Oscillatory Networks with Hebbian Matrix of Connections

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The systems of symmetrically coupled limit cycle oscillators admit the design of recurrent associative memory networks with Hebbian matrix of connections. Unlike the similar neural networks this matrix proved to be the complex-valued Hermitian one with nonzero diagonal. In the case of strong interaction in oscillatory system the memory vectors of the network are slightly perturbed properly normalized eigenvectors of matrix of connections. They can be calculated by perturbation method on the appropriate small parameter. The self-consistent analysis of dynamical system fixed points in the case of homogeneously all-to-all connected oscillators is presented. It is proved that for positive values of connection strength only a single memory vector can be stored. Some questions concerning the "extraneous " memory of the networks are discussed.

1 Introduction

Large systems of coupled oscillators [1-4] in the regime of synchronization (phase locking) have an ability to memorize information [5-8]. So the problem of neural oscillatory system of associative memory design arises. The design includes determination of matrix of connections and proper choice of other modifiable parameters of the corresponding dynamical system to provide effective retrieval characteristics of the network.

One of the most attractive features of oscillatory models is undoubtedly possible numerous physical implementations. In contradiction with well known optoelectronic and nonlinear optical implementations based on the idea of vector-matrix multiplier oscillatory models promise direct - and by this reason much more effective - implementations. When one analyzes neural network implementations based on photon-echo effect [9], it becomes clear that the potentialities of this effect that have been used so far are exceedingly greater than those already used in the known schemes.

As for theoretical study of oscillatory systems from the viewpoint of associative memory modeling there is a number of various ways of the modeling based on systems of coupled oscillators. One of them is the encoding of memory patterns by two subpopulations of oscillatory system in the vicinity of phase transition into synchronized state - subpopulations of synchronized and unsynchronized states - that has been developed in a number of works (see, for instance, [5]).

The modeling of *recurrent* associative memory oscillatory network in the state of *synchronization* is still at the very beginning [6-8]. Up to now only the special kind of oscillatory system with the simplest kind of interaction - limit-cycle interacting oscillators with linear interactions of pairs - have been studied.

It has been found out that such special oscillatory systems are closely related to the systems of magnetic spins on the plane (clock spin glasses or phasor systems). The associative memory networks with Hebbian matrix of connections have been designed for the clock spin systems and the phase transition of memory "overloading", permitting to obtain the retrieval characteristics of the network, has been analyzed [10]. However, the important problem of "extraneous" memory for clock spin networks is not studied at all.

An attempt has been done to study phasor networks with asymmetrical complex-valued matrix of connections and non-zero thresholds (clearly, the Hopfield model is imbedded into it). This model is the natural generalization of the phasor network model studied in [14]. The further study of this model would be quite desirable.

As far as we know, the present work is the first attempt to design and to begin study of recurrent associative memory oscillatory network with Hebbian matrix of connections.

2 The Dynamical Equations of the Model of Phase Oscillators.

We consider the system of N limit-cycle oscillators on the plane with symmetrical nonhomogeneous coupling, the state of each being defined as a complex-value a point $z_j = r_j exp(i\theta_j)$ of complex plane. In appropriate parametric domain the dynamical system governing the dynamics of oscillatory system can be reduced to "phase" dynamical system

$$\dot{\theta}_j = \omega_j + K \sum_{k=1}^{N} \mathcal{W}_{jk} sin(\theta_k - \theta_j + \beta_{jk}), \quad j = 1, ..., N. \tag{1}$$

where $\omega_j, \quad j = 1, ...N$, are the natural frequencies on the cycles and complex-valued Hermitian $N \times N$ matrix $W = [W_{jk}] = [\mathcal{W}_{jk} exp(i\beta_{jk})], \quad W = \bar{W}^\top \equiv W^+$ specifies the weights of connections of oscillators in the network, the real value K defines the absolute value and the sign of interaction strengths in the system [10].

The dynamical system (1) defines the model of system of "phase oscillators" which corresponds to the approximation that interaction of network oscillators does not influence on the amplitudes of oscillations, the last being constant. So, the state vector of the network of phase model is

$$z = (z_1, ..., z_N)^\top, \quad z_j = exp(i\theta_j),$$

Note that the matrix W in (1) should not have the zero diagonal in difference with the case of neural networks. This is just the consequence of the form of representation of "operator" of interaction of amplitude-phase dynamical system that is reduced to phase system (1).

Any Hermitian matrix W can be represented in a form

$$W = N^{-1}\sum_{m=1}^{M}\lambda^m V^m (V^m)^+, \quad M = rankW, \tag{2}$$

where $\{V^m\}$ is the set of mutually orthogonal eigenvectors of W corresponding to the set of its nonzero eigenvalues [12]:

$$WV^m = \lambda^m V^m, \quad (V^s)^+ V^m = N\delta_{ms}, \quad m = 1, ...N. \tag{3}$$

where δ_{ms} denotes the Kronecker symbol. With the help of expansion (2) the dynamical system (1) can be rewritten in the form

$$\dot{\theta}_j = \omega_j + (K/N)\sum_{m=1}^{M}\sum_{k=1}^{N}\lambda^m sin([\theta_k - \beta_k^m] - [\theta_j - \beta_j^m]). \tag{4}$$

One more form of system (1) can be obtained if one uses the expansion of state vector z in eigenbasis $\{V^m\}$ of matrix W

$$z = \sum_{m=1}^{M} Z^m V^m, \quad Z^m = N^{-1}(V^m)^+ z = N^{-1}\sum_{j=1}^{N} exp(i[\theta_j - \beta_j^m]) = R^m exp(i\psi^m), \tag{5}$$

The variables Z^m, the inner products of current state vector z and the basis vectors V^m, are the macrovariables (in the case of high dimension N of the dynamical system). They are just the "overlaps" which are usually used in asymptotical analysis of retrieval characteristics of associative memory neural networks. For the case of oscillatory networks the "overlaps" have the additional sense: the "order parameters", governing the phase transition of oscillatory system into synchronized state.

Being rewritten in terms of macrovariables $Z_m R^m exp(i\psi^m)$, the system (1) has the form of N independent equations.

$$\dot{\theta}_j = \omega_j + K\sum_{m=1}^{M}\lambda^m R^m sin(\psi^m + \beta_j^m - \theta_j), \tag{6}$$

System (3) provides the "self-consistent field" description of oscillatory network. It proved to be very convenient for the analysis of fixed points of the phase dynamical system.

3 The "Hebbian" Solution to Associative Memory Network Design Problem

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As it is very well known from the theory of associative memory neural networks, the matrix of connections that is the sum of outer products by orthogonal set of memory vectors just provides the simplest solution to network design problem. The outer-product matrices of connections themselves are usually regarded as "Hebbian" because of the relation to Hebbian learning algorithm.

As it follows from (2),(3), the "Hebbian" solution to the associative memory design problem exists for the model of system of "phase oscillators". More exactly, we have the following result.

I. The case $\omega_j = 0$ (phasor networks).

Let the set of M, $M \le N$, of linearly independent vectors $\{\mathcal{V}^m\}$ is given as $\mathcal{V}_j^m = exp(i\beta_j^m)$ is given. Then the "Hebbian" solution to the problem can be realized in the following steps:

- Find the orthogonal system of vectors $\{V^m\}$, corresponding to the set $\{\mathcal{V}^m\}$.
- Define the matrix W by formulas (2),(3), where λ^m are some real values.

Then vectors V^m are just the stable fixed points of phase dynamical system (4). Thus, the memory vectors of oscillatory network coincide exactly with V^m .

The question of proper choice of $\{\lambda^m\}$ should be the subject of special analysis. It's worth to recall from the theory of neural networks with Hebbian matrix of connections that there is serious disadvantage in the choice of equal $\{\lambda^m\}$. Such a choice just leads to drastically great extraneous memory (this is quite natural from the viewpoint of linear algebra). So the choice of close, but different $\{\lambda^m\}$ seems to be preferable.

The existence of "extraneous" memory should be also a subject of special analysis. In any case, it is very plausible that the "extraneous" memory exists in the subspace $kerW$ of complex unitary space of network state vectors [12].

II. The case $\omega_j \neq 0, \quad \sum_{j=1}^{N}\omega_j = 0$. It can be shown that at arbitrary ω_j in the case of "strong" interaction in the oscillatory network there exist the set $\tilde{V}^m$, close to V^m, which is the set of memory vectors of oscillatory network with the matrix of connections defined by formulas (2),(3).

To formulate the result more exactly, first of all note that the parameter $\gamma = \Omega/K$, where $\Omega = max_j|\omega_j|$, is the essential parameter of the system. For instance, the simple sufficient condition of synchronization of oscillatory network is $\gamma \le 1$. The case of $\omega_j = 0$ can be considered as the limit case of infinitely strong interaction of oscillators in the network ($K \to \infty$). When K is great, but finite value, γ is the small parameter, and the perturbation method for the system of equation defining the fixed points of network dynamics can be derived. It is just the system (6) that proved to be the most convenient for this purpose. The perturbation method provides the following result.

At sufficiently small $\epsilon = \gamma = \Omega/K$ the memory vectors $\tilde{V}^m$ of oscillatory network belong to small vicinities of vectors V^m and the following estimations take place:

$$\tilde{V}_j^m = V_j^m + \epsilon(\lambda^m)^{-1}\omega_j + O(\epsilon^2), m = 1, ...M, j = 1, ...N. \tag{7}$$

These facts permit to conclude that the retrieval characteristics of clock spin networks (phasor networks) obtained in [10] (storage capacity $\alpha \sim 0.037$, the limit value of "overlap" equals to ~ 0.9) are simultaneously the retrieval characteristics of oscillatory networks in the case of strong interaction.

The fact that the memory vectors of the network under strong interaction are slightly perturbed eigenvectors of matrix W is confirmed in computer experiments. The last ones also show that in the process of further gradual increase of γ the stable fixed points $\tilde{V}^m$ disappear one after another. This process stops at $\gamma = \gamma^*$ where γ^* is the threshold of synchronization. In small vicinity of γ^* the dynamical system (4) has a single stable fixed point. So, in principle, the parameter γ can be used as the parameter controlling the memory storage abilities of oscillatory networks.

4 The Oscillatory Networks Containing a Single Memory Vector

The oscillatory networks with homogeneous all-to-all connections can be regarded as the networks containing a single memory vector. Indeed, their matrices of connections $[W_{jk}] = N{-}1$ contain only the single term of the expansion (2):

$$W = N^{-1}VV^{\top}, \quad V = (1, ...1)^{\top}, \tag{8}$$

and the dynamical system in the form (6), which one is especially simple, can be written as:

$$\dot{\theta}_j = \gamma\tilde{\omega}_j + Rsin(\psi - \theta_j), \tag{9}$$

where

$$Z = N^{-1} < z, V >= N^{-1}\sum_{j=1}^{N} exp(i\theta_j) = Rexp(i\psi), \tag{10}$$

and $\tilde{\omega}_j \equiv \omega_j/\Omega$. Note that in this case the single macrovariable Z coincides with well known order parameter which was elsewhere used for investigation of phase transition into synchronized state for the systems of uniformly all-to-all coupled phase oscillators [1-4]. The functional self-consistent equation for Z in closed form, which together with the equations for fixed points

$$\gamma\tilde{\omega}_j + Rsin(\psi - \theta_j) = 0 \tag{11}$$

delivers the self-consistent analysis of network equilibria, can be obtained. The self-consistent analysis shows that the phase ψ can be chosen arbitrary (the consequence of rotational symmetry of the system) and the self-consistency equation for R can be written in the forms:

$$R^2 = N^{-1}\sum_{j=1}^{N}(R^2 - \gamma^2\tilde{\omega}_j^2)^{1/2}, \tag{12}$$

or

$$\gamma = N^{-1}u\sum_{j=1}^{N}(1 - \tilde{\omega}^2u^2)^{1/2}, \quad u \equiv \gamma/R. \tag{13}$$

The analysis of fixed points of phase dynamical system on the base of (11), (12) and (13) gives the following results (for $K > 0$):

1. There exists the single stable fixed point of the network $\bar{z} = \tilde{V}$. At $\gamma = 0 \quad \tilde{V} = V$, at $\omega_j \neq 0$ $\tilde{V}$ is slightly perturbed V at small γ in accordance with (7). Gradual increase of γ leads to greater deviations of $\tilde{V}$ from V. The point $\tilde{V}$ exists up to $\gamma = \gamma^*$ being the threshold of synchronization. The later can be calculated exactly from (13) and in the case of three oscillator network is equal to .588.

2. The condition $R = 0$ defines N unstable fixed points of the network that are symmetrical ones and can be complicated equilibrium states of the dynamical system.

5 Concluding Remarks

The following results have been obtained.

- The associative memory oscillatory network with Hebbian Matrix of connections is designed. In the case of strong oscillatory interaction ($\gamma = \Omega/K$ is small) the memory vectors of the oscillatory network are slightly perturbed properly normalized eigenvectors of matrix of connections. The retrieval characteristics (in the same case) coincide with those ones of a clock spin network [10]

- An example of self-consistent analysis of total fixed points of the network is given in the case of the network, containing a single memory vector (of the network with uniform all-to-all connections).

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