Phase Memory in Oscillatory Networks

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As it was shown earlier, oscillatory networks consisting of limit-cycle oscillators interacting via complex-valued connections can be used for associative memory design. Phase memory as a special type of associative memory in oscillatory networks has been invented and studied. Detailed analysis of phase memory features of phasor networks related to oscillatory networks has been performed. It has been found that under special choice of parameters the oscillatory networks possess high memory storage capacity and low extraneous memory.

The designed networks can be interpreted as networks consisting of complex-valued neurons. They could be useful in problems of invariant pattern recognition and in recognition of colored patterns.

1. Introduction

Dynamical system governing the dynamics of nonlinear coupled limit cycle oscillators [1-3] is studied here from the viewpoint of associative memory modelling.

Similar dynamical systems and their limit cases ("phase" systems) were studied in different aspects for a long time, refs in [1-3] . The main interest was concentrated around phase transition into the state of synchronization.

As it is known, in the problem of associative memory network design the following subproblems arise:

- development of an algorithm for calculation of the whole set of stable equilibria of the dynamical system at given W (network memory);
- development of an algorithm of imposing a prescribed set of stable equilibria with natural, large enough basins of attraction;

- calculation of maximum number M of equilibria at a given finite number N of processing elements in a network;
- estimation of "loading ratio" $\alpha = M/N$ in the limit $N \to \infty$, $M \to \infty$, $M/N < \infty$ (memory storage capacity);
- study of "extraneous" memory (additional equilibria arising in a network together with the desired).

In addition, a proper learning algorithm should be developed.

As it was shown [1-3], the oscillatory networks of associative memory with Hebbian matrix of connections (W is taken in the form of proper outer product on memory vectors) can be designed. A number of questions from the listed subproblems have been elucidated.

The designed oscillatory networks can be implemented in semiconductor laser systems [4], since the dynamics in these systems is governed by the same equations as in oscillatory networks. At the present time such implementation is under development.

2. Oscillatory Networks and Phase Memories

Let us recall the dynamical system of N coupled limit cycle oscillators [1-3]:

$$\dot{z}_j = (1 + i\omega_j - |z_j|^2)z_j + \kappa \sum_{k=1}^{N} W_{jk}(z_k - z_j), \qquad j = 1, ..., N. \tag{1}$$

Here $z(t)$ is a complex-valued N-dimensional vector representing the states of oscillators as functions of independent variable t, $z_j(t) = r_j exp(i\theta_j) = x_j + iy_j$. Each oscillator has a natural frequency ω_j. Complex-valued $N \times N$ Hermitian matrix $W = [W_{jk}]$ specifies the weights of connections. Non-negative parameter κ defines the absolute value of interaction strength in oscillatory system. Matrix W satisfies the following natural restrictions:

$$W = W^+, \qquad |W_{jk}| \leq 1, \qquad \sum_{k=1}^{N} |W_{jk}| = 1. \tag{2}$$

Matrix W is constant in the phase space C^N.

The system (1) can be rewritten in matrix form:

$$\dot{z} = (A - D_z)z, \tag{3}$$

$A = D_0 + \kappa W$. The diagonal matrix $D_0 = diag(D_{01}, \ldots, D_{0N})$,

$$D_{0j} = 1 + i \cdot \omega_j - \kappa \eta_j, \quad \eta_j = \sum_{k=1}^{N} W_{jk},$$

is constant. In contrast, the diagonal matrix $D_z = diag(|z_1|^2, \ldots, |z_N|^2)$ depends on absolute values of z_j.

In Cartesian coordinates the matrix A can be rewritten as follows:

$$A = \begin{pmatrix} g_1 + ih_1 & b_{12} + ic_{11} & \ldots & b_{1N} + ic_{1N} \\ b_{12} - ic_{11} & g_2 + ih_2 & \ldots & b_{2N} + ic_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1N} - ic_{1N} & b_{2N} - ic_{2N} & \ldots & g_N + ih_N \end{pmatrix} \tag{4}$$

Here $b_{jk} + ic_{jk}$ denote the weights κW_{jk}, $g_j = 1 - \kappa Re(\eta_j)$, $h_j = \omega_j - \kappa Im(\eta_j)$.

Dynamical system (1-3) demonstrates a great variety of complicated dynamical regimes at different values of its parameters ω_j, κ, W. Among them the regime of mutual synchronization at some threshold value κ^* of interaction strength exists. Parametric domain for synchronization regime can be roughly specified as $\kappa > \Omega$, where $\Omega = max_j|\omega_j|$. The dynamical regime is simple in the domain of synchronization: if $\sum_j \omega_j = 0$, this is relaxation to stable equilibria. This condition for frequencies can be always satisfied by proper rescaling of (1-3), and below we assume it satisfied.

As it was found out ([1-3] and refs.), the system (1), being imposed into parametric domain of synchronization, can possess sufficiently rich set of stable equilibria. So, the problem of recurrent associative memory design was posed. Let us remind the problem.

Given M arbitrary points $V^1, \ldots, V^M$ in the phase space C^N of the dynamical system, it is necessary to point out the parameters (ω_j, κ, and matrix W) to provide the following properties:

1) the points $V^1, \ldots, V^M$ are stable fixed points of dynamics;

2) basin of attraction for each V^k is as large as possible;

3) total number of extraneous stable fixed points (other than $V^1, \ldots,$ V^M) is as small as possible.

The set $V^1, \ldots, V^M$ is just the network memory.

Obviously, if we fix amplitudes r_j of oscillators, then the corresponding equilibrium points of eq.(3) are defined by the linear system with constant coefficients:

$$(A - D_z)z = 0.$$

Hence, the determinant of the matrix $(A - D_z)$ must be zero to provide at least one non-zero equilibrium point.

Jacobian of the system (3) in a point z can be written as follows:

$$J = \begin{pmatrix} G_1 & BC_{12} & \dots & BC_{1N} \\ BC_{12}^T & G_2 & \dots & BC_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ BC_{1N}^T & BC_{2N}^T & \dots & G_N \end{pmatrix} \tag{5}$$

Here G_j is the matrix of order 2:

$$G_j = \begin{pmatrix} g_j - 3x_j^2 - y_j^2 & -h_j - 2x_j y_j \\ h_j - 2x_j y_j & g_j - x_j^2 - 3y_j^2 \end{pmatrix},$$

and BC_{jk} is also the matrix of order 2:

$$BC_{jk} = \begin{pmatrix} b_j & -c_j \\ c_j & b_j \end{pmatrix},$$

BC_{jk}^T denotes the transposed matrix.

An equilibrium point is stable iff one eigenvalue of J is zero (due to invariance of solutions relative to multiplication with $e^{i\varphi}$, where φ is any constant angle), and all others are negative.

We call the stable points that have constant amplitudes $r_j \equiv const.$ *phase memories*. By convention, we call the stable points with non-constant amplitudes *amplitude memories*. In general, both phase and amplitude stable points exist in the system (3).

Phase memories have peculiar properties as associative memory and can be studied exhaustively. First of all, let us note that if some phase vector is an eigenvector of the matrix A with a positive eigenvalue, then, taking this vector with an appropriate normalizing factor, we obtain an equilibrium point of the system (3). Conversely, a phase equilibrium point is an eigenvector of the matrix A. Thus, the one-to-one correspondence exists between the eigenvectors and equilibrium points of the system (3). Consequently, we can design a matrix A with a prescribed set of eigenvectors so that all extraneous memories will be amplitude ones. This property looks very promising, because this is probably that a hardware method might discriminate amplitude from phase memories.

Now, we describe the method permitting to load up to $N-1$ phase memories. Aiming at this goal, let us introduce the “phase basis”.

If we take N phases $(0, \beta_2^1, \dots, \beta_N^1)$ with arbitrary $\beta_2^1, \dots, \beta_N^1$ and calculate recurrently:

$$\begin{bmatrix} 0 \\ \beta_2^m \\ \beta_3^m \\ . \\ . \\ . \\ \beta_N^m \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \beta_2^m + \varphi \\ \beta_3^m + 2\varphi \\ . \\ . \\ . \\ \beta_N^m + (N-1)\varphi \end{bmatrix} \qquad (6)$$

$\varphi = 2\pi/N$, then we obtain N linearly independent orthogonal phase vectors $V_j^m, m = 1, \dots, N, j = 1, \dots, N$ in C^N, if N is a prime number.

Any subset of these vectors can be loaded as phase memories.

Noteworthy is that extraneous memories (that can be easily revealed as memories with different amplitudes) are more or less abundant for one or another combinations of M vectors from the phase basis for the same values of M. Examples with $N \leq 20$ can be shown where M is close to $M/2$ and extraneous memories do not appear in computer simulation of the retrieval process (this means that if extraneous memories exist in these cases, then they have very small basins of attraction).

3. Phase Memories in Oscillatory Networks of Low Dimensions

1. $N = 2$. In this case the strict analytical analysis of dynamical system (3) with arbitrary parameters has been fulfilled and exact solution of the system has been found.

Only one stable point can exist in the network of two coupled oscillators. Its polar coordinates: $(r_1, r_2 e^{\theta_2})$, $r_1 = r_2 = \sqrt{g + \sqrt{d - h^2}}$, $\theta_2 = -iln\left(\frac{b-ic}{ih+\sqrt{d-h^2}}\right)$. Here $g = 1 - b_{12}$, $h = \omega - c_{12}$. So, in this case, only phase memory exists.This point exists and is stable iff

$$d > h^2, \quad AND \quad |g-1| \leq \sqrt{d} \quad AND \quad (g > 0 \quad OR \quad g^2 + h^2 < d).$$

Conversely, for an arbitrary point $(r_1, r_2 e^{\theta_2})$ the parameters providing its stability can be presented.

2. $N = 3$. In this case amplitude stable points exist. One arbitrary phase vector can be loaded and two orthogonal phase vectors can be loaded as well. The values g_j in (4) have to be real and equal, if phase vectors are loaded. Consequently, b_{jk} are equal as well. This is not

true for $N \geq 4$, but in any case the matrices W providing storage of a prescribed set of the vectors from the phase basis (8) are quite special.

4. Phasor Networks Related to Oscillatory Networks

Analysis of dynamical system (1) with arbitrary frequencies ω_j and arbitrary matrix W represents a complicated mathematical problem. Only a few number of rigorous results was obtained for the system (1) and for its limit case — so-called phase model. Those results concern mainly the case of the special architecture of connections — homogeneous all-to-all connections ($W_{jk} = N^{-1}(1 - \delta_{jk})$).

Eq. (1) with $\omega_j \equiv 0$ represents an important special case of oscillatory system which can be regarded as phasor networks. These phasor networks can be viewed as natural generalization of the known "clock" neural networks.

The equilibria of oscillatory networks and corresponding phasor networks proved to be closely related. Namely, the following proposition is valid.

> Let $\mathcal{N}(\{\omega_j\}, \kappa, W)$ be an oscillatory network with arbitrary frequencies ω_j satisfying the condition $\sum_j \omega_j = 0$.
> Let $\mathcal{N}(\{0\}, \kappa, W)$ be the corresponding phasor network possessing the collection of M phase memory vectors $\{U^1, \ldots U^M\}$.
> Define $\tilde{\kappa} > \kappa$ satisfying the condition: $\gamma \equiv \Omega/\tilde{\kappa} \ll 1$, where $\Omega = max_j|\omega_j|$.
> Then oscillatory network $\mathcal{N}(\{\omega_j\}, \tilde{\kappa}, W)$ has phase memory vectors $\tilde{U}^1, \ldots, \tilde{U}^M$, which represent slight perturbations of the corresponding $U^1, \ldots, U^M$.

The proof of this proposition has been obtained using the perturbation method on small parameter γ. This proposition is also confirmed by computer studies of phase portraits of the dynamical system (1) for small N.

5. The Class of Phasor Networks with Guaranteed Memory Characteristics.

Phasor networks governed by dynamical system (1) at $\omega_j = 0$ can be considered as basic ones among all oscillatory networks of given architecture (defined by the same matrix W). Their memory has the most symmetrical structure preserving at the same time all the features inherent to oscillatory network memory.

As it was shown [2], special class of phasor networks with Hebbian matrix of connections W^H, possessing the guaranteed memory of high storage capacity, can be designed. The construction of W^H is based on the important property of interaction term in eq. (1). As one can see, the interaction between two oscillators of the network has the form $W_{jk}(z_j - z_k)$. This means that the matrices with nonzero diagonal are admissible for specification of network connections (unlike to the case of neural networks). This permits to use the matrices of projection operators in construction of Hebbian-like matrices of connection.

The most essential step in architecture design is introduction of a special set of orthogonal vectors in N-dimensional complex space C^N — "phase" basis (6):

$$\mathcal{B}_{\mathcal{N}} = \{ \ V^m \ \mid \ (V^s)^+ V^m = N\delta_{sm} \ \ m, s = 1, \ldots, N.\}$$

The phase basis is defined by single generating vector $V^0 = (1, ...1)^\top$ and the single parameter $\varphi = 2\pi/N$. All other vectors are of $\mathcal{B}_{\mathcal{N}}$ can be calculated with the help of recurrent transformation or, the same, by multiple action on vector V^0 of irreducible group representation operator

$$T_g = diag(1, exp(i\varphi), \ldots, exp(i(N-1)\varphi).$$

The basis $\mathcal{B}_{\mathcal{N}}$ is an eigenbasis of any weight Hermitian $N \times N$ matrix W satisfying the conditions (2). Therefore, any W can be represented in the form

$$W = N^{-1} \sum_{m=1}^{N} \lambda^m V^m (V^m)^+,$$

where λ^m, $m = 1, \ldots N$, are real numbers, V^m is column-vector $(V_1^m, ...V_N^m)^\top$ and $(V^m)^+$ is the corresponding conjugated row-vector: $(V^m)^+ = (\bar{V}_1^m, ...,$ $\bar{V}_N^m)$. For zero-diagonal W, obviously, $\sum_{m=1}^{N} \lambda^m = 0$.

The matrix W^H of rank M,

$$W^H = \sum\nolimits_{m=1}^{M} V^m (V^m)^+, \qquad M = rankW, \tag{7}$$

is the matrix of the projection operator into M-dimensional subspace of C^N spanned on $V^1, \ldots, V^M$.

Note, that both the basis $\mathcal{B}_{\mathcal{N}}$ and the matrices W^H are cyclical.

The following results are valid for phasor networks with matrices of connections W^H.

1. Let N to be a prime number.

Define basis $\mathcal{B}_{\mathcal{N}}$ and choose any subset of $M \leq N$ vectors from this basis $\{V^1, \ldots V^M\}$. Construct W^H in accordance with (7).
Then phasor network has memory vectors

$$U^1, \ldots U^M, \quad U^m = cV^m,$$

where $c = 1$ if $V^0 \in \{V^1, \ldots, V^M\}$ and $c = (1+\kappa)^{1/2}$ if $V^0 \notin \{V^1, \ldots, V^M\}$.
All memory vectors $U_1, \ldots U^M$ have equal basins of attraction.

The sizes of the basins can be controlled if weighted Hebbian matrix

$$\tilde{W}^H = \sum_{m=1}^{M} \lambda^m V^m (V^m)^+$$

is used. The values of λ^m should be slightly different and all close to unit.

It should be noted that all matrices W^H are irreducible if N is prime.

2. Let the number of oscillators N to be not prime.

The main feature of the network memory in this case is that the memory is not completely controllable unlike to the previous case. Namely, only special odd numbers M of vectors from the basis $\mathcal{B}_{\mathcal{N}}$ can be imposed into network memory. If M is different from mentioned special numbers, recalling process is impossible at all: the dynamical system (1) demonstrates continual set of degenerated equilibria.

The matrices W^H are reducible in this case. Under the interaction specified by these matrices the phasor system is decomposed into non-interacting subsystems.

Conclusions

The special type of oscillatory associative memory have been designed. The class of oscillatory and corresponding phasor networks of high performance can be pointed out. It is characterized by fully controllable memory of high storage capacity: up to $N-1$ memory vectors from some specific set ("phase" basis) can be loaded into the memory of the network consisting of N processing units. The weight matrix is designed in Hebbian form generalized to complex-valued connections.

Extraneous memory exists, but it can be easily discriminated due to its non-phase character.

Oscillatory networks are promising from many viewpoints, in particular, in view of possibility of physical (optical) implementations.

Acknowledgments

This work is partially funded by Russian Foundation for Fundamental Researches, grant n. 96-01-00084.

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