OSCILLATORY NETWORKS WITH GUARANTEED MEMORY CHARACTERISTICS

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The paper continues our investigation of associative memory structure in recurrent oscillatory networks with Hebbian matrix of connections. The suggested new approach is aimed at elucidation of algebraic and geometric properties of the memory. The main result is that oscillatory networks with guaranteed memory characteristics can be designed only for prime numbers of oscillators N. Extraneous memory exists in the networks, as a rule. Cyclical group of order N related to oscillatory system acts on extraneous memories. Examples of discrete and continuous sets of stable equilibria of network dynamics have been presented.

As it has been shown in [1], the systems of synchronized coupled oscillators admit the design of associative memory networks with Hebbian matrix of connections. Here the properties of associative memory of oscillatory networks and also of related phasor networks have been studied in more detail. The methods of dynamical system theory combined with algebraic analysis have been used for the further treatment of the associative memory structure.

Relations exist between the considered oscillatory networks and discrete Hopfield-like models, and also other continuous-time models (refs. in [1]). So, the results obtained earlier in that fields can be interpreted from the new point of view. Besides, the oscillatory networks are attractive objects from the following viewpoints:

- biological modelling: studies of well-known synchronization phenomena,
- mathematics: as nonlinear dynamical systems with compact phase space that are close to linear,
- physics: for understanding and interpretation of phenomena in laser systems, charge-density waves, Josephson junctions etc.

Phasor networks [1] represent the most symmetrical type of the oscillatory networks and can be interpreted as magnetic systems. They are close to well known "clock" neural networks. The results obtained on memory structure are exact for the phasor networks.

The associative memory model based on systems of coupled oscillators is worthy of physical implementation. For instance, an implementation in semiconductor laser systems with optical feedback is possible.

To conclude introductory notes, it should be mentioned that surely the idea suggested in [5] about utilization of extraneous memory could be realized in the oscillatory systems, for the structure of the extraneous memory becomes transparent in these systems.

1. OSCILLATORY NETWORKS AND RELATED PHASOR NETWORKS

Recurrent associative memory networks we are dealing with are designed basing on coupled limit cycle oscillators in synchronization regimes. The dynamical system governing oscillatory network dynamics, derived in [1], can be represented in the form:

$$\dot{z} = (D_0 - D_z + \kappa W)z \tag{1}$$

Here $z(t)$ is a complex-valued N-dimensional vector representing the states of oscillators as functions of independent variable t, $z_j = r_j exp(i\theta_j)$. Complex-valued $N \times N$ matrix $W = [W_{jk}]$ specifies the weights of connections between pairs of oscillators in the network. In our present studies we are assuming it to be Hermitian. (In development of physical implementations of the networks, matrices of connections *close* to Hermitian should be studied.)

Nonnegative parameter κ defines the absolute value of interaction strength in oscillatory system. Matrix W satisfies the natural restrictions:

$$W = W^+, \quad |W_{jk}| \leq 1, \quad \sum_{k=1}^{N} |W_{jk}| = 1. \tag{2}$$

Matrix W is constant in the phase space C^N. Also the diagonal matrix $D_0 = diag(D_{01}, \ldots, D_{0N})$,

$$D_{0j} = 1 + i \cdot \omega_j - \kappa\eta_j, \quad \eta_j = \sum_{k=1}^{N} W_{jk},$$

is constant. Conversely, the diagonal matrix $D_z = diag(|z_1|^2, \ldots, |z_N|^2)$ depends on absolute values of z_j.

Thus, the equilibrium points of eq.(1) are defined by the linear system with constant coefficients:

$$(D_0 - D_z + \kappa W)z = 0,$$

if one considers fixed absolute values of z. Stability of the points is defined by eigenvalues of Jacobian:

$$J(z) = D_0 - 2D_z + \kappa W \tag{3}$$

Consequently, all equilibrium points with constant absolute values of z are simultaneously stable or unstable, for their Jacobian is the same.

Therefore, the associative memory design is reduced to purely algebraic problem. For a given set of vectors $V^1, \ldots, V^M$, it is necessary to point out a matrix $D_0 + \kappa W$, possessing the following property: the vectors belong to zero linear subspace of the matrix $D_0 - D_z + \kappa W$, and after subtraction of D_z from this matrix, its spectrum moves to the left in such a way that its real parts become negative. Geometrically, to find the equilibrium points, it is necessary to determine an intersection of torus T^N, defined by the absolute values of vectors V^m, with the zero linear subspace. Usually, in addition to the desired vectors V^m one obtains extra stable points of eq.(1): extraneous, or so-called spurious memories.

It is worth noting that the eigenvalues of $D_0 - D_z + \kappa W$ can satisfy the resonance conditions [3], chapter 5. But, as it turned out, this fact does not complicate our problem, because stability in linear approximation occurs.

As it has been shown [1], under the condition $\omega_j = 0$, the dynamical system (1) corresponds to phasor network, which, in some sense, can be considered as basic for oscillatory networks due to its maximum memory capacity. Besides, the phasor network governed by eq.(1) can be viewed as natural generalization of the well known clock neural networks [2]. Indeed, phasor networks admit mechanico-magnetic interpretation (as systems of heavy magnetic spins in external gravitational field with purely magnetic interaction). This interpretation has proved to be helpful for estimation of total number of equilibria in the systems.

2. THE CLASS OF NETWORKS WITH GUARANTEED MEMORY CHARACTERISTICS

As it has been shown [1], the memory vectors, which can be imposed into oscillatory network with Hebbian matrix W, are not arbitrary. They must be chosen from a special set of orthogonal vectors in complex space C^N, which may be called the "phase" basis:

$$\mathcal{B} = \{ \; V^m \; | \; (V^s)^+ V^m = N\delta_{sm} \;\; m, s = 1, \ldots, N. \}$$

The phase basis is defined by one generating vector V^1. All other vectors are produced from it with the use of recurrent transformation. The basis $\mathcal{B}$ is the eigenbasis of the weight Hermitian matrix. That is, any W satisfying the conditions (2) can be represented in the form

$$W = N^{-1} \sum_{m=1}^{N} \lambda^m V^m (V^m)^+ .$$

The matrix W^H of rank M,

$$W^H = \sum\nolimits_{m=1}^{M} V^m (V^m)^+, \quad M = rank W, \tag{4}$$

is the matrix of the projection operator into M-dimensional subspace of C^N spanned on $V^1, \ldots, V^M$.

All the cases are reduced to the vector basis with unit generating vector $V^1 = (1, \ldots, 1)^\top$, because the arbitrary "phase" matrices (2) can be transformed into that one using an reversible matrix. Below only this kind of basis will be kept in mind. An essential property of eq. (1) is that it admits the cyclic group of symmetry - N-polygonal pyramid group. By this reason the basis $\mathcal{B}$ and the matrices W^H are cyclical. Two essentially different cases exist.

1. The number of oscillators N is prime.

In this case the basis $\mathcal{B}$, consisting of N vectors, is unique to numbering. All its vectors can be obtained as the results of multiple acting of irreducible group representation operator

$$T_g = diag(1, exp(i\phi), \ldots, exp(i(N-1)\phi), \;\; , \phi = 2\pi/N$$

on the vector V^1.

The following results concerning the memory features of oscillatory and phasor network with Hebbian matrix of connections (3) are valid.

- If $M < N/2$, then vectors $V^1, \ldots, V^M$ from $\mathcal{B}$ are memory vectors with an accuracy of constant coefficients, $c^1V^1, \ldots, c^MV^M$, in the phasor network ($\omega_j = 0$), with the matrix $W = W^H$ defined by eq.(4). The coefficients c^m can be easily calculated.
- In a network with an arbitrary frequency distribution it is necessary to choose κ to provide the smallness of parameter $\gamma = \Omega/\kappa$, where $\Omega \equiv max_j|\omega_j|$. If this is true, then a similar set of vectors $V^1, \ldots, V^M$ from $\mathcal{B}$ defines perturbed M vectors with perturbed coefficients c^m as memorized in oscillatory network with the matrix of connections W^H.

It is worth noting that matrices W^H are always irreducible if N is prime.

2. The number of oscillators N is not prime.

In this case the basis $\mathcal{B}$ is not unique, the group representation is reducible, the matrices W^H are reduced into matrices of orders equal to divisors of N. As a rule, the dynamical system (1) possesses continual set of stable equilibria. Physical sense of the phenomenon: spin system splits into the set of spin pairs, in stable equilibrium state each, all the pairs are in neutral equilibrium.

3. EXTRANEOUS MEMORY

The important problem is the analysis of extraneous memory in the networks - an additional set of stable equilibria, which arises in the phase space simultaneously with the memory vectors. Basins of attraction of memory vectors decrease due to existence of extraneous memory. Therefore the network performance decreases as well. Under some conditions, in Hopfield neural networks the number of extraneous memories can grow exponentially as a function of M (the number of imposed memory vectors), [4].

1. The situation is the most transparent for prime numbers N. The rigorous result is that together with every extraneous vector $N-1$ other, different from this one, are present in the memory. This fact is the direct consequence of cyclicity of W.

There is some evidence that memory capacity is $[N/2]/N$. If M is close to $[N/2]$, then extraneous memory exists as a rule. Confirming computer experiments have been executed.

For every prime number N a value of M^* exists and also M^* vectors from $\mathcal{B}$, defining the network without any extraneous vectors. Computer experiments show that this value M^* can be rather large. An interesting moment is that this property essentially depends on specific choice of the vectors. Some subsets of M^* vectors of $\mathcal{B}$ can deliver networks with extraneous memory, whereas other ones - without it. The values M^* have been estimated for prime numbers $N \leq 29$: for $N = 5$ and $N = 7$ $M^* \geq 3$, for $N = 11$ and $N = 17$ $M^* \geq 4$, for $N = 19, N = 23$ and $N = 29$ $M^* \geq 5$. It should be emphasized that these values are only lower bounds of M^*. An examples of 7 extraneous vectors for $N = 7$, $M = 4$ can be shown. The extraneous vectors form a cycle of length N, they can be produced from one of them by rotations in C^N. Similarly, for $N = 19$, $M = 5$ an example with 38 extraneous vectors consisting of two cycles of length 19 can be presented. Inside one cycle, all vectors can be produced from one, using cyclical permutation of its coordinates.

Imposed memory vectors have the same amplitudes c^m. Unlikely, the absolute values of the components for extraneous vectors can be different. It means that the corresponding points in C^N belong to tori with different radii. This behavior is observed for $N \geq 17$. Beginning from

$N \geq 17$, the coordinates of extraneous vectors usually have different absolute values.

2. For composite numbers N the system (1) has sets of degenerated equilibria. Examples of networks with continuous sets of stable equilibria can be easily constructed. The whole set of stable points consists of linear subvarieties combined with discrete points.

SUMMARY

The following results of the work are the most essential:

- The class of oscillatory and phasor networks with Hebbian matrix of connections and guaranteed memory characteristics is pointed out - these are the networks of N oscillators, where N is a prime number. In these networks M memory vectors, $M < N/2$, can be stored. The networks with composite numbers have extremely poor retrieval characteristics.
- The characteristic feature of the networks with prime N is that an existence of extraneous memory depends not only on the number of stored vectors M, but also on the choice of the vectors themselves. For rather large M, the set of vectors defining memory without extraneous memories can be found, as computer simulations has confirmed.
- The structure of extraneous memory of the networks has been clarified using the group analysis. The approach permits an exhaustive study of total set of stable equilibria for arbitrary Hermitian matrix of connections.
- From the viewpoint of associative memory design, the considered oscillatory systems have still unrealized potentialities. For instance, "amplitude-phase" memory could be designed.

Discrete models of neural networks consisting of multi-state neurons can be produced from the considered phasor networks. The models can be viewed as the improved variant of the networks, equivalent to systems of Potts magnetic spins. The models seem to be promising, for instance, in some problems of object classification.

References.

1. M. G. Kuzmina, E. A. Manykin, I. I. Surina, Associative memory oscillatory networks with Hebbian and pseudo-inverse matrices of connections, - *EUFIT'95* pp.392-395.
2. J. Cook, The mean-field theory of a Q-state neural network model - *J. Phys. Math. Gen.,* Vol.22, pp. 2057-2067, 1989.
3. V. I. Arnold, Additional chapters of the theory of ODE, - Moscow, Nauka, 1978, 304p., (in Russian).
4. J. Bruck, V P. Roychowdhury, On the number of spurious memories in the Hopfield model, - *IEEE Trans. on IE*, v. 36, No 2, 1990, pp.393-397.
5. E. R. Caianiello, M. Ceccarelli, M. Marinaro, Can spurious states be useful? - *Complex Systems*, Vol.6, 1992, pp.1-12.