Pattern formation in active oscillatory media and its relation to associative memory networks

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ABSTRACT

We continue to study the arrays of nonlinear coupled oscillators. The networks of associative memory based on limit-cycle oscillators connected via complex-valued Hermitian matrices were previously designed. Another class of networks consisting of locally connected nonlinear oscillators, closely related to so-called cellular neural networks, is the subject of study in the present paper. In spatially continual limit these oscillatory networks can be considered as oscillatory media governed by the system of reaction-diffusion equations. Formation of spatio-temporal dissipative structures (wave trains, standing waves, targets and shock structures, spiral waves, stripe patterns, cluster states) in various nonlinear active media is widely used for modelling of complicated nonlinear phenomena in physics, chemistry, biology, neurophysiology.

Here the results of analytical study of 1D oscillatory media corresponding to closed and unclosed chains of limit-cycle oscillators are presented. Conditions of existence of some spatio-temporal regimes inherent to nonlinear active media (diffusion instability caused by coupling, formation of wave trains and standing waves) have been clarified.

1. INTRODUCTION

Systems of nonlinear coupled oscillators deliver a remarkable tool for modelling of critical phenomena and phase transitions in large-scale systems of various nature. Some interdisciplinary links for oscillatory systems are presented in Fig.1.

Spatially-continual analogies of discrete oscillatory systems - nonlinear oscillatory media - are very useful because they permit to obtain strict mathematical results concerning asymptotical properties and types of behavior of discrete systems [1].

It is worth to remind that the adequate continual analogy of some given large-scale discrete system consisting from interacting processing units is crucially dependent on the range of interaction in the system. In the case of global connections between the units (long range interaction) the kinetic equations, providing the description in terms of distribution function, deliver the continual description.

In the case of local or quasi-local connections the representation of discrete system in the form of equivalent continuous medium with proper local interaction is relevant.

Nonlinear Fokker-Planck equation was derived for globally connected oscillatory systems and successfully used for analysis of the types of dynamics and phase transitions in equivalent spatially continuous systems [2-4]. In the case of homogeneously globally connected oscillators of Ginzburg-Landau type it was shown [2] that two phases of oscillatory system state exist. These phases are analogous to paramagnetic (disordered) and ferromagnetic (ordered) phases for magnetic systems. Phase transition that oscillatory system undergoes from paramagnetic phase into ferromagnetic one is similar to phase transition of second kind and can be appropriately described in terms of proper macrovariable - "order parameter". For the system of Ginzburg-Landau oscillators the order parameter is two-component macrovariable [2]. The self-consistency equation for order parameter, that can be derived from Fokker-Planck equation, permits to extract the essential parameters of the model and to clarify completely the character of bifurcations of Fokker-Planck equation.

In inhomogeneously connected oscillatory system coupled via special nonsymmetrical random-valued matrix of connection (van Hemmen matrix) there exist four phases: paramagnetic, ferromagnetic, spin glass and mixed ones [4]. The order parameter is three-component macrovariable, but in continual limit two-component order parameter, defining all the phase transitions, can be extracted.

Homogeneously locally connected oscillatory networks can be considered as 1D, 2D or nD spatially distributed arrays of oscillators. Similarly to locally connected neural networks known as cellular neural networks (CNN) [5,6], locally connected oscillatory networks may be naturally regarded as cellular oscillatory networks.

Cellular oscillatory networks were already used and can be used further for modelling of a variety of phenomena in physics, chemistry, biology, neurophysiology. The attempts of modelling synchronization and desynchronization phenomena in brain visual cortex are worth of special mention. The matter is that the synchronization is a flexible dynamical mechanism for binding separated objects into a coherent one, for formation of dynamical links in brain structures. Since the discovery of stimulus-dependent oscillations and long-range synchronization in cat visual cortex, many theoretical and modelling attempts have been enterprised to understand and interpret the origin of oscillations and the role of synchronization in visual cortex. One of important questions concerns the character of connections in synchronizable visual cortex networks. There exists the evidence of local and quasi-local character of visual cortex connections, and so some initial attempts to use 1D and 2D networks of locally connected Wilson-Cowan oscillators for modelling brain cortical oscillations were already fulfilled. The single oscillator was formed by a couple of connected excitatory and inhibitory neurons [7].

At last, it is worth to mention another interesting and promising direction of research - the usage of cellular neural networks in the development of the model of so called "bionic eye" [8].

From mathematical viewpoint various models of nonlinear media governed by the systems of reaction-diffusion equations were studied since 70s. [1,9-13]. Belousov-Zhabotinskii oscillating chemical reaction in a thin layer of fluid and oscillatory media of Ginzburg-Landau oscillators belong to the most familiar examples of active media.

Considerable scope of oscillatory media studies exists. The strict mathematical [9-13], physical level [14-18] and computer modelling [6,14] results could be mentioned as examples.

Generalized Ginzburg-Landau oscillatory medium models were also used for studying of spatio-temporal radiative dissipative structures in spatially distributed wide-aperture laser systems. These are characterized by bistability and hysteresis phenomena under proper character of nonlinear dependence of medium refraction index and also absorption on radiation inside the medium. Formation and interaction of localized impulse wave trains (regarded as localized dissipative structures, of auto-solitons) and wave front trains (switch over waves) in 1D and 2D multistable wide aperture laser systems with hysteresis were studied theoretically (physical level) and confirmed in computer experiments [16-18].

Here we present some results of qualitative mathematical analysis of dissipative structures in 1D oscillatory media corresponding to homogeneous closed and unclosed chains of limit-cycle oscillators of Ginzburg-Landau type.

The results on 1D media that are interesting by itself, could be also considered as an initial step in further study of the most interesting 2D oscillatory media.

2. OSCILLATORY NETWORKS OF ASSOCIATIVE MEMORY

Associative memory in the networks of coupled oscillators was studied analytically and by computer modelling methods [19,20].

We consider oscillatory network model consisting of limit-cycle oscillators possessing two degrees of freedom. Limit cycle of a single oscillator is the circle of unit radius in the plane. Dynamical equations governing the dynamics of the network of N coupled oscillators are:

$$\dot{u}_j = (1 + i\omega_j - |u_j|^2)u_j + \sum_{k=1}^{N} W_{jk}(u_k - u_j), \quad j = 1, ..., N. \tag{1}$$

Here the variable $u_j(t) = r_j(t)\exp(i\theta_j(t))$ defines the state of j-th oscillator (r_j and θ_j are the amplitude and phase of oscillations, respectively), ω_j is its cycle frequency. The first term in the right-hand side of (1) defines the intrinsic dynamics of a free isolated oscillator, while the second one, responsible for interaction, is specified by the matrix of connections $W = [W_{jk}]$.

2.1 The definition of associative memory (AM) in oscillatory networks (ON).

The system (1) can always be transformed to the same system with $\sum \omega_j = 0$ by the shift of variables. Therefore, we assume $\sum \omega_j = 0$. The oscillatory system governed by eq.(1) may relax to synchronized states with constant amplitudes $\{r_j\}$ and phase-locked phases $\{\varphi_j\}$.

For a wide range of parameters $\{\omega_j, W_{jk}\}$ two principal phenomena were observed for the system (1): 1) all initial states $\{r_j(t_0), \varphi_j(t_0)\}$ evolve to synchronized states and 2) these synchronized states form a finite set of isolated points, if we identify all the points in the phase space of (1), which have the same phase-locked phases (in other words, we reduce the phases of all points to the points with $\varphi_1(t) = 0$, subtracting $\varphi_1(t)$ from all other phases. Such identified sets are named fixed points here. These two phenomena are basic in design of oscillatory AM.

Thus, the AM in the system (1) is defined as an algorithm prescribing the calculation of parameters $\{\omega_j, W_{jk}\}$ for a given set of memories $\{r_j^m exp(i\varphi_j^m)\}$, providing these memories to be isolated fixed points of the system (1).

It would be noted that such AM is a generalization of Q-state Potts neural networks. In ON the states of oscillatory "neuron" run over a circle, the phase space of the system is the N-dimensional torus. This model is more convenient for analysis than discrete models, because the phase space is a continuous compact variety. Moreover, the symmetry is its inherent property. In addition, this model looks more natural in the context of physical implementations — in nonlinear optics, for example.

2.2 Hebbian learning in ON.

In the problem of phase AM in ON the "phase basis" is an important object. It is defined as follows. Consider N phase points in N-dimensional complex space C^N:

$$z_k = (1, e^{i\psi_k}, e^{2i\psi_k}, \ldots, e^{(N-1)i\psi_k}), \quad \psi_k = 2\pi k/N, \quad k = 0, \ldots, (N-1).$$

This set of points is the phase basis in C^N.

It was shown in [19] that Hebbian learning works in the system (1) and permits to design phase AM. The basic case is $\omega_j \equiv 0$ (phasor networks). In this case we may take the matrix W in the form:

$$W = N^{-1} \sum_{m=1}^{M} \lambda^m V^m (V^m)^+, \tag{2}$$

where $\{V^m\}$ is the set of mutually orthogonal vectors from the above phase basis, V^+ is the Hermitian conjugate to V.

Obviously, all the vectors V^m are equilibrium points of the system (1). Varying the coefficients λ^m, we can make some subset of these vectors the stable points of the system (1). Computer experiments show that at least $N/2$ points can be made stable, if N is a prime number. It is worth noting that here spurious memory exists and is highly symmetrical.

If ω_j are not identically zero it can be shown that at arbitrary ω_j in the case of strong interaction in the oscillatory network there exist the set $\tilde{V}^m$ close to V^m, which is the set of memory vectors of oscillatory network with the matrix of connections defined by formula (2).

To formulate the result more exactly, note first of all that the parameter $\gamma = \Omega/K$, where $\Omega = max_j|\omega_j|$, is the essential parameter of the system. The case $\omega_j \equiv 0$ can be considered as the limit case of infinitely strong interaction of oscillators in the network ($K \to \infty$). When K is a great, but finite value, γ is the small parameter, and the perturbation method for the system of equations defining the fixed points of network dynamics can be derived. The perturbation method provides the following result.

At sufficiently small $\gamma = \Omega/K$ the memory vectors $\tilde{V}^m$ of oscillatory network belong to small vicinities of vectors V^m and the following estimations take place:

$$\tilde{V}_j^m = V_j^m + \gamma(\lambda^m)^{-1}\omega_j + O(\gamma^2), m = 1, ...M, j = 1, ...N.$$

The fact that the memory vectors of the network under strong interaction are slightly perturbed eigenvectors of matrix W was confirmed in computer experiments.

2.3 The results obtained for low-dimensional ON

The system of two coupled oscillators was solved completely [20]. It was demonstrated that depending on parameters, two or four equilibrium points can exist, and only one of them can be stable. The structural portrait of two-oscillatory dynamical system is performed in [20]. The domains of synchronization, so-called "amplitude death" and multi-frequency oscillations are specified there.

The system of three coupled oscillators was studied in the case of zero ω_j and homogeneous connections, i.e., when the connection in one direction is a complex value $(a + ib)$ and in the opposite direction — its complex conjugate $(a - ib)$.

Evidently, only the points of "phase basis" can be the fixed points of such systems.

It was shown that the plane (a, b) is separated into six domains, in three of which only one stable fixed point exists, and in three others two fixed points are simultaneously stable. The curves separating the domains are the straight lines beyond the vicinity of zero.

The system of three coupled oscillators with homogeneous connections is the simplest closed chain of oscillators. The cases of closed chains with homogeneous connections for $N = 4, 5, 6$ were also studied analytically and similar situation was observed: the plane (a, b) is divided into domains containing from one to two stable points from the phase basis for $N = 4, 5$, and from one to three stable points for $N = 6$. Moreover, the curves separating the domains are also the straight lines beyond the vicinity of zero.

2.4 The remarks on homogeneous closed chains and network architecture

Computer modelling demonstrated that with increase in the number of oscillators the overlap of the domains of stability for a chosen fixed point becomes wider and, therefore, the maximum number of fixed points, which are simultaneously stable, grows rather fast. It looks very likely that in homogeneous closed chains only the points of phase basis can be stable. Neither extra phase points, nor amplitude memories (i.e., the points with different amplitudes r_j at different j) exist.

Analysis of the results obtained for homogeneous closed chains has shown that the properties of associative memory in the chains depend on the property of N to be a prime number or not. In some sense the memory in a chain of N oscillators is a combination of memories in the chains of oscillators of the orders that are the factors of N.

Associative memory properties for the system (1) strongly depend on the architecture of the network. For instance, the following hypothesis can be proposed: if the matrix W corresponds to "tree-like" connections, which means the absence of closed chains in the graphs of the network connections, then only one fixed point can be stable. It was rigorously proved for purely phase system corresponding to (1) and was confirmed in computer experiments for the system (1).

The value of homogeneous closed chains is wider than just to be a special type of ON. If N is a prime number and we combine matrix W from the vectors of phase basis according to (2), then it can be shown that W can be presented as a sum of matrices corresponding to homogeneous closed chains. Computer modelling shows that associative memory in the system (1) with the matrix W is closely related to the memory in the component chains. Probably, this is an adequate method to study the phase associative memory in general system (1): to represent the matrix W of oscillatory network with Hebbian connections as a sum of matrices of closed chains and tree-like unclosed subsets, and describe the memory in the whole network via the memory in the separate items.

3. HOMOGENEOUSLY CONNECTED DISCRETE OSCILLATORY SYSTEMS AND RELATED OSCILLATORY MEDIA

In the case of homogeneously locally connected oscillatory chains the matrix of interaction in (1) can be written as

$$W_{jk} = \begin{cases} d = \kappa e^{i\chi} = d_1 + id_2 & \text{if } k = j-1, j+1 \\ 0 & \text{if } k \neq j-1, j+1 \end{cases} \tag{3}$$

where $d = \kappa e^{i\chi}$ is the coupling strength in the chain. To reduce the dynamical system (1) to spatially continuous description, one should introduce a spatial variable $x \in [0, l] \subset R^1$ and a complex-valued function $u(x,t)$ instead of $u_j(t)$. Then the reaction-diffusion equation, representing spatially continual limit of dynamical description (1), can be easily derived:

$$u_t = (1 + i\omega(x) - |u|^2)u + d \cdot u_{xx}, \tag{4}$$

where $u(x,t) = u_1(x,t) + iu_2(x,t)$ and $u_{xx} \equiv \partial^2 u/\partial x^2$ is 1D Laplacian $\triangle u$ in spatially 1D case. Below we consider oscillatory media with $\omega(x) = \omega = const, x \in [0, l]$.

The equation (3) can be rewritten in terms of real-valued two-component vector-function $\mathbf{u} = (u_1, u_2)^\top$:

$$\mathbf{u}_t = \hat{F}(\mathbf{u})\mathbf{u} + \hat{D}\mathbf{u}_{xx}, \tag{5}$$

where

$$\hat{F}(\mathbf{u}) = \begin{bmatrix} 1 - u_1^2 - u_2^2 & -\omega \\ \omega & 1 - u_1^2 - u_2^2 \end{bmatrix} \quad \hat{D} = \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}. \tag{6}$$

Oscillatory media governed by reaction-diffusion equation (RDE) (4) represent a kind of Ginzburg-Landau oscillatory media [10,11,16-18].

4. SPATIO-TEMPORAL DISSIPATIVE STRUCTURES IN OSCILLATORY MEDIA

As one can easily obtain, RDE (4) possesses the following properties.

1. In the case $\omega(x) = \omega = const, x \in [0, l]$, the RDE (4) can be reduced to RDE with $\omega = 0$ for the function $w(x,t) = u(x,t)e^{-i\omega t}$. So, if $\omega(x) = const$, this is sufficient to analyse only the RDE with $\omega = 0$.

2. The function $u_0(x,t) = e^{i\theta_0}$ is the spatially homogeneous solution to RDE (4) at $\omega = 0$.

3. To analyse the properties of nonlinear RDE it is often quite helpful to use an expansion of its solutions into the series on orthonormalized system of eigenfunctions $\{X_m(x)\}$ of the corresponding linear scalar diffusion operator. For RDE (5) at $\omega = 0$ we put:

$$u_1(x,t) = \sum_{m=1}^{\infty} X_m(x) P_m(t), \quad u_2(x,t) = \sum_{m=1}^{\infty} X_m(x) Q_m(t). \tag{7}$$

For the medium corresponding to unclosed chain the boundary conditions for RDE are: $u_{1t}(0,t) = u_{2t}(l,t) = 0$. It gives $X_m(x) = \cos(\sigma_m x)$, $\sigma_m = \pi m / l$. The system of coupled ODE for $\{P_m(t), Q_m(t)\}$ has been obtained and used in the analysis of types of dissipative structures inherent to oscillatory media.

Now the behavior of some types of RDE solutions can be discussed.

4.1 Diffusion instability of spatially homogeneous solution

Spatially homogeneous solution $u_0(x,t) = e^{i\theta_0}$ can lose the stability for some parameters of the diffusion operator under some types of spatial structure of perturbations. This kind of instability inherent to nonlinear media is known as diffusion instability (because it is caused by the presence of diffusion). Elucidation of diffusion instability parametrical domain can be reduced to the analysis of RDE linearized around $u_0(x,t)$. Let us consider oscillatory medium corresponding to unclosed chain:

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}, \quad \mathbf{u}_0 = (1, 0)^\top,$$

and use the expansion (7) for the solution $\tilde{\mathbf{u}}$ of linearized RDE. Then we obtain the following second order ODE for $T_k(t) = (P_k(t), Q_k(t))^\top$, defining time behavior of k-th spatial harmonics:

$$\dot{T}_k = \hat{B}(\sigma_k) T_k, \quad \hat{B}(\sigma_k) = \begin{bmatrix} -(2 + d_1 \sigma_k^2) & d_2 \sigma_k^2 \\ -d_2 \sigma_k^2 & d_1 \sigma_k^2 \end{bmatrix} \tag{8}$$

The eigenvalues of $\hat{B}(\sigma_k)$, that can be easily calculated in the explicit form, provide the information on diffusion instability with respect to perturbation of the spatially homogeneous state by k-th spatial harmonics. In particular, the following result can be obtained: the diffusion instability with respect to perturbation of arbitrary spatial structure occurs in parametrical range $\chi \in [3\pi/4, \pi]$ of angles χ, defining oscillatory interaction according to (2).

4.2 Wave trains

Plane wave trains are RDE solutions of the form $u(x,t) = U(z)$, where $z = \omega t - kx$. Strict results on the existence of small amplitude wave train solution were obtained in [10]. These wave trains arise as a result of bifurcation from a uniform spatially homogeneous state. One-parametrical family of wave trains was shown to exist in the case of a special class of RDE systems — so called $(\lambda - \omega)$-systems. The RDE (5) belongs to the class of $(\lambda - \omega)$-systems in the case of diagonal diffusion operator, i.e., at real-valued interaction.

4.3 Target patterns, spiral waves, shock structures

In the case of 2D oscillatory media the well known target patterns and rotating spiral waves exist. Strict analysis of these structures is based on the theory of "slowly varying waves" [11], which — locally in space and time — are close to plane wave trains. This study demands the derivation and analysis of dispersion relations. Impringing wave trains (analogous to converging target patterns) and shock structures that accompany target patterns were also studied in detail [11].

4.4 Standing Waves. Cluster States

Modulated standing waves are special RDE solutions with separated variables x and t. In the case of oscillatory media related to unclosed chains these are the solutions of the form

$$u(x,t) = T_0 e^{-i\omega t} + T_k e^{-i(\omega t+\gamma)} \cos(kx) \tag{9}$$

The existence of standing waves for RDE (5) can be established by direct substitution of (9) into the RDE. In this way one can obtain four equations: two equations for T_0^2, T_k^2, the dispersion equation reflecting the relation between ω and k and the algebraic equation for $\tan(\gamma)$. Analysis of the algebraic equation shows the existence of real-valued solutions for $\tan(\gamma)$. Therefore, standing wave solutions to RDE (5) exist. The parametrical domain of their existence still remains to be evaluated.

Cluster states are RDE solutions with separated variables of another type: they correspond to medium decomposition into synchronously oscillating subdomains (clusters). The own amplitude, phase shift and frequency of oscillations are inherent to each cluster. Irregular oscillations of clusters are possible as well.

All the listed types of spatio-temporal patterns were confirmed experimentally in CO oxidation oscillating reaction on platinum crystal surface [15].

5. CONCLUSIVE REMARKS

I. The designed networks of associative memory demonstrate the advantages:

• the special class of the designed oscillatory networks have memory characteristics exceeding those of Hopfield neural networks (storage capacity up to $1/2N$, low "extraneous" memory);

• being interpreted as the networks consisting of complex-valued neurons, the designed phasor networks can be applied in the field of pattern recognition (patterns of complicated topological structure);

• the necessary property for design of amplitude memory also exists: the amplitudes in synchronized states are often splitted (in our studies we only observed this property, but did not analyse it);

• there exist promising perspectives in modelling of oscillatory networks in near-synchronization regime. These models would be capable both to relaxational and oscillatory dynamics, that is essential in modelling of some phenomena in biological neural networks (for instance, cortical oscillations);

• the designed oscillatory networks admit nonlinear optical implementations.

There are some shortcomings specific of the designed associative memory:

• only in the case of prime N the associative memory demonstrates good memory characteristics.

• the control over Hebbian learning in oscillatory networks is more complicated than in conventional neural network associative memories.

II. Models of cellular oscillatory networks belong to the class of cellular neural networks (CNN) that are of essential interest in natural sciences and AI. Some CNN models admit impressive applications ("bionic eye"). Therefore, the study of pattern formation in oscillatory media, representing spatially continuous limit for CNN, is of importance.

We have obtained several analytical results concerning spatio-temporal behavior of 1D oscillatory media.

The results of qualitative mathematical analysis of RDE governing the formation of spatio-temporal structures in 1D active oscillatory media are presented. The study of 1D media should be considered as an initial step to study of dissipative structures in 2D media. The ability of 2D nonlinear media to form a rich variety of spatio-temporal dissipative structures seems to be promising from the viewpoint of modelling of 2D locally connected networks of visual cortex. To attain this objective the model of oscillatory network consisting of limit-cycle oscillators with modifiable cycle radius and center location, governed by natural generalization of dynamical system (1), can be proposed.

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