Pattern Formation in Locally Connected Oscillatory Networks

Margarita Kuzmina

Keldysh Institute of Applied Mathematics, RAS
Moscow, Russia
kuzmina@spp.keldysh.ru

Irina Surina

RRC Kurchatov Institute
Moscow, Russia
surina@isssph.kiae.ru

Abstract

The subject of our study is a class of networks consisting of locally connected nonlinear oscillators. In spatially continual limit these oscillatory networks can be considered as oscillatory media governed by a system of reaction-diffusion equations. Formation of spatio-temporal patterns in nonlinear active media (wave trains, standing waves, targets and shock structures, spiral waves, stripe patterns, cluster states) is the subject of interest in physical, chemical, biological problems.

Here the results of analytical study of 1D oscillatory media corresponding to closed and unclosed chains of limit-cycle oscillators are presented. Diffusion instability (caused by coupling) has been analysed. The analysis is reduced to the problem of existence of growing solutions of the second order ODE system for arbitrary spatial harmonics. The conditions of standing waves existence has been clarified as well.

1 Introduction

Our previous studies on oscillatory systems were devoted to the networks of limit-cycle oscillators coupled via arbitrary Hermitian matrix of connections. Associative memory networks of Hopfield type were designed and their main characteristics were analysed [1].

In the present paper we study similar oscillatory networks from another viewpoint: as a model of active oscillatory medium. Depending on a local coupling template (defining the neighbors of each network unit) these networks can be considered as 1D, 2D or nD spatially distributed arrays of large number of processing units. Similarly to locally connected neural networks known as cellular neural networks (CNN) [2, 3], locally connected oscillatory networks may be naturally regarded as cellular oscillatory networks.

Cellular oscillatory networks were already used and can be used further for modelling of a variety of phenomena in physics, chemistry, biology, neurophysiology. In particular, 1D and 2D networks of locally connected Wilson-Cowan oscillators were successfully used for modelling of brain cortical oscillations. A single oscillator of the model network is formed by a couple of connected excitatory and inhibitory neurons [10].

Pattern formation in discrete cellular networks is closely related to formation of dissipative structures in the corresponding nonlinear media. Various models of nonlinear media governed by the systems of reaction-diffusion equations were studied since 70s. [4 - 8]. Belousov-Zhabotinskii oscillating chemical reaction in a thin layer of fluid and oscillatory media of Ginzburg-Landau oscillators belong to the most familiar examples of active media. A considerable scope of oscillatory media studies exists. The strict mathematical results [5 - 7], physical level results [8, 9] and computer modelling [3, 8] could be mentioned as examples.

Here we present qualitative mathematical analysis of 1D media representing spatially continual limit for closed and unclosed chains of limit-cycle oscillators of Ginzburg-Landau type. The significant feature of the governing reaction-diffusion system is that the diffusion operator is not diagonal and in some parametrical range is not positively defined.

2 Homogeneously Connected Oscillatory Chains and Related 1D Oscillatory Media

We consider oscillatory network model consisting of limit-cycle oscillators possessing two degrees of freedom. Limit cycle of a single oscillator is the circle of unit radius in the plane. Dynamical equations governing the dynamics of the network of N coupled oscillators are:

$$\dot{u}_j = (1 + i\omega_j - |u_j|^2)u_j + \sum_{k=1}^{N} W_{jk}(u_k - u_j), \qquad j = 1, ..., N. \tag{1}$$

Here the variable $u_j(t) = R_j(t)\exp(i\theta_j(t))$ defines the state of j-th oscillator (R_j and θ_j are the amplitude and the phase of oscillations, respectively), ω_j is its cycle frequency. The first term in the right-hand side of (1) defines the intrinsic dynamics of a free isolated oscillator, while the second one, responsible for interaction, is specified by the matrix of connections $W = [W_{jk}]$.

In the case of homogeneously locally connected oscillatory chains the matrix of interaction in (1) can be written as

$$W_{jk} = \begin{cases} d = \kappa e^{i\chi} = d_1 + id_2 & \text{if } k = j-1, j+1 \\ 0 & \text{if } k \neq j-1, j+1 \end{cases} \tag{2}$$

where $d = \kappa e^{i\chi}$ is the coupling strength in the chain. To transfer from dynamical system (1) to spatially continuous description, one should introduce a spatial variable $x \in [0, l] \subset R^1$ and a complex-valued function $u(x, t)$ instead of $u_j(t)$. Then the reaction-diffusion equation, representing spatially continual limit of dynamical description (1), can be easily derived:

$$u_t = (1 + i\omega(x) - |u|^2)u + d \cdot u_{xx}, \tag{3}$$

where $u(x,t) = u_1(x,t) + iu_2(x,t)$ and $u_{xx} \equiv \partial^2 u/\partial x^2$ is 1D Laplacian $\triangle u$ in spatially 1D case. Below we consider oscillatory media with $\omega(x) = \omega = const, x \in [0,l]$.

The equation (3) can be rewritten in terms of real-valued two-component vector-function $\mathbf{u} = (u_1, u_2)^\top$:

$$\mathbf{u} = \hat{F}(\mathbf{u})\mathbf{u} + \hat{D}\mathbf{u}_{xx}, \tag{4}$$

where

$$\hat{F}(\mathbf{u}) = \begin{bmatrix} 1 - u_1^2 - u_2^2 & -\omega \\ \omega & 1 - u_1^2 - u_2^2 \end{bmatrix} \quad \hat{D} = \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}. \tag{5}$$

Oscillatory media governed by reaction-diffusion equation (RDE) (3) represent a special case of Ginzburg-Landau oscillatory media [7, 8]. However, the diffusion operator is of more general type for RDE (3).

3 Diffusion Instability. Types of Spatio-Temporal Patterns

As one can easily obtain, RDE (3) possesses the following properties.

1. In the case $\omega(x) = \omega = const, x \in [0,l]$, the RDE (3) can be reduced to that one with $\omega = 0$ for the function $w(x,t) = u(x,t)e^{-i\omega t}$. So, if $\omega(x) = const$, it is sufficiently to analyse only the RDE with $\omega = 0$.

2. The function $u_0(x,t) = e^{i\theta_0}$ is the spatially homogeneous solution to RDE (3) at $\omega = 0$.

3. To analyse the properties of nonlinear RDE it is often quite helpful to use an expansion of its solutions into the series on orthonormalized system of eigenfunctions $\{X_m(x)\}$ of the corresponding linear scalar diffusion operator. For RDE (4) at $\omega = 0$ we put:

$$u_1(x,t) = \sum_{m=1}^{\infty} X_m(x)P_m(t), \quad u_2(x,t) = \sum_{m=1}^{\infty} X_m(x)Q_m(t). \tag{6}$$

For medium corresponding to unclosed chain the boundary conditions for RDE are: $u_{1t}(0,t) = u_{2t}(l,t) = 0$. It gives $X_m(x) = \cos(\sigma_m x)$, $\sigma_m = \pi m/l$. In the case the following system of coupled ODE for $\{P_m(t), Q_m(t)\}$ can be derived:

$$\dot{P}_0 = P_0 - 1/2 P_0 R_0 + \sum_{m=1}^{\infty} P_m R_m \tag{7}$$

$$\dot{Q}_0 = Q_0 - 1/2 Q_0 R_0 + \sum_{m=1}^{\infty} Q_m R_m \tag{8}$$

$$\dot{P}_k = P_k - \sigma_k^2(d_1P_k - d_2Q_k) - 1/2\sum_{k=1}^{m} P_{k-m}R_m - 1/2\sum_{m=1}^{\infty}(P_{k+m}R_m + P_mR_{k+m}) \quad (9)$$

$$\dot{Q}_k = Q_k - \sigma_k^2(d_2P_k + d_1Q_k) - 1/2\sum_{k=1}^{m} Q_{k-m}R_m - 1/2\sum_{m=1}^{\infty}(Q_{k+m}R_m + Q_mR_{k+m}), \quad (10)$$

where

$$R_0 = 1/2[P_0^2 + Q_0^2 + \sum_{m=1}^{\infty}(P_m^2 + Q_m^2)], \quad (11)$$

$$R_m = 1/2\sum_{l=1}^{m}(P_mP_{m-l} + Q_mQ_{m-l}) + \sum_{l=1}^{\infty}(P_mP_{m+l} + Q_mQ_{m+l}). \quad (12)$$

The "moment" system (7)-(12) is in complete agreement with the analogous system derived in [7] for the case of oscillatory medium of Ginzburg - Landau oscillators with real-valued interaction.

Now the behavior of some types of RDE solutions can be discussed.

3.1 Diffusion instability of spatially homogeneous solution

Spatially homogeneous solution $u_0(x,t) = e^{i\theta_0}$ can lose the stability for some parameters of diffusion operator under some types of spatial structure of perturbations. This kind of instability inherent to nonlinear media is known as diffusion instability (because it is caused by the presence of diffusion). Elucidation of diffusion instability parametrical domain can be reduced to the analysis of RDE linearized around $u_0(x,t)$. Let us consider oscillatory medium corresponding to unclosed chain, put

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}, \quad \mathbf{u}_0 = (1,0)^\top,$$

and use the expansion (6) for the solution $\tilde{\mathbf{u}}$ of linearized RDE. Then we obtain the following second order ODE for $T_k(t) = (P_k(t), Q_k(t))^\top$, defining time behavior of k-th spatial harmonics:

$$\dot{T}_k = \hat{B}(\sigma_k)T_k, \quad \hat{B}(\sigma_k) = \begin{bmatrix} -(2 + d_1\sigma_k^2) & d_2\sigma_k^2 \\ -d_2\sigma_k^2 & d_1\sigma_k^2 \end{bmatrix} \quad (13)$$

The eigenvalues of $\hat{B}(\sigma_k)$, that can be easily calculated in the explicit form, provide the information on diffusion instability with respect to perturbation of the spatially homogeneous state by k-th spatial harmonics. In particular, the following result can be obtained: the diffusion instability with respect to perturbation of arbitrary spatial structure occurs in parametrical range $\chi \in [3\pi/4, \pi]$ of angles χ, defining oscillatory interaction accordingly to (2).

3.2 Wave trains

Plane wave trains are RDE solutions of the form $u(x,t) = U(z)$, where $z = \omega t - kx$. Strict results on small amplitude wave train solution existence were obtained in [5]. These wave trains arise as a result of bifurcation from a uniform spatially homogeneous state. One-parametrical family of wave trains was shown to exist in the case of a special class of RDE systems — so called $(\lambda - \omega)$-systems. The RDE (4) belongs to the class of $(\lambda - \omega)$-systems in the case of diagonal diffusion operator, i.e., at real-valued interaction.

3.3 Target patterns, spiral waves, shock structures

In the case of 2D oscillatory media the well known target patterns and rotating spiral waves exist. Strict analysis of these structures is based on the theory of "slowly varying waves" [6], which — locally in space and time — are close to plane wave trains. This study demands the deriving and analysis of dispersion relations. Impringing wave trains (analogous to converging target patterns) and shock structures that accompany target patterns were also studied in detail [6].

3.4 Standing Waves. Cluster States

Modulated standing waves are special RDE solutions with separated variables x and t. In the case of oscillatory media related to unclosed chains these are the solutions of the form

$$u(x,t) = T_0 e^{-i\omega t} + T_k e^{-i(\omega t + \gamma)} \cos(kx) \qquad (14)$$

The existence of standing waves for RDE (4) can be established either with the help of moment system (7)-(12) or by direct substitution of (14) into the RDE. In this way one can obtain four equations: two equations for T_0^2, T_k^2, the dispersion equation reflecting the relation between ω and k and the algebraic equation for $\tan(\gamma)$. Analysis of the algebraic equation shows the existence of real-valued solutions for $\tan(\gamma)$. Therefore, standing wave solutions to RDE (4) exist. The parametrical domain of their existence still remains to be revealed.

Cluster states are RDE solutions with separated variables of another type: they correspond to medium decomposition into synchronously oscillating subdomains (clusters). The own amplitude, phase shift and frequency of oscillations are inherent to each cluster. Irregular oscillations of clusters are possible as well.

All the listed types of spatio-temporal patterns were confirmed experimentally in CO oxidation oscillating reaction on platinum crystal surface [9].

Conclusive Remarks

The results of qualitative mathematical analysis of RDE governing the formation of spatio-temporal structures in 1D active oscillatory media are presented. The study of 1D media should be considered as an initial step of study of dissipative structures in 2D media. The ability of 2D nonlinear media to form a rich variety of spatio-temporal patterns seems to be promising from the viewpoint of modelling of 2D locally connected networks of visual cortex. To attain this objective the model of oscillatory network consisting of limit-cycle oscillators with modifiable cycle radius and center location, governed by natural generalization of dynamical system (1), can be proposed.

Acknowledgment

This study was partially funded by Russian FFR, grant n. 96-01-00084.

References

[1] Kuzmina M. G., Manykin E. A., Surina I. I.: Recurrent associative memory network of nonlinear coupled oscillators. In: Artificial Neural Networks (ICANN'97), Springer-Verlag, Berlin, 1997, pp.433-238 (Lecture notes in computer science, no.1327)

[2] Chua L.O., Roska T. Stability of class of nonreciprocal cellular neural networks. IEEE Trans. CS, 1990; 37:1520-1527

[3] Thiran P., Crounse K.R., Chua L.O., Hasler M. Pattern formation properties of autonomous cellular neural networks. IEEE Trans. CS, 1995; 42:757-774.

[4] Kuramoto Y., Tsuzuki T. On the formation of dissipative structures in reaction-diffusion systems. Progr. of Theor Phys., 1975; 54, no.3

[5] Howard L.N., Kopell N. Wave trains, shock fronts and transition layers in reaction-diffusion equations. SIAM-AMS Proc., 1974; 8:1-12.

[6] Howard L.N., Kopell N. Slowly varying waves and shock structures in reaction-diffusion equations. Studies in Appl.Math., 1977; 56:95-145.

[7] Ahromeeva T.S., Malinetskii G.G. New properties of nonlinear dissipative systems. KIAM Preprint, 1983, no.118 (in Russian)

[8] Falcke M., Engel H., Neufeld M. Cluster formation, standing waves and stripe patterns in oscillatory active media with local and global coupling. Phys. Rev. E, 1995; 52:763-771.

[9] Jakubith S., Rotermund H.H., Engel W., von Oertzen A., Ertl G. Spatiotemporal concentration patterns in a surface reaction: propagating and standing waves, rotating spirals, and turbulence. Phys. Rev. Lett., 1990; 65:3013-3016.

[10] Wang D.L., Terman D. Locally Excitatory Globally Inhibitory Oscillator Networks. Transactions on Neural Networks, 1995; 6:283-286