty$ and $w(t) = \nu'(t) = \frac{\Gamma(\frac{m}{2} n)}{2^{\frac{m}{2} n - 1} \Gamma(\frac{m}{2}(n-1)) \Gamma(\frac{m}{2})} (1-t)^{\frac{m}{2}(n-1)-1} \, (1+t)^{\frac{m}{2}-1}$. Hence, in the case $\mathbf{T}_m P^{n-1}$, $m = 1, 2$, and 4, $n = 3, 4, \ldots$, with this substitution,

$$Q_i(t) = P_i^{\frac{m}{2}(n-1)-1, \frac{m}{2}-1}(t).$$

Example 3.6. *Consider the n-dimensional torus $T^n = ([0,1)^n, d(\mathbf{x},\mathbf{y}), \mu)$ with metric*

$$d(\mathbf{x},\mathbf{y}) = \max_{1\le i\le n} \min(|x_i - y_i|, 1 - |x_i - y_i|)$$

and the n-dimensional cube $I^n = ([0,1]^n, d(\mathbf{x},\mathbf{y}), \mu)$ with metric

$$d(\mathbf{x},\mathbf{y}) = \max_{1\le i\le n} |x_i - y_i|.$$

In both cases μ is the normalized n-dimensional volume. It is clear that T^n is a distance invariant space while I^n is not, and $D(T^n) = \frac{1}{2}$, $D(I^n) = 1$. In the case of the n-dimensional torus T^n we have $\mu(\mathbf{x}, \rho) = (2\rho)^n$ for any $\mathbf{x} \in T^n$ and any ρ, $0 \le \rho \le D(T^n) = \frac{1}{2}$. The linear substitution $\sigma(d) = 1 - 4d$ is standard and by (3.3) $\nu(t) = 1 - \left(\frac{1-t}{2}\right)^n$ and, hence, $w(t) = \nu'(t) = n2^{-n}(1-t)^{n-1}$. According to Theorem 3.1 and (3.16) in this case $r_i = \frac{n+2i}{n}\binom{n-1+i}{i}^2$ and

$$Q_i(t) = P_i^{n-1,0}(t).$$

For the n-dimensional cube I^n, $\mu(\mathbf{x},\rho)$ in general depends on $\mathbf{x} \in I^n$. However, it is possible to calculate that $\overline{\mu}(\rho) = \int_{I^n} \mu(\mathbf{x},\rho) d\mu(\mathbf{x}) = (2\rho - \rho^2)^n$ for any ρ, $0 \le \rho \le D(I^n) = 1$. The substitution $\sigma(d) = 1 - 2(2d - d^2)$ is standard and by (3.3) $\nu(t)$ also equals $1 - \left(\frac{1-t}{2}\right)^n$. Hence, the polynomials $Q_i(t)$ and constants r_i coincide with those for the torus T^n but under the latter standard substitution.

4. An universal bound on the size of τ-designs

We formulate an extremum problem for a system of orthogonal polynomials. This problem has a unique optimal solution. An application to the system Q introduced above gives a universal bound on the size of weighted τ-designs in a compact metric space X.

Let $F[t]$ be the set of polynomials in a real t. Any $f \in F[t]$ of degree at most $s = s(X)$ can be uniquely represented in the form $\sum_{i=0}^{s} f_i Q_i(t)$ where

$$f_i = f_i(Q) = r_i \int_{-1}^{1} f(t) Q_i(t) d\nu(t). \tag{4.1}$$

We also introduce the notation

$$\Omega(f) = \Omega_Q(f) = \frac{f(1)}{f_0} \quad \text{when } f_0 \ne 0. \tag{4.2}$$

Remark 4.1. *We can extend definitions (4.1) and (4.2) to polynomials $f(t)$ of degree more than s when $s = s(X)$ (and X) are finite. It is correct because in this case the polynomials $f(t)$ and $\sum_{i=0}^{s} f_i Q_i(t)$, where f_i are defined in (4.1), coincide at all $t = \sigma(d)$ when $d \in \Delta(X)$.*

Notice that by (3.2), (3.7)-(3.10) and (4.1)

$$f_0 = \int_X \int_X f(\sigma(d(x,y)))d\mu(x)d\mu(y)$$

and f_0 is equal to zero when $f(t) = Q_i(t)$, $i = 1, ..., s(X)$. This gives the following corollary.

Corollary 4.2. *A weighted set $C = (C, m)$ is a weighted τ-design if and only if*

$$\sum_{x \in C} \sum_{y \in C} Q_i(\sigma(d(x,y)))m(x)m(y) = 0, \quad i = 1, ..., \tau.$$

Consider an arbitrary weighted set $C = (C, m)$. A polynomial $f \in F[t]$ is called *annihilating* for C if $f(\sigma(d(x,y))) = 0$ for any $x, y \in C$, $x \neq y$. A polynomial, annihilating for C, of minimal degree (i.e., $s(C)$), is called *minimal.* Let $f_C(t)$ denote the minimal polynomial for C such that $f_C(1) = 1$. Let

$$\Delta'(C) = \{i : i \in \{0, 1, ..., s(X)\}, \quad B'_i(C) \neq 0\}.$$

where

$$B'_i(C) = \frac{r_i}{|C|} \sum_{x,y \in C} Q_i(\sigma(d(x,y)))m(x)m(y), \quad i = 0, 1, ..., s(X) \tag{4.3}$$

(the normalization (2.7) is used). Notice that

$$B'_0(C) = |C| \tag{4.4}$$

and hence $0 \in \Delta'(C)$. Let $\Delta'_0(C) = \Delta'(C) \backslash \{0\}$. From the definition (2.9) of the dual distance $d'(C)$ and Corollary 4.2 it follows (cf. (2.4)) that

$$d'(C) = \min \Delta'_0(C)$$

when the set $\Delta'_0(C)$ is not void.

For arbitrary weighted set $C = (C, m)$ and $f \in F[t]$, by (4.3) we have

$$\sum_{x,y \in C} f(\sigma(d(x,y)))m(x)m(y) = |C| \sum_{i=0}^{s(X)} \frac{f_i(Q)}{r_i} B'_i(C). \tag{4.5}$$

Theorem 4.3. *For any weighted τ-design $C = (C, m)$ and any polynomial $f \in F[t]$ of degree at most τ such that $f_0(Q) > 0$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$,*

$$|C| = \sum_{x \in C} m(x) \geq \Omega_Q(f) \tag{4.6}$$

with equality if and only if C is a simple (i.e., $m(x) \equiv 1$ for all $x \in C$) τ-design and $f(t)$ is annihilating for C.

Proof. By Corollary 4.2 and (4.3), $B_i'(C) = 0$ for $i = 1, ..., \tau$ and hence (4.5), (4.4), (3.1) and (2.7) imply the following inequalities

$$f_0 \geq f(\sigma(0)) \frac{\sum_{x \in C} m^2(x)}{\left(\sum_{x \in C} m(x)\right)^2} \geq \frac{f(1)}{|C|}$$

with the above mentioned conditions of attainability. ■

This theorem extends Delsarte's results [11] to the case of weighted designs in compact metric spaces and also gives rise to the following extremum problem for the system Q of orthogonal polynomials whose solution ensures the best bound in (4.6).

A restricted[1] τ-design problem: In the class of polynomials $f \in F[t]$ of degree at most τ such that $f_0(Q) > 0$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$ find a polynomial which maximizes $\Omega_Q(f)$.

Note that any polynomial f from this class implies the bound (4.6). Delsarte [11] used a certain polynomial $f = g^{(\tau)}$ defined below for τ-designs (τ even) in polynomial association schemes. Later Dunkl [15] gave the definition of $g^{(\tau)}$ for odd τ, calculated $\Omega_Q(g^{(\tau)})$ for some systems Q of a discrete variable, and remarked that by the result of Schoenberg and Szegö [33] for all τ the polynomial $f = g^{(\tau)}$ is a unique (up to a constant factor) solution of the restricted τ-design problem. Proofs of these results can also be found in [24], [25] as special cases of the solution of the restricted σ-packing problem for systems of orthogonal polynomials.

To describe the optimal polynomial $g^{(\tau)}$ and calculate $\Omega_Q(g^{(\tau)})$ for some compact metric spaces we define systems $Q^{a,b}$, $a, b \in \{0, 1, ... \}$, of orthogonal polynomials which in a certain sense are *adjacent* to an arbitrary system Q satisfying the conditions (3.8) and (3.9). As before we assume that $\nu(t)$ is a left continuous step function or continuously differentiable, and hence (3.11) or (3.12) holds. First we choose a function $\nu^{a,b}(t)$ and constants $c^{a,b}$ as follows. If $\nu(t)$ is continuously differentiable and $\nu'(t) = w(t)$, then so is $\nu^{a,b}(t)$ and $(\nu^{a,b}(t))' = c^{a,b}(1-t)^a(1+t)^b w(t)$. If $\nu(t)$ is a step function, then so is $\nu^{a,b}(t)$ and it is produced from $\nu(t)$ by multiplying its steps w_i at points $t_i = \sigma(d_i)$ $(i = 0, 1, ..., s)$ by $c^{a,b}(1-\sigma(d_i))^a(1+\sigma(d_i))^b$ (thus the number $s^{a,b} + 1$ of steps of $\nu^{a,b}(t)$ will be $s - 1 + \delta_{a,0} + \delta_{b,0}$ since $\sigma(d_0) = 1$ and

[1] The term "restricted" means that we only consider polynomials of degree at most τ; later we remove the restriction for polynomial metric spaces.

$\sigma(d_s) = -1$; we put $s^{a,b} = \infty$ in the continuous case). Constants $c^{a,b}$ are chosen so that the Lebesgue-Stieltjes measure $\nu^{a,b}$ on $[-1,1]$ generated by the function $\nu^{a,b}(t)$ (similar to (3.5)-(3.6)) would be also normalized, i.e.,

$$\int_{-1}^{1} d\nu^{a,b}(t) = c^{a,b} \int_{-1}^{1} (1-t)^a (1+t)^b d\nu(t) = 1. \tag{4.7}$$

Similar arguments as in Theorem 3.1 show that there exists a unique system $Q^{a,b}$ of orthogonal polynomials $Q_i^{a,b}(t)$ of degree i on interval $[-1,1]$ and a unique system of positive constants $r_i^{a,b} \quad (i = 0, 1, ..., s^{a,b})$ such that

$$r_i^{a,b} \int_{-1}^{1} Q_i^{a,b}(t) Q_j^{a,b}(t) d\nu^{a,b}(t) = \delta_{i,j}, \tag{4.8}$$

$$Q_i^{a,b}(1) = 1, \quad i = 0, 1, ..., s^{a,b}. \tag{4.9}$$

It is easily seen that $Q_0^{a,b}(t) = 1$, $r_0^{a,b} = 1$, $Q_i^{0,0}(t) = Q_i(t)$, $r_i^{0,0} = r_i$, $c^{0,0} = 1$. Sometimes we shall omit the indices a, b if they both equal 0. Let

$$T_i^{a,b}(x,y) = \sum_{j=0}^{i} r_j^{a,b} Q_j^{a,b}(x) Q_j^{a,b}(y), \ i = 0, 1, ..., s^{a,b}. \tag{4.10}$$

By Christoffel-Darboux formula (see [5]),

$$Q_i^{a,b+1}(t) = \frac{T_i^{a,b}(t,-1)}{T_i^{a,b}(1,-1)}, \quad Q_i^{a+1,b}(t) = \frac{T_i^{a,b}(t,1)}{T_i^{a,b}(1,1)}. \tag{4.11}$$

Let $t_i^{a,b}$ denote the largest root of polynomial $Q_i^{a,b}(t)$ and put

$$d_i^{a,b} = \sigma^{-1}(t_i^{a,b}). \tag{4.12}$$

To define $g^{(\tau)}$ we represent τ as an odd or even integer in the form

$$\tau = 2l - \theta \quad \text{where } l = l(\tau) \in \{1, 2, \dots\} \text{ and } \theta = \theta(\tau) \in \{0, 1\}.$$

It is clear that $l = \lfloor \frac{\tau+1}{2} \rfloor$ and $\theta = 2\lfloor \frac{\tau+1}{2} \rfloor - \tau$. Let

$$g^{(\tau)}(t) = (1+t)^{\theta} \left(\sum_{j=0}^{l-\theta} r_j^{0,\theta} Q_j^{0,\theta}(t) \right)^2. \tag{4.13}$$

In accord with (4.8)

$$r_i^{0,\theta} c^{0,\theta} \int_{-1}^{1} Q_i^{0,\theta}(t) Q_j^{0,\theta}(t) (1+t)^{\theta} d\nu(t) = \delta_{i,j},$$

and, hence, by (4.1) and (4.7)

$$g_0^{(\tau)}(Q) = \int_{-1}^{1} g^{(\tau)}(t)d\nu(t) = (c^{0,\theta})^{-1}\sum_{i=0}^{l-\theta} r_i^{0,\theta},$$

where $c^{0,0} = 1$ and

$$(c^{0,1})^{-1} = \int_{-1}^{1}(1+t)d\nu(t) = \frac{-2Q_1(-1)}{1-Q_1(-1)},$$

since $1+t = 2\frac{Q_1(t)-Q_1(-1)}{1-Q_1(-1)}$. Using the polynomial $g^{(\tau)}$ in Theorem 4.3 and taking (4.9)-(4.11) into account, we get the following theorem.

Theorem 4.4. *For any weighted τ-design $C = (C, m)$ in a compact metric space X,*

$$|C| \geq \Omega_Q(g^{(\tau)}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta}\sum_{i=0}^{l-\theta} r_i^{0,\theta}, \tag{4.14}$$

where $l = \lfloor\frac{\tau+1}{2}\rfloor$ and $\theta = 2l-\tau$ with equality in (4.14) holding if and only if C is a simple τ-design and $(1+t)^{\theta}Q_{l-\theta}^{1,\theta}(t)$ is annihilating for C.

A τ-design C for which the bound (4.14) is attained is called a *tight* design. The following two statements of Delsarte [11] are also valid for codes in compact metric spaces.

Lemma 4.5. *For any code C in a compact metric space X*

$$\tau(C) \leq 2s(C) - \beta(C). \tag{4.15}$$

Proof. For any code C there exists a polynomial $g(t)$ of degree $s-\beta$ where $s = s(C)$ and $\beta = \beta(C)$, such that the polynomial $f(t) = (1-t)(1+t)^{\beta}(g(t))^2$ is annihilating for C and, moreover, the left-hand side of (4.5) equals zero. The polynomial $f(t)$ is nonnegative for $-1 \leq t \leq 1$ and by (4.1), $f_0(Q) > 0$ (if its degree $2s-\beta+1$ does not exceed $s(X)$ in the case of finite spaces X). Therefore, the inequalities

$$s(X) \geq \tau(C) \geq 2s(C) - \beta(C) + 1$$

imply that $B_i'(C) = 0$, $i = 1, ..., 2s-\beta+1$, and contradict the equality (4.5) for the polynomial $f(t)$ of degree $2s-\beta+1 \leq s(X)$. ■

Lemma 4.6. *A code C in a compact metric space X is a tight design if and only if $\tau(C) = 2s(C) - \beta(C)$ and $f_C(t) = (\frac{1+t}{2})^{\beta(C)}Q_{s(C)-\beta(C)}^{1,\beta(C)}(t)$.*

Proof. If the polynomial $(1+t)^\theta Q_{l-\theta}^{1,\theta}(t)$ where $\theta \in \{0,1\}$ is annihilating for a code C, then $s(C)-\beta(C) \leq l-\theta$, and $\beta(C) \leq \theta$ since $\sigma(D(X)) = -1$ and $Q_l^{1,0}(-1) \neq 0$. Therefore for a tight $(2l-\theta)$-design C,

$$\tau(C) \geq 2l-\theta \geq 2s(C)-2\beta(C)+\theta \geq 2s(C)-\beta(C)$$

and hence by Theorem 4.4 and Lemma 4.5 $\tau(C) = 2s(C)-\beta(C)$, $l = s(C)$, $\theta = \beta(C)$. On the other hand, if the conditions of the lemma are satisfied, then using $f(t) = f_C(t)$ in (4.5) shows that $|C| = \Omega(f_C)$. However, taking (4.9)-(4.11) into account again, it is easy to verify that $\Omega(f_C) = \Omega(g^{(\tau(C))})$, and we have equality in (4.14). ■

The systems $Q^{0,1}$ for the Hamming space H_v^n and for the Johnson space J_w^n coincide with systems Q for spaces H_v^{n-1} and J_{w-1}^{n-1} respectively (see, for example, [24]). For the Euclidean sphere S^{n-1}, the projective spaces $\mathbf{T}_m P^{n-1}$, the torus T^n, and the cube I^n, the polynomials $Q_i(t)$ are equal to Jacobi polynomials $P_i^{\alpha,\beta}(t)$ with some α and β ; hence, $Q_i^{0,1}(t) = P_i^{\alpha,\beta+1}(t)$ in these cases. It is known (see, for example, [5]) that for normalized Jacobi polynomials $P_i^{\alpha,\beta}(t)$ (see (3.15)-(3.16)),

$$\sum_{j=0}^{i} r_j^{\alpha,\beta} = \binom{i+\alpha+\beta+1}{i}\binom{i+\alpha+1}{i} \Big/ \binom{i+\beta}{i} \tag{4.16}$$

and

$$1 - \frac{1}{P_1^{\alpha,\beta}(-1)} = \frac{\alpha+\beta+2}{\beta+1}. \tag{4.17}$$

Therefore, for $Q_i(t) = P_i^{\alpha,\beta}(t)$ the right-hand side of (4.14) equals

$$\binom{l+\alpha+\beta+1}{l}\binom{l+\alpha+1-\theta}{l-\theta} \Big/ \binom{l+\beta}{l} \tag{4.18}$$

We can use Examples considered and Theorem 4.4 to obtain the lower bounds on the size of a weighted $(2l-\theta)$-design given in Table 4.1. In the connection with these bounds for the projective spaces the following references are relevant: [11], [12], [13], [22], [26], [21], and [3].

5. Designs and codes in polynomial spaces

Now we put a restriction on the system Q and, hence, on the compact metric space $X = (X, d(x,y), \mu)$ which the system corresponds to. This allows us to simplify the definition (2.8) of a τ-design and apply the system Q for obtaining a universal bound on the size of codes with a given minimum distance.

For a compact metric space $X = (X, d(x,y), \mu)$ we discuss real- and complex-valued functions $F(x,y)$ in two variables $x, y \in X$. In the case of a finite X, $F(x,y)$ might be considered as an element of a matrix of order $|X| \times |X|$ in row $x \in X$ and column $y \in X$. A function $F(x,y)$ is called *Hermitian if* $F(x,y) = \overline{F(y,x)}$ for

Space X	Standard substitution $\sigma(d)$	Lower bound on the size of a $(2l-\theta)$-design	References
H_v^n	$1-\frac{2d}{n}$	$v^\theta \sum_{j=0}^{l-\theta} \binom{n-\theta}{j}(v-1)^j$	[30]
J_w^n	$1-\frac{2d}{w}$	$\left(\frac{n}{w}\right)^\theta \binom{n-\theta}{l-\theta}$	[31] when $\theta=0$ [15] when $\theta=1$
S^{n-1}	$1-\frac{d^2}{2}$	$\binom{n+l-2}{l-1}+\binom{n+l-1-\theta}{l-\theta}$	[13]
$\mathbf{R}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\binom{n+2l-1-\theta}{2l-\theta}$	
$\mathbf{C}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\binom{n+l-1}{l}\binom{n+l-1-\theta}{l-\theta}$	
$\mathbf{H}P^{n-1}$	$2\left(1-d^2\right)^2-1$	$\frac{1}{l+1}\binom{2n+l-1}{l}\binom{2n+l-2-\theta}{l-\theta}$	
T^n	$1-4d$	$\binom{n+l}{l}\binom{n+l-\theta}{l-\theta}$	
I^n	$1-2d(2-d)$	$\binom{n+l}{l}\binom{n+l-\theta}{l-\theta}$	

Table 4.1: Lower Bounds on the Size of $(2l-\theta)$-Designs

any $x, y \in X$ where $\bar{a}$ is the conjugate of a. A Hermitian function $F(x,y)$ is called *nonnegative definite on* X if for any finite set (code) $C \subseteq X$ and any complex function $\mathbf{v}(x)$, $x \in X$:

$$\sum_{x,y \in C} F(x,y)\mathbf{v}(x)\overline{\mathbf{v}(y)} \geq 0 \tag{5.1}$$

(since $F(x,y)$ is Hermitian, the left-hand side of (5.1) is real). For a finite X, a matrix $F(x,y)$, $x, y \in X$, is nonnegative definite if and only if there exists some number h of functions $\mathbf{w}_1(x), ..., \mathbf{w}_h(x)$ on X such that

$$F(x,y) = \sum_{j=1}^{h} \mathbf{w}_j(x)\overline{\mathbf{w}_j(y)}. \tag{5.2}$$

For our purposes in the case of infinite metric spaces it is also sufficient to consider only nonnegative definite functions of this type. We shall call a function which can be represented in the form (5.2) with continuous nonzero functions $\mathbf{w}_j(x)$, $j = 1, ..., h$ ($h \geq 1$), a *finite dimensional*[2] *nonnegative definite function (FDNDF) on* X. The most significant example of a FDNDF on the Euclidean complex space $\mathbf{C}^n$ is the usual inner product $(x,y) = \sum_{i=1}^{n} x_i \overline{y_i}$ of vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. To discuss simultaneously FDNDFs on finite and compact infinite metric spaces X with normalized measure μ we consider the (possibly, finite dimensional) Hilbert space $L_2(X, \mu)$ of complex-valued functions $\mathbf{u}$ on X satisfying

$$\|\mathbf{u}\|^{\mathbf{2}} = \int_X |\mathbf{u}(x)|^{\mathbf{2}} d\mu(x) < \infty$$

with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_X \mathbf{u}(x)\overline{\mathbf{v}(x)} d\mu(x) \tag{5.3}$$

(Functions are equivalent if they differ on a set A with $\mu(A) = 0$, and hence equivalent continuous functions coincide by our assumption (2.1).) We associate with every FDNDF $F(x,y)$ a linear operator F which transforms $\mathbf{u} \in L_2(X, \mu)$ to the function

$$F\mathbf{u} = \int_X F(x,y)\,\mathbf{u}(y)\,d\mu(y) = \sum_{j=1}^{h} \mathbf{w}_j(x) \int_X \overline{\mathbf{w}_j(y)}\mathbf{u}(y) d\mu(y) \tag{5.4}$$

of a subspace $V(F)$ of continuous functions generated by $\mathbf{w}_1(x), \ldots, \mathbf{w}_h(x)$. This finite dimensional operator F is self-conjugate, since

$$\langle F\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, F\mathbf{v} \rangle = \sum_{j=1}^{h} \langle \mathbf{u}, \mathbf{w}_j \rangle \overline{\langle \mathbf{v}, \mathbf{w}_j \rangle}. \tag{5.5}$$

[2] Below we consider the function $F(x,y)$ as the kernel of a linear finite dimensional operator on $L_2(X, \mu)$.

From (5.4), (5.5) it follows that all eigenvalues of F are nonnegative, all eigenfunctions with positive eigenvalues belong to $V(F)$, and all eigenfunctions with zero eigenvalue are orthogonal to $V(F)$. It follows that for any FDNDF $F(x,y)$ on X, there exists in $V(F)$ an orthonormal basis of $m = \dim V(F)$ continuous eigenfunctions $\mathbf{e}_1(x), \ldots, \mathbf{e}_m(x)$ with positive eigenvalues $\lambda_1, \ldots, \lambda_m$ (i.e., $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, $F\mathbf{e}_i = \lambda_i \mathbf{e}_i$) such that (5.2) can be reduced to the following diagonal form:

$$F(x,y) = \sum_{i=1}^{m} \lambda_i \mathbf{e}_i(x) \overline{\mathbf{e}_i(y)}. \tag{5.6}$$

We shall call a compact metric space X *FDNDF-polynomial (with respect to a standard substitution* $\sigma(d)$) if $Q_i(\sigma(d(x,y)))$ is a FDNDF on X for any $i = 0, 1, ..., s(X)$. For a FDNDF-polynomial space X (which, in general, is not distance invariant) let $F_i(x,y) = Q_i(\sigma(d(x,y)))$, $V_i = V(F_i)$, $m_i = \dim V_i$, and $F_i(x,y)$ has the following diagonal form (5.6):

$$F_i(x,y) = Q_i(\sigma(d(x,y))) = \sum_{j=1}^{m_i} \lambda_{i,j} \mathbf{e}_{i,j}(x) \overline{\mathbf{e}_{i,j}(y)}. \tag{5.7}$$

The functions $\mathbf{e}_{i,j}(x)$, $i = 0, 1, ..., s(X)$, $j = 1, ..., m_i$, form an orthonormal system with respect to the inner product (5.3), since by (3.8), (3.7) and (3.2),

$$r_i \int_{-1}^{1} Q_i(t) Q_j(t) d\nu(t) = r_i \sum_{k=1}^{m_i} \sum_{l=1}^{m_j} \lambda_{i,k} \lambda_{j,l} |\langle \mathbf{e}_{i,k}, \mathbf{e}_{j,l} \rangle|^2 = \delta_{i,j}. \tag{5.8}$$

Note that by (3.9) and (5.7) for any $x \in X$,

$$1 = Q_i(\sigma(d(x,x))) = \sum_{j=1}^{m_i} \lambda_{i,j} |\mathbf{e}_{i,j}(x)|^2 \tag{5.9}$$

and hence $\sum_{j=1}^{m_i} \lambda_{i,j} = 1$. (We used that $\langle \mathbf{e}_{i,j}, \mathbf{e}_{i,j} \rangle = 1$.) Furthermore, from (5.8) it follows that $r_i \sum_{j=1}^{m_i} \lambda_{i,j}^2 = 1$. Since

$$1 = r_i \sum_{j=1}^{m_i} \lambda_{i,j}^2 = \left(\sum_{j=1}^{m_i} \lambda_{i,j} \right)^2 \leq m_i \sum_{j=1}^{m_i} \lambda_{i,j}^2,$$

then $r_i \leq m_i$ with equality if and only if all $\lambda_{i,j}$, $j = 1, ..., m_i$, are equal and, hence, equal to $\frac{1}{r_i}$. In this important case, when for any i, $i = 0, 1, .., s(X)$, all eigenvalues $\lambda_{i,j}$, $j = 1, ..., m_i$, are equal, we shall call a FDNDF-polynomial space X *polynomial* (cf. [26], [18]). Thus, for polynomial spaces the constants r_i in (3.8) are integers and equal to the dimensions of the subspaces V_i, and (5.7) takes the form

$$Q_i(\sigma(d(x,y))) = \frac{1}{r_i}\sum_{j=1}^{r_i} \mathbf{e}_{i,j}(x)\overline{\mathbf{e}_{i,j}(y)}. \tag{5.10}$$

Fortunately, some metric spaces of significant interest in coding theory, in particular, the Hamming space, the Johnson space, the unit Euclidean sphere, and the projective spaces (with the standard substitutions given above) turn out to be polynomial. Moreover, the theory of group representation applied to the isometry groups of the spaces gives the explicit description of the subspaces V_i and the basis functions $\mathbf{e}_{i,j}(x)$ (see [11], [13], [22], [23], [37], [39]).

The representation (5.7) (and (5.10)) allows us to extend some results of Delsarte [11] on codes in Q-polynomial association schemes to the general case.

Theorem 5.1. *For any weighted set $C = (C, m)$ in a FDNDF-polynomial space X with the standard substitution $\sigma(d)$ and any $i = 0, 1, ..., s(X)$,*

$$B_i'(C) = \frac{r_i}{|C|}\sum_{x,y\in C} Q_i(\sigma(d(x,y)))m(x)m(y)$$

$$= \frac{r_i}{|C|}\sum_{j=1}^{m_i}\lambda_{i,j}\Big|\sum_{x\in C}\mathbf{e}_{i,j}(x)m(x)\Big|^2 \geq 0. \tag{5.11}$$

Corollary 5.2. *A weighted set $C = (C, m)$ in a FDNDF-polynomial space X is a weighted τ-design if and only if*

$$\sum_{x\in C}\mathbf{e}_{i,j}(x)m(x) = 0 \text{ for any } i \text{ and } j,\ i = 1, ..., \tau,\ j = 1, ..., m_i, \tag{5.12}$$

or, equivalently, if and only if for any $u(x) \in \bigcup_{i=0}^{\tau} V_i$,

$$\int_X u(x)d\mu\,(x) = \frac{1}{|C|}\sum_{x\in C} u(x)m\,(x)\,. \tag{5.13}$$

In particular, in the case of $X = S^{n-1}$ with $\sigma(d) = 1 - d^2/2$ the set $\bigcup_{i=0}^{\tau} V_i$ consists of all functions $u(\mathbf{x})$, which are polynomials in coordinates of $\mathbf{x} = (x_1, ..., x_n) \in S^{n-1}$ of the total degree at most τ, and (5.13) coincides with the classical definition of a spherical τ-design (see [13]). In the case of Hamming space $X = H_v^n$ and Johnson space $X = J_w^n$ (with linear substitutions) Delsarte [11] proved that for a code C (i.e., $m(x) \equiv 1$) the conditions (5.12) are satisfied if and only if C is an orthogonal array $OA_\lambda(\tau, n, v)$ with $\lambda = |C|v^{-\tau}$or a block design $S_\lambda(\tau, w, n)$ with $\lambda = |C|\binom{w}{\tau}\binom{n}{\tau}^{-1}$, respectively.

For any weighted set $C = (C, m)$ and any polynomial $f(t)$ consider (4.5). Taking into account that $B'_0(C) = |C|$, $B'_i(C) \geq 0$ for $i = 1, ..., s(X)$, and that $f(\sigma(d(x, y))) = f(1)$ if $d(x, y) = 0$ and $\sigma(d(x, y)) \leq \sigma(d)$ if $d(x, y) \geq d$, we obtain the following statements.

Theorem 5.3. *For any weighted τ-design $C = (C, m)$ in a FDNDF-polynomial space X and any polynomial $f(t)$ such that $f_0(Q) > 0$, $f_i(Q) \leq 0$ for $i = \tau + 1, ..., s(X)$ and $f(t) \geq 0$ for $-1 \leq t \leq 1$,*

$$|C| = \sum_{x \in C} m(x) \geq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{5.14}$$

with equality if and only if C is a simple τ-design, the polynomial $f(t)$ is annihilating for C and

$$f_i(Q) B'_i(C) = 0 \text{ for } i = \tau + 1, ..., s(X). \tag{5.15}$$

Theorem 5.4. *For any code C with minimal distance $d(C) = d$ in a FDNDF-polynomial space X and any polynomial $f(t)$ such that $f_0(Q) > 0$, $f_i(Q) \geq 0$ for $i = 1, ..., s(X)$ and $f(t) \leq 0$ for $-1 \leq t \leq \sigma = \sigma(d)$,*

$$|C| \leq \Omega_Q(f) = \frac{f(1)}{f_0(Q)} \tag{5.16}$$

with equality if and only if the polynomial $f(t)$ is annihilating for C and

$$f_i(Q) B'_i(C) = 0 \text{ for } i = 1, ..., s(X). \tag{5.17}$$

The comparison of Theorem 5.3 with Theorem 4.3 shows that one can use polynomials of degree more than τ to obtain a lower bound on the size of τ-designs in FDNDF-polynomial spaces. This was really used to improve the bound (4.14) for sufficiency large τ (see [25] and [41]). Moreover, for FDNDF-polynomial spaces the condition $\tau(C) = 2s(C) - \beta(C)$ is already sufficient for C to be a tight design (cf. Lemma 4.6).

At the same time Theorem 5.4 allows us to use the system Q for obtaining universal (upper) bounds on the size of codes with given minimum distance d in compact metric spaces, which are FDNDF-polynomial. A review of the known results in this direction can be found in [24], [25]. Now we only give an universal bound for codes, which is, in a sense, dual to the bound of Theorem 4.4. For any positive integer l and $\theta \in \{0, 1\}$ put $\sigma = t^{1,\theta}_{l-\theta} = \sigma(d^{1,\theta}_{l-\theta})$ (see (4.12)) and $\tau = 2l - \theta$ and consider the polynomial

$$f^{(\sigma)}(t) = \frac{t+1}{t-\sigma} g^{(\tau)}(t)$$

where $g^{(\tau)}$ is defined in (4.13). It is possible to show (see [24] or [25]) that this polynomial $f(t) = f^{(\sigma)}(t)$ satisfies all conditions of Theorem 5.4 and $\Omega_Q(g^{(\tau)}) = \Omega_Q(f^{(\sigma)})$. This implies the following universal bound.

Theorem 5.5. *For any code C in a FDNDF-polynomial space X such that $d(C) \geq d_{l-\theta}^{1,\theta}$ for some positive integer l and $\theta \in \{0,1\}$,*

$$|C| \leq \Omega_Q(g^{(\tau)}) = \left(1 - \frac{1}{Q_1(-1)}\right)^{\theta} \sum_{i=0}^{l-\theta} r_i^{0,\theta}, \tag{5.18}$$

where $\tau = 2l - \theta$, with equality in (5.18) holding if and only if $d(C) = d_{l-\theta}^{1,\theta}$ and C is a tight τ-design.

In conclusion we should note that in the paper the problems of the existence of (tight) τ-designs were not considered, although numerous examples are known for association schemes, the Euclidean sphere, and the projective spaces (see [11], [13], [12], [21], [35], [2], [6], [8], [4]). It is also interesting to consider connections of this concept of a design with the calculation theory, in particular, with codes for Monte Carlo method (see [36], [29]).

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